Lie groups an Lie algebras

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Group $G = (\{g_i\}, \circ)$; $g_i \circ g_j \in G$ $\exists e \in G$

$e \circ g_i = g_i \circ e = g_i$, $\forall g_i \; \exists g_i^{-1}$, $g_i \circ g_i^{-1} = e = g_i^{-1} \circ g_i$

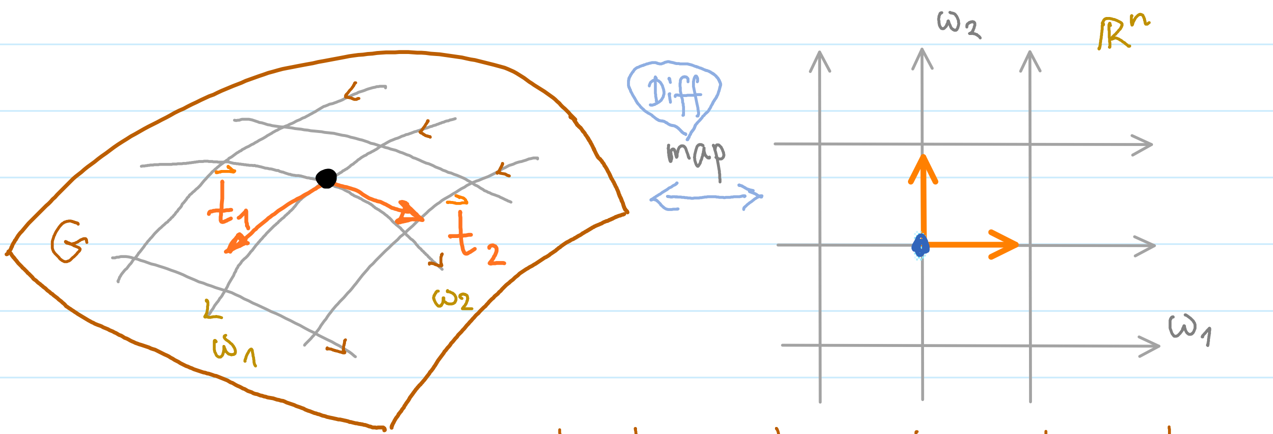
G is abelian if $\forall g_1, g_2 \in G$ $g_1 \circ g_2 = g_2 \circ g_1$

Lie group (G) is a group whose elements form smooth manifold and $\circ$ is smooth map!

- we shall call multiplication and usually suppress it

smooth manifold $n = \dim G$

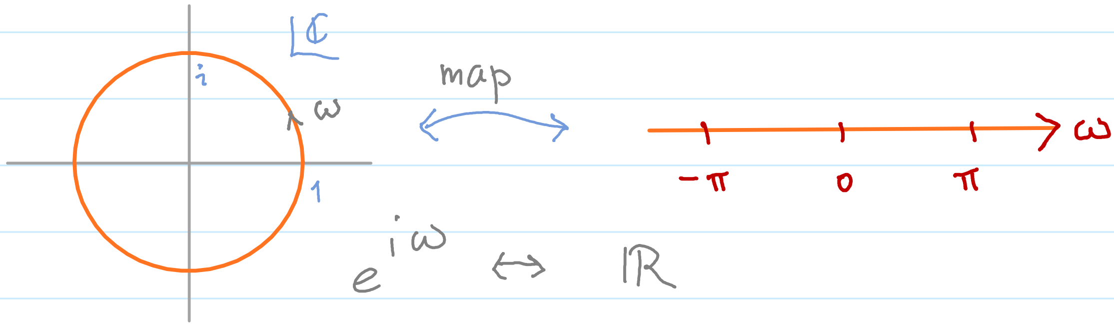
$\longrightarrow$ locally is diffeomorphic (m to $\mathbb{R}^n$



coordinate system tangent vectors

Example

$U(1) \sim \{ g \in \mathbb{C} \,,\, |g| = 1 \} \simeq S^1$



$e^{i\omega} \leftrightarrow \mathbb{R}$

$U(1) = \left(\{ e^{i\omega} \} \,,\, \omega \in \mathbb{R} \,,\, \circ \right)$

$\circ$ multiplication $\qquad e^{i\omega_1} \cdot e^{i\omega_2} = e^{i(\omega_1 + \omega_2)}$

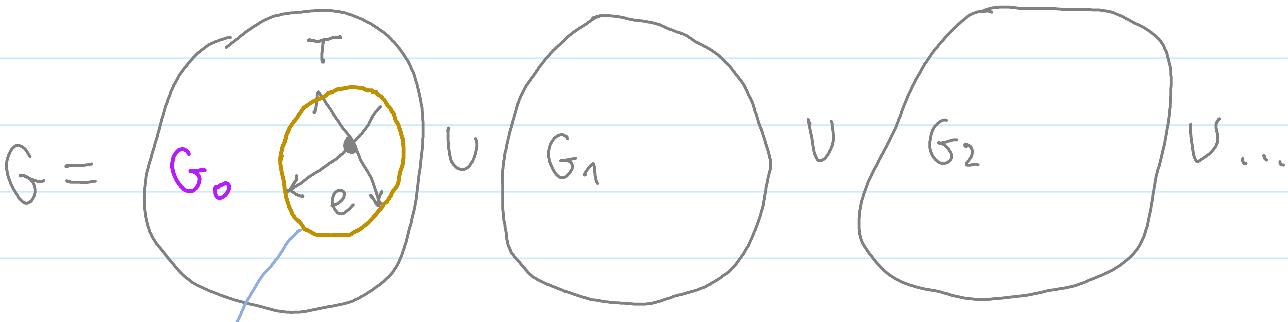
$SU(2) = \left(\left\{ g = \begin{pmatrix} a & b \\ -b^* & d \end{pmatrix} \,,\, a, b \in \mathbb{C} \,,\, |a|^2 + |b|^2 = 1 \right\} \,,\, \circ \right) \sim S^3$

$\circ$ = matrix multiplication

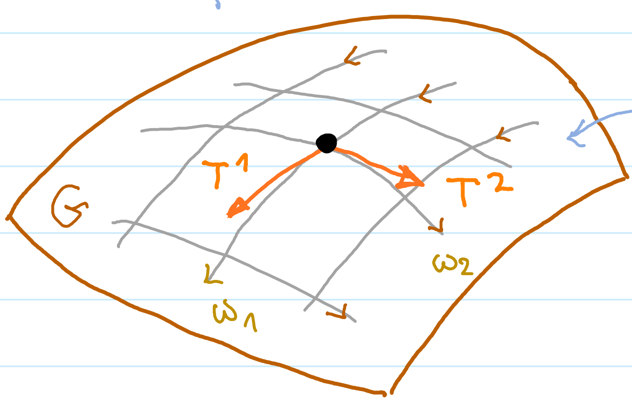
$a = e^{i\alpha} \cos\gamma \,,\quad b = e^{i\beta} \sin\gamma$

Topology:

Compact Lie groups : G is a compact manifold



Description close to $e \in G_0 \subset G$



$g(\omega_1, \omega_2, \dots) \in G$

we set $g(0,0,0,\dots) = e$

We can expand $g(\omega)$ around $\omega = 0$ $\quad (\omega_a \in \mathbb{R},\ |\omega_a| \ll 1)$

($\pm$)

$$g(\omega) \approx e + i \cdot \omega_1 \partial_1 g\big|_{\omega=0} + i\omega_2 \partial_2 g\big|_{\omega=0} + \dots + O(\omega)^2$$

Let $\boxed{\partial_a g\big|_{\omega=0} =: i T^a}$ – tangent vector in the direction „a".

T^a: linearly independent vectors tangent to lines of the coord. system (picture)

$$g(\omega) \approx e + i(\omega_1 T^1 + \omega_2 T^2 + \dots \omega_n T^n) + O(\omega)^2$$

$$\approx e + i \sum_a \omega_a T^a + O(\omega)^2 = e + i\omega T$$

$e \approx 1 \qquad g(\omega) \approx (1 + i\omega T)$

$$g_n(\omega) \approx \left(1 + i\frac{\omega}{n}T\right) \in G_0$$

$$\left(1 + i\frac{\omega}{n}T\right) \circ \left(1 + i\frac{\omega}{n}T\right) \in G_0$$

$$\lim_{n\to\infty} \left(1 + i\frac{\omega}{n}T\right)^n = e^{i\omega T} = g(\omega) \in G_0$$

$$g(\alpha) \cdot g(\beta) \in G_0$$

l.h.s. $\quad e^{i\alpha_a T^a} \quad e^{i\beta_a T^a}$

BCH formula $\quad e^A \cdot e^B = e^{A + B + \frac{1}{2}[A,B] + [[A,..],B] + \dots}$

$A = i\alpha_a T^a \qquad B = i\beta_a T^a$

$A + B = i(\alpha_a + \beta_a)T^a \qquad [A,B] = -\alpha_a \beta_b [T^a, T^b]$

$$\Rightarrow \boxed{[T^a, T^b] = i f^{ab}{}_c T^c} \qquad f^{ab}{}_c = \text{const}$$

$$\Rightarrow \quad e^A \cdot e^B = e^C \qquad C = i\omega_a(\alpha, \beta) T^a$$

Lie algebra

Vector space $\{T^a\}$ (i.e. elements of the form $\omega_a T^a$) with $[\cdot,\cdot]$ is called **Lie algebra** of the group G : $\underset{\sim}{G}$

if $A, B \in \underline{G}$ $\Rightarrow$ $[A,B] \in \underline{G}$, $[[A, \dots], B] \in \underline{G}$

$\Rightarrow \quad e^A e^B = e^C \qquad C \in \underline{G}$

FACT: Any element of G_0 can be represented by:

$$\boxed{g = e^{iT}} \qquad T = \sum_a \omega_a T^a \qquad \omega_a \in \mathbb{R}$$

we understand r.h.s. as: $e^{iT} = 1 + \sum_{n=1}^{\infty} \frac{(iT)^n}{n!}$

We define: (see lie algebra)