From flat pencils to Frobenius manifolds and hyperplane arrangements

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§1 Local Hamiltonian structures

Dubrovin Novikov Hamiltonian structures

$$F = \int_{S^1} f(\underline{u})\, dx \,, \quad G = \int_{S^1} g(\underline{u})\, dx$$ local functionals (hydrodynamic)

$\underline{u} = (u^1, \text{—}, u^n)$

$$\{F, G\} = \int_{S^1} \frac{\delta F}{\delta u^i} \Big[\underbrace{g^{ij} \frac{d}{dx} - g^{is} \Gamma^j_{sk} u^k_x}_{A^{ij}} \Big] \frac{\delta G}{\delta u^j}\, dx$$

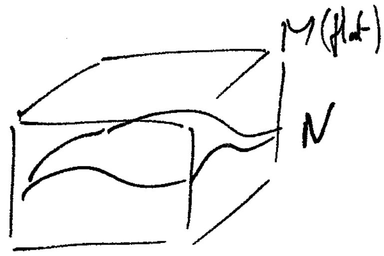
Hamiltonian [assume $\det g^{ij} \neq 0$ throughout]

- g^{ij} - symmetric tensor - defines a (pseudo) Riemannian metric
- Γ - Levi Civita connection of g
- Riem $= 0$

[so there exists a distinguished coordinate system - flat coordinates in which $g^{ij} \sim$ constants (so $\Gamma = 0$)]

Constraint to submanifold : Dirac reduction (Ferapontov)

$$A^{ij} \longrightarrow \underbrace{A^{ij}}_{\text{with induced connection}} \underbrace{+ \sum_{\alpha} w^{i}_{\alpha q} u^{q}_{x} \left(\nabla^{\perp}\right)^{-1} w^{j}_{\alpha q} u^{q}_{x}}_{\text{non-local tail}}$$



(g, Γ, w) satisfy : g, Γ - induced metric and connection on N

w - Weingarten maps

$$\text{Riem} = ww - ww \qquad \left[\text{Gauss - Codazzi eq}^{\text{n}}\text{s} \right]$$

where $\nabla^{\perp}$ is normal connection

<u>Outline of talk</u>   <u>local</u>   <u>non-local</u>

<u>Dubrovin '98</u>

bi-Hamiltonian structure of DN type ⟶ (extra conditions) ⟶ Frobenius manifold

mirror construction of Dubrovin - - - ⟶

non-local bi-Hamiltonian structure of above type ⟶ $\boxed{?}$

(joint with Liana David)

<u>N.B.</u> no flat metrics, so no distinguished coordinate systems to simplify calculations

Notation

Two non-local structures $(g, \nabla, R) \longmapsto \{\, ,\}$

$(\tilde{g}, \tilde{\nabla}, \tilde{R}) \longmapsto \{\, ,\}^{\sim}$

want $\{\, ,\} + \lambda \{\, ,\}^{\sim}$ to be Hamiltonian:

Define $g_{(\lambda)}^{-1} = g^{-1} + \lambda \tilde{g}^{-1}$ (note g^{ij} appears in $\{,\}$, not g_{ij})

with associated ${}^{(\lambda)}\nabla$, ${}^{(\lambda)}R$.

Def: (Mokhov) – almost compatible iff: $g_{(\lambda)}^{-1}\, {}^{(\lambda)}\nabla_X \alpha = g^{-1} \nabla_X \alpha + \lambda \tilde{g}^{-1} \tilde{\nabla}_X \alpha$

– compatible iff, in addition

$$g_{(\lambda)}^{-1}\, {}^{(\lambda)}R_{X,Y}\alpha = g^{-1} R_{X,Y}\alpha + \lambda \tilde{g}^{-1} \tilde{R}_{X,Y}\alpha$$

$\forall\, X, Y \in TM$

$\forall\, \alpha \in T^*M$

$\forall\, \lambda$

Result: (Mokhov) • almost compatible $\Longrightarrow N_A = 0$, Nijenhuis tensor on $A = \tilde{g}^{-1} g$

• semi-simple (eigenvalues of A distinct) a.c. $\Rightarrow$ c

From now on do not assume any special coordinate systems: everything coordinate free.

§2 Multiplication on T^*M

Aside

Old idea [Vaisman '66] : define multiplication in terms of difference of two connections

$$X * Y = \nabla_X Y - \tilde{\nabla}_X Y$$

relate algebraic properties of $*$ with geometric properties of $(\nabla, \tilde{\nabla})$

e.g. $*$ commutative $\iff$ $T^\nabla = T^{\tilde{\nabla}}$ (same torsion)

$*$ associative $\iff$ curvature conditions.

Here the natural multiplication is on T^*M

Def: [Dubrovin] $\alpha \circ \beta = \nabla_{g^{-1}\alpha}(\beta) - \tilde{\nabla}_{g^{-1}\alpha}(\beta)$

N.B. g^{-1}, not $\tilde{g}^{-1}$

Thm a) $g^{-1}(\alpha \circ \beta, \gamma) = g^{-1}(\alpha, \gamma \circ \beta)$ always

b) if $g, \tilde{g}$ almost compatible: $\tilde{g}^{-1}(\alpha \circ \beta, \gamma) = \tilde{g}^{-1}(\alpha, \gamma \circ \beta)$

c) if $g, \tilde{g}$ compatible $(\beta \circ \gamma) \circ \alpha = (\beta \circ \alpha) \circ \gamma$

$\forall \alpha, \beta, \gamma \in T^*M$

Let $C_\alpha(\beta) = \beta \circ \alpha$:

c)' if $g, \tilde{g}$ compatible $\iff$ $[C_\alpha, C_\beta] = 0$

(c.f. notation of a Higgs field).

§3 Homogeneity conditions

Without further conditions, $[C, C] = 0$ is all one gets.

<u>Add weak homogeneity</u> — $E \in TM$ s.t.

$$\mathcal{L}_E\, g = (1-d)\, g$$

$$\mathcal{L}_E\, \tilde{g} = D\, \tilde{g}$$

<u>Def:</u> $T(u) = \left(\frac{d-1}{2}\right) u + u \hat{\nabla} E$ an automorphism of T^*M s.t. $T(u) = g(E) \circ u$

[Difficult to motivate: In Frobenius manifold case, a natural operator (cf. isomonodromy problems) Idea works still in <u>curved</u> situation]

with this <u>extra</u> structure:

Define $u \bullet v = u \circ T^{-1}(v)$ and use $\tilde{g}$ to define multiplication on TM

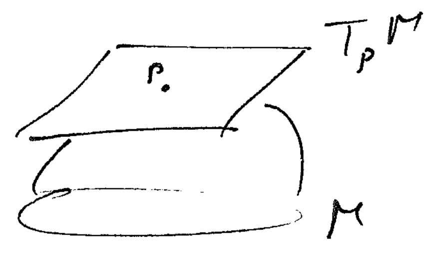
then

a) $\bullet$ associative, commutative multiplication with unity $e = g(E)$

b) $\tilde{g}(X \bullet Y, Z) = \tilde{g}(X, Y \bullet Z)$

(3,0) tensor totally symmetric

$\Bigg\}$ T_pM a <u>Frobenius algebra</u>



<u>and</u>

c) define (4,0) tensor $\tilde{\nabla}(\circ)(X,Y,Z,V) = \tilde{g}(\nabla_X(\circ)(Y,Z), V)$

<u>then</u> $\tilde{\nabla}(\circ)(X,Y,Z,E) = \tilde{\nabla}(\circ)(E,X,Y,Z)$

Euler

<u>N.B.</u> All known examples have:

(c′) $\tilde{\nabla}(\circ)$ <u>totally</u> symmetric (4,0) tensor

and hence, by a result of Hertling, get an F-manifold structure.

(<u>N.B.</u> no flatness conditions used).

To get further conditions (and so edging towards a Frobenius manifold)

homogeneity = weak homogeneity + $\mathcal{L}_e \overset{-1}{g} = \overset{\sim -1}{g}$

$$\mathcal{L}_e \overset{\sim -1}{g} = 0$$

$$[e, E] = e$$

<u>Then</u> $\tilde{\nabla} e = 0$, $\mathcal{L}_E(\circ) = \circ$

<u>Finally</u> can impose $R = \tilde{R} = 0$ to recover Frobenius manifold

<u>So?</u> Many of the key features of a Frobenius manifold (ass, com. multiplication compatible with metric, symmetry etc) are <u>independent</u> of the notion of flatness – and hence locality of Hamiltonian structures. Such structures are coming from compatibility / bi-Hamiltonian structure <u>not</u> from flatness.

§4 Examples

Two classic ways to introduce curvature:

→ conformal curvature;

→ submanifold theory:

Basic example of Frobenius manifolds on orbit space [c.f. K. Saito '80]

$V = \mathbb{R}^N$, W - Coxeter group

Construct W-invariant objects: $g = \sum dz_i^2$ invariant metric

$s_i(\underline{z})$ invariant polynomials

Chevalley Th: $\mathbb{C}^W[\underline{z}] = \mathbb{C}[s_i]$

Euler field $E = \sum_i z^i \frac{\partial}{\partial z^i} = \frac{1}{h} \sum d^i s^i \frac{\partial}{\partial s^i}$

Define $e = \frac{\partial}{\partial s^1}$ [s^1 has maximal d^i]

Define $\mathcal{L}_e g^{-1}$

Thm (K. Saito) • $\mathcal{L}_e g^{-1}$ non-degenerate. Define $\tilde{g}^{-1} = \mathcal{L}_e g^{-1}$.

• $\tilde{g}$ flat

So get a flat pencil $g^{-1} + \lambda \tilde{g}^{-1}$ and hence local biHamiltonian structure

homogeneous etc. $\Rightarrow$ Frobenius manifold on: $V \otimes \mathbb{C} / W$

[s^i not quite flat coordinates]

Conformal curvature : Let $h = \Omega^2(\underline{t})\, g$
$\tilde{h} = \Omega^2(\underline{t})\, \tilde{g}$

Thm • $(h, \tilde{h})$ compatible

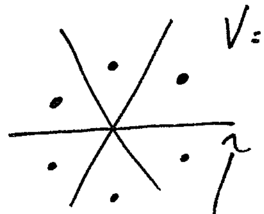
• weak homogeneity $\Rightarrow \Omega = \Omega(t^n)$

• $\Omega(t) = \frac{1}{ct+d}$: $\tilde{h}$ - flat
h - constant sectional curvature $4cd$.

Thus get local / nonlocal Hamiltonian pair.

Submanifold Discriminant $\mathcal{D}$.

In general:



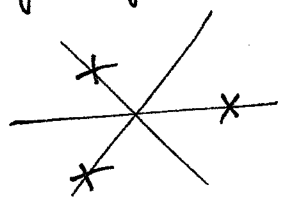
$V = \mathbb{R}^2$

$W = I_2(3) \simeq A_2$

orbit (finite) of a point •

reflection hyperplane

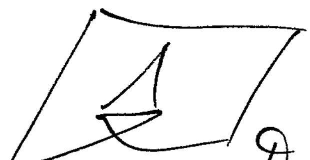
However, special point has a lower number of reflections



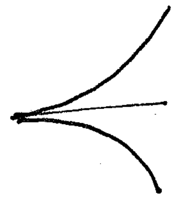
orbit irregular.

$\mathcal{D}$ – manifold of irregular orbits: union of its reflection hyperplanes in V

In orbit space $V \otimes \mathbb{C} / W$:



$\mathcal{D}$,



, etc.

Can restrict Frobenius manifold structure onto $\mathcal{D}$:

$g|_{\mathcal{D}}$ is flat [g restrict to a plane]

$\tilde{g}|_{\mathcal{D}}$ curved [c.f. above picture]

This gives a local / nonlocal biHamiltonian structure:

Since $\mathcal{D}$ has flat normal connection $\nabla^{\perp} = d_x$.

Hyperplane arrangement of a discriminant configuration given by a set, finite, of vectors $\mathcal{U} = \{ \alpha^i \}$, where α^i are the normals to the reflection hyperplanes (all planes go through origin).

Geometrically: everything determined by a finite set of vectors.

Question What hyperplane arrangements (or what algebraic conditions on the set $\mathcal{U}$) give rise to a biHamiltonian structure?

Related question [Veselov]: What algebraic conditions on the set $\mathcal{U}$ are required so that the function

$$F = \frac{1}{4} \sum_{\alpha \in \mathcal{U}} (\alpha . z)^2 \log (\alpha . z)$$

[c.f. almost dual Frobenius manifold]

defines a solution to the WDVV eq^{n}s? The conditions are known as V-conditions.

Examples : $\mathcal{U} : R_W \leftarrow$ root system of a Coxeter group

$\mathcal{U}$: Discriminant arrangement.

How is the notion of a V-system related to biHamiltonian structures? if at all!

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