Central extensions of universal hierarchy: (2+1)-dimensional bi-Hamiltonian systems

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(2+1)-dimensional 'universal' system

Consider a (2+1)-dimensional system of hydrodynamic type

$$u_t = \partial_x^{-1} u_{yy} + u u_y - u_x \partial_x^{-1} u_y, \tag{1}$$

or if $u = w_x$: $w_{xt} - w_{yy} + w_{xx} w_y - w_x w_{xy} = 0$.

Pavlov (2003), Martinez Alonso & Shabat (2003)

Lax representation for (1) is

$$l_t = [\Omega, l] + \Omega_y \qquad\qquad [a, b] = a b_x - a_x b$$

where

$$l = \lambda + u \qquad \Omega = \lambda^2 + u\lambda + \partial_x^{-1} u_y.$$

Ferapontov & Khusnutdinova, Dunajski, Manakov & Santini, Grant, Strachan

Hereditary recursion operator: $\Phi = u_x \partial_x^{-1} - u + \partial_y \partial_x^{-1} \quad u_{t_n} = \Phi^n u_x$.

(*no-go theorem Zakharov & Konopelchenko (1984)*)

Motivation

- Consider the semidirect product of the Virasoro algebra and the dual and central extensions of this product. Then the Euler equation represents a (2+1)-dimensional bi-Hamiltonian system which contains the system (1) as a sub-system.
  Ovsienko & Roger (2007)
- We consider the loop algebra over the Lie algebra of Ovsienko and Roger and central extensions of the former, but we use the approach in spirit of the paper by *A.G. Reyman & Semenov-Tian-Shansky (1988)*

Universal hierarchy

The universal hierarchy is a set of commuting flows of the form

$$G_{t_n} = [(\lambda^n G)_+, G], \quad n \in \mathbb{N}$$

where

$$G = 1 + g_1 \lambda^{-1} + +g_2 \lambda^{-2} + \dots$$

or

$$G = g_0 + g_1 \lambda^1 + g_2 \lambda^2 + \dots \; .$$

For $a = \sum_{i \in \mathbb{Z}} a_i \lambda^i$ we set $a_+ = \sum_{i \geq 0} a_i \lambda^i$, and the commutator $[a, b] = a b_x - b a_x$.

A finite-field reductions can be obtained upon setting

$$G = \frac{1}{\lambda^N} \prod_{i=1}^{N} (\gamma_i + \lambda).$$

Vector fields on the circle

Consider $\mathfrak{v} = \mathrm{Vect}(\mathbb{S}^1)[\![\lambda, \lambda^{-1}]\!]$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then,

$$[a, b] \equiv \mathrm{ad}_a b := ab_x - ba_x \qquad a, b \in \mathfrak{v}.$$

We have the standard pairing of $\mathfrak{v}$ and $\mathfrak{v}^*$

$$\langle u, a \rangle = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \mathrm{res}\,(au)\ dxdy \qquad u \in \mathfrak{v}^* \quad a \in \mathfrak{v},$$

where $\mathrm{res} \sum_i \varphi_i \lambda^i := \varphi_{-1}$.

Thus

$$\langle \mathrm{ad}_a^* u, b \rangle = -\langle u, \mathrm{ad}_a b \rangle \quad \Rightarrow \quad \mathrm{ad}_a^* u = 2a_x u + a u_x.$$

Adjoint invariance

Consider the Lie-Poisson bracket on some Lie algebra $\mathfrak{g}$

$$\{H, F\}(\eta) = \langle \eta, [dF, dH]_R \rangle ,$$

where $\eta \in \mathfrak{g}^*$, $H, F \in \mathcal{C}^\infty(\mathfrak{g}^*)$ and $dH, dF \in \mathfrak{g}$.

Then the Hamiltonian equations take the form

$$\eta_t = \mathrm{ad}^*_{RdH}\eta.$$

One can identify ad^* with ad if there exists a symmetric nondegenerate bilinear form $(\cdot,\cdot) : \mathfrak{v} \times \mathfrak{v} \to \mathbb{K}$ which is ad-invariant, i.e.

$$(a, [b, c]) + ([b, a], c) = 0.$$

Semi-direct product of $\mathfrak{v}$ and $\mathfrak{v}^*$

Let $\mathfrak{w} = \mathfrak{v} \ltimes \mathfrak{v}^*$, with the Lie bracket defined by

$$[(a,u),(b,v)] := \left([a,b]\,, \mathrm{ad}^*_a v - \mathrm{ad}^*_b u\right), \tag{2}$$

where $a, b \in \mathfrak{v}$ and $u, v \in \mathfrak{v}^*$.

There is a natural nondegenerate bilinear symmetric product on $\mathfrak{w}$

$$((a,u),(b,v))_{\mathfrak{w}} = \langle v, a\rangle + \langle u, b\rangle = \int_{\mathbb{S}^1\times\mathbb{S}^1} \mathrm{res}(av + bu)\, dx dy$$

which is ad-invariant with respect to (2).

Classical R-matrices

There is a natural decomposition of $\mathfrak{w}$ into Lie subalgebras

$$\mathfrak{w} = \mathfrak{w}_+ \oplus \mathfrak{w}_-,$$

where $(\sum_i \varphi_i \lambda^i)_+ = \sum_{i \geqslant 0} \varphi_i \lambda^i$ and $(\sum_i \varphi_i \lambda^i)_- = \sum_{i<0} \varphi_i \lambda^i$.

Thus, we can define the classical R-matrix on $\mathfrak{w}$ as

$$R = \frac{1}{2}(P_+ - P_-) = P_+ - \frac{1}{2} = \frac{1}{2} - P_-,$$

which defines an additional Lie bracket on $\mathfrak{w}$

$$[\boldsymbol{u}, \boldsymbol{v}]_R := [R\boldsymbol{u}, \boldsymbol{v}] + [\boldsymbol{u}, R\boldsymbol{v}]\,.$$

Infinite family of R-matrices

Actually, we have an infinite family of classical R-matrices

$$R_n = R\lambda^n \qquad n \in \mathbb{Z} \tag{3}$$

and the corresponding new Lie brackets on $\tilde{\mathfrak{w}}$

$$[\boldsymbol{u}, \boldsymbol{v}]_{R_n} := [R_n \boldsymbol{u}, \boldsymbol{v}] + [\boldsymbol{u}, R_n \boldsymbol{v}] \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathfrak{w}. \tag{4}$$

The R-matrices (3) and the Lie brackets (4) are well-defined since λ^n is a so-called intertwining operator, i.e. satisfy

$$\lambda^n [\boldsymbol{u}, \boldsymbol{v}] = [\lambda^n \boldsymbol{u}, \boldsymbol{v}] = [\boldsymbol{u}, \lambda^n \boldsymbol{v}].$$

Cotangent universal hierarchy

Let $\boldsymbol{l}$ be an element of $\mathfrak{w} = \mathfrak{v} \ltimes \mathfrak{v}^*$, i.e.

$$\boldsymbol{l} = (l_1, l_2) = \sum_{i \in \mathbb{Z}} (u_i, v_i) \lambda^i.$$

Now we can write down the *cotangent universal hierarchy*

$$\boldsymbol{l}_{t_n} = [R(\lambda^n \boldsymbol{l}), \boldsymbol{l}] = [(\lambda^n \boldsymbol{l})_+, \boldsymbol{l}], \qquad n = 0, 1, 2, \dots \quad . \tag{5}$$

In the component form we can write (5) as

$$\begin{aligned} (l_1)_{t_n} &= [(\lambda^n l_1)_+, l_1] \\ (l_2)_{t_n} &= \mathrm{ad}^*_{(\lambda^n l_1)_+} l_2 - \mathrm{ad}^*_{l_1} (\lambda^n l_2)_+. \end{aligned}$$

Central extension

Let $\mathfrak{g}$ be an arbitrary Lie algebra with respect to $[\cdot,\cdot]$.

Consider $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$.

The bracket

$$[(a,\eta),(b,\xi)] = ([a,b],\omega(a,b)),$$

where $a,b,c \in \mathfrak{v}$, $\eta,\xi \in \mathbb{K}$, for a bi-linear form $\omega : \mathfrak{v} \times \mathfrak{v} \to \mathbb{C}$, is a Lie bracket on $\tilde{\mathfrak{v}}$ provided ω is two-cocycle, i.e.

- $\omega(a,b) = -\omega(b,a)$
- $\omega([a,b],c) + \omega([b,c],a) + \omega([c,a],b) = 0$.

Then one can consider Lie-Poisson bracket on $\mathfrak{g} \equiv \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \{\alpha\}$

$$\{H,F\}(\eta) = \langle \eta, [dF,dH]_R \rangle + \alpha\, \omega(a,b),$$

where $\alpha \in \mathbb{K}$.

Central extensions

In analogy with *Ovsienko & Roger* define the following two-cocyles on $\mathfrak{w}$.

- An analogue of the Maurer-Cartan two-cocycle:

$$\omega_1\left((a,u),(b,v)\right) = \left((a,u),(b,v)_y\right)_{\mathfrak{w}} = \int_{\mathbb{S}^1\times\mathbb{S}^1} \operatorname{res}(av_y - bu_y)\, dxdy.$$

- The generalization of the Gelfand-Fuchs two-cocycle:

$$\omega_2\left((a,u),(b,v)\right) = \left((a,u),(0,b)_{3x}\right)_{\mathfrak{w}} = \int_{\mathbb{S}^1\times\mathbb{S}^1} \operatorname{res}(ab_{3x})\, dxdy.$$

Centraly extended cotangent universal hierarchy

Let $\boldsymbol{l} = (l_1, l_2) = \sum_i (u_i, v_i)\lambda^i$ be some element from $\mathfrak{w} = \mathfrak{v} \ltimes \mathfrak{v}^*$.

The Lax hierarchy of pairwise commuting flows is

$$(l_1)_{t_q} = [(\Omega_1^q)_+, l_1] + \alpha\partial_y(\Omega_1^q)_+$$
$$(l_2)_{t_q} = \mathrm{ad}^*_{(\Omega_1^q)_+} l_2 - \mathrm{ad}^*_{l_1}(\Omega_2^q)_+ + \alpha\partial_y(\Omega_2^q)_+ + \beta\partial_x^3(\Omega_1^q)_+,$$

where $(\sum_i \varphi_i\lambda^i)_+ = \sum_{i\geqslant 0}\varphi_i\lambda^i$ and α, β are constants.

The Lax hierarchy is generated by $\Omega_q = (\Omega_1^q, \Omega_2^q)$ satisfying

$$[\Omega_1^q, l_1] + \alpha(\Omega_1^q)_y = 0$$
$$\mathrm{ad}^*_{\Omega_1^q} l_2 - \mathrm{ad}^*_{l_1}\Omega_2^q + \alpha(\Omega_2^q)_y + \beta(\Omega_1^q)_{3x} = 0.$$

Functionals on $\mathfrak{w}$.

The space $\mathcal{C}^\infty(\mathfrak{w}^* \cong \mathfrak{w})$ consists of functionals

$$H(\boldsymbol{l}) = \int_{\mathbb{S}^1\times\mathbb{S}^1} h\left(\ldots, \vec{u}_{xy}, \vec{u}_x, \vec{u}_y, \vec{u}, \partial_x^{-1}\vec{u}_y, \partial_y^{-1}\vec{u}_x, \ldots\right)\, dxdy,$$

where $\vec{u}$ includes all fields from $\boldsymbol{l} = (l_1, l_2) = \sum_i (u_i, v_i)\lambda^i \in \mathfrak{w}$.

Then, the differential dH reads

$$\tilde{\mathfrak{w}} \ni dH = \left(\frac{\delta H}{\delta l_2}, \frac{\delta H}{\delta l_1}\right) = \sum_i \left(\frac{\delta H}{\delta v_i}, \frac{\delta H}{\delta u_i}\right)\lambda^{-i-1},$$

such that

$$(\boldsymbol{l}_t, dH)_{\tilde{\mathfrak{w}}} = \int_{\mathbb{S}^1\times\mathbb{S}^1} \sum_i \left((u_i)_t \frac{\delta H}{\delta u_i} + (v_i)_t \frac{\delta H}{\delta v_i}\right)\, dxdy.$$

Natural Lie-Poisson bracket

The natural Lie-Poisson bracket on $\mathcal{C}^\infty(\mathfrak{w})$ is

$$\{H, F\}(\boldsymbol{l}) = (\boldsymbol{l}, [dF, dH])_{\mathfrak{w}} + \alpha\, \omega_1 (dF, dH) + \beta\, \omega_2 (dF, dH)\,,$$

where $\boldsymbol{l} \in \mathfrak{w}$ and $H, F \in \mathcal{C}^\infty(\mathfrak{w})$.

Let $dH = (\delta_2 H, \delta_1 H)$, where $\delta_i H \equiv \frac{\delta H}{\delta l_i}$ for $i = 1, 2$.

Then the related Poisson tensor π, such that

$$\{H, F\} = (dF, \pi dH)_{\mathfrak{w}} \qquad \pi dH = ((\pi dH)_1, (\pi dH)_2)\,,$$

is given by the formulas

$$(\pi dH)_1 = [\delta_2 H, l_1] + \alpha(\delta_2 H)_y$$
$$(\pi dH)_2 = \mathrm{ad}^*_{\delta_2 H} l_2 - \mathrm{ad}^*_{l_1} \delta_1 H + \alpha(\delta_1 H)_y + \beta(\delta_2 H)_{3x}.$$

Family of new Lie-Poisson brackets

The R-brackets R_n induce a family of new Lie-Poisson brackets on $\mathcal{C}^\infty(\mathfrak{w})$

$$\{H,F\}_n(\boldsymbol{l}) = \left(\boldsymbol{l},[dF,dH]_{R_n}\right)_{\mathfrak{w}} + \alpha\,\omega_1^{R_n}(dF,dH) + \beta\,\omega_2^{R_n}(dF,dH)\,,$$

where

$$\omega_i^{R_n}(\boldsymbol{u},\boldsymbol{v}) := \omega_i(R_n\boldsymbol{u},\boldsymbol{v}) + \omega_i(\boldsymbol{u},R_n\boldsymbol{v}) \qquad \boldsymbol{u},\boldsymbol{v}\in\tilde{\mathfrak{w}} \qquad i=1,2. \tag{6}$$

The related Poisson tensors such that $\{H,F\}_n = (dF,\pi_n dH)_{\tilde{\mathfrak{w}}}$ are

$$(\pi_n dH)_1 = [R_n\delta_2 H, l_1] + R_n^*[\delta_2 H, l_1] + (R_n + R_n^*)\left(\alpha(\delta_2 H)_y\right)$$
$$(\pi_n dH)_2 = \mathrm{ad}^*_{R_n\delta_2 H} l_2 - \mathrm{ad}^*_{l_1} R_n\delta_1 H + R_n^*\left(\mathrm{ad}^*_{\delta_2 H} l_2 - \mathrm{ad}^*_{l_1}\delta_1 H\right)$$
$$+ (R_n + R_n^*)\left(\alpha(\delta_1 H)_y + \beta(\delta_2 H)_{3x}\right),$$

where $R_n^* = -\lambda^n R_n$.
All the Poisson tensors π_n are compatible.

Lax representation

The annihilators of natural Lie-Poisson bracket, such that $\pi\Omega_q = 0$ and $\Omega_q = (\Omega_1^q, \Omega_2^q)$, satisfy

$$0 = [\Omega_1^q, l_1] + \alpha(\Omega_1^q)_y$$

$$0 = \mathrm{ad}^*_{\Omega_1^q} l_2 - \mathrm{ad}^*_{l_1}\Omega_2^q + \alpha(\Omega_2^q)_y + \beta(\Omega_1^q)_{3x}.$$

Then Ω_q generated Lax hierarchy of pairwise commuting symmetries

$$(l_1)_{t_q} = [(\Omega_1^q)_+, l_1] + \alpha\partial_y(\Omega_1^q)_+$$

$$(l_2)_{t_q} = \mathrm{ad}^*_{(\Omega_1^q)_+} l_2 - \mathrm{ad}^*_{l_1}(\Omega_2^q)_+ + \alpha\partial_y(\Omega_2^q)_+ + \beta\partial_x^3(\Omega_1^q)_+.$$

Multi-Hamiltonian Lax hierarchy

Then the Lax hierarchy is multi-Hamiltonian

$$\boldsymbol{l}_{t_q} = \ldots = \pi_{-1} dH_{q+1} = \pi_0 dH_q = \pi_1 dH_{q-1} = \ldots.$$

The respective Hamiltonians H_q can be reconstructed from Ω_n by the homotopy formula

$$H_q(\boldsymbol{l}) = \int_0^1 (\boldsymbol{l}, dH_q(\mu\,\boldsymbol{l}))_{\mathfrak{w}}\, d\mu.$$

The Poisson tensors π_n form a proper subspace of $\mathfrak{w}$ with respect to (7) if $N \geqslant n \geqslant -m$ for $N \geqslant 0$ and if $0 \geqslant n \geqslant -m$ for $N = -1$.

Generic reductions

In the generic case the appropriate Lax operators have the form

$$\boldsymbol{l} = (u_N, v_N)\lambda^N + (u_{N-1}, v_{N-1})\lambda^N + \ldots + (u_{-m}, v_{-m})\lambda^{-m}, \quad (7)$$

where $N \geqslant -1$, $m \geqslant 0$ and u_N is nonzero constant and v_N is constant.

We are looking for the generating functions in the form

$$\Omega_q = \left(a_q\lambda^q + a_{q-1}\lambda^{q-1} + \ldots, b_q\lambda^q + b_{q-1}\lambda^{q-1} + \ldots\right),$$

where a_q and b_{q+k} are constants and $a_q \neq 0$.

Note that

$$\Omega_q = \lambda^q\Omega_0.$$

Example

Consider the Lax operator in the form

$$\boldsymbol{l} = (\lambda + u, v).$$

We have

$$\Omega_0 = \left(1 + u\lambda^{-1} + \alpha\partial_x^{-1}u_y\lambda^{-2} + A\lambda^{-3} + \ldots, v\lambda^{-1} + B\lambda^{-2} + C\lambda^{-3} + \ldots\right),$$

where

$$A = \alpha\partial_x^{-1}v_y + \beta u_{2x}$$

$$B = \alpha^2\partial_x^{-2}u_{2y} - 2\alpha\partial_x^{-1}(uu_y) + \alpha u\partial_x^{-1}u_y$$

$$C = \alpha^2\partial_x^{-2}v_{2y} + \alpha\partial_x^{-1}(uv)_y - 2\alpha u\partial_x^{-1}v_y + \alpha v\partial_x^{-1}u_y + 2\alpha\beta u_{xy} - \frac{1}{2}\beta u_x^2 - \beta uu_x.$$

Example

The first nontrivial member of the Lax hierarchy is

$$\begin{aligned} u_{t_2} &= \alpha^2 \partial_x^{-1} u_{yy} - \alpha u u_y + \alpha u_x \partial_x^{-1} u_y \\ v_{t_2} &= \alpha^2 \partial_x^{-1} v_{yy} + 2\alpha u_y v - \alpha u v_y - 2\alpha u_x \partial_x^{-1} v_y + \alpha v_x \partial_x^{-1} u_y \\ &\quad + 2\alpha\beta u_{xxy} - 2\beta u_x u_{xx} - \beta u u_{xxx}. \end{aligned}$$

If $u = w_x$ and $\alpha = -1$ the first equation is

$$w_{xt} = w_{yy} + w_x w_{xy} - w_{xx} w_y,$$

i.e. the system (1).

Example

The above two-field system is bi-Hamiltonian

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \pi_0 dH_2 = \pi_1 dH_1$$

with respect to the compatible Poisson tensors

$$\pi_0 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \qquad \pi_1 = \begin{pmatrix} 0 & \partial_x u - 2u\partial_x + \alpha\partial_y \\ -2\partial_x u + u\partial_x + \alpha\partial_y & \beta\partial_x^3 + \partial_x v + v\partial_x \end{pmatrix}$$

and Hamiltonians

$$H_1 = \int_{\mathbb{S}^1\times\mathbb{S}^1} \left(\alpha u\partial_x^{-1} v_y + \frac{1}{2}\beta u u_{xx}\right) dxdy$$

$$H_2 = \int_{\mathbb{S}^1\times\mathbb{S}^1} (\alpha^2 u\partial_x^{-2} v_{yy} + \alpha uv\partial_x^{-1} u_y - \alpha u^2\partial_x^{-1} v_y + \alpha\beta u u_{xy} - \frac{1}{4}\beta^2 u^2 u_{xx})\, dxdy.$$

Conclusions

We considered the loop algebra over the semi-direct product of $\mathrm{Vect}(\mathbb{S}^1)$ and its dual. Then we constructed the central extensions of the loop algebra in question.

- Using the above structures we suggested a systematic construction of a wide class of (2+1)-dimensional integrable bi-Hamiltonian systems.
- The Lax hierarchies we considered provide multi-component generalization of the well-known universal hierarchy.