Lorentz group

18 października 2010 11:16

groups-gutowski
spinor&s

space-time coordinates

$$x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3) = (x^0, x^i) = (x^0, \vec{x})$$

Space-time interval

$$\Delta s^2 = (x_2^0 - x_1^0)^2 - (\vec{x}_2 - \vec{x}_1)^2$$

infinitezimal $ds^2 = (dx^0)^2 - (d\vec{x})^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$

Minkowski space $= (\mathbb{R}^4, \eta) = \mathbb{M}^4$

4- vectors $v^\mu, w^\mu, dx^\mu \in \mathbb{M}^4$

scalar product defined with $\eta_{\mu\nu}$

$$v \circ w = v_0 w_0 - \vec{v} \cdot \vec{w} = \eta_{\mu\nu} v^\mu w^\nu = v^T \eta w$$

lower index vectors from v^μ : $v_\mu \equiv \eta_{\mu\nu} v^\nu$

$$v \cdot w = v^\mu w_\mu = v_\mu w^\mu$$

Linear transf.

$$w^\mu \to w'^\mu = \Lambda^\mu{}_\nu w^\nu$$

$$w \to w' = \Lambda w \qquad \Lambda \in Mat(4\times 4, \mathbb{R})$$

Lorentz group

= set of transformations $w \to w'$ leaving $v \cdot w$ invariant

$$\boxed{v^T \eta w = v'^T \eta w' \quad \Rightarrow \quad \Lambda^T \eta \Lambda = \eta}$$

$$\{\Lambda \in Mat(4\times 4, \mathbb{R});\ \Lambda^T \eta \Lambda = \eta\} = O(3,1) = O(1,3)$$

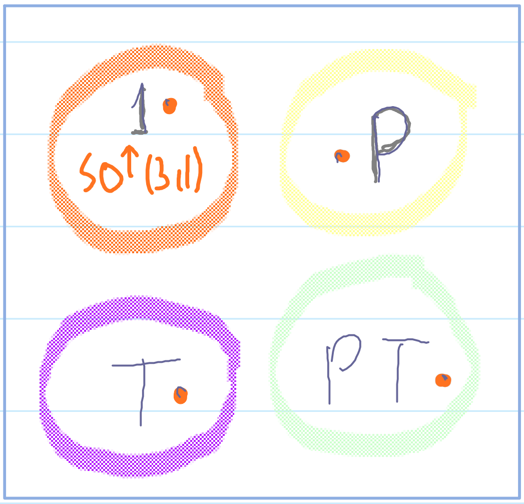
notation follows that of $O(n)$ but here

scalar product involves $\eta = diag(\underset{\uparrow 1}{1}, \underbrace{-1,-1,-1}_{3})$

Lorentz group

Has four disconnected components   Patrz → L.A.

$O(3,1) =$



$$P = diag(1,-1,-1,-1)$$

$$T = diag(-1,1,1,1)$$

$$P\cdot T = diag(-1,-1,-1,-1)$$

$(P \cdot x) = (x^0, -x^1, -x^2, -x^3)$ „reflexion" Parity

proper Lorentz group $= SO^+(3,1) \equiv SO^\uparrow(3,1)$

Notation : (upper and lower index objects)

Let vector v^μ, w_μ

we define $v_\mu = \eta_{\mu\nu} v^\nu$ $\quad w^\mu = \eta^{\mu\nu} w_\nu$

where $\eta^{\mu\nu} \eta_{\nu\rho} = \delta^\mu{}_\rho \equiv (\eta^{-1} \cdot \eta = \mathbb{1})$

Invariant scalar product: $\eta_{\mu\nu} v^\mu w^\nu = v^\mu w_\mu$

$W^{\mu\nu}$, V_μ, b_M $\longrightarrow$ $W^{\mu\nu} V_\mu b_\nu$ is L. inv.

Invariance of 4d measure

$d^4 x' = |\det(\Lambda)| d^4 x = d^4 x$

Lie algebra $SO(3,1)$

$\Lambda = e^{iM}$

$$M^T \eta + \eta M = 0 \quad \equiv \quad (\eta M)^T = -(\eta M)$$

i.e. ηM span $SO(4)$.

common notation

$$i M = \frac{i}{2} \omega_{\mu\nu} M^{\mu\nu} \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$= i\left(\omega_{01} M^{01} + \dots \right) = i \sum_{\mu<\nu} \omega_{\mu\nu} M^{\mu\nu}$$

sl(2,C)

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$\underline{SL(2, \mathbb{C})}$

$$SL(2, \mathbb{C}) = \{ g \in Mat(2\times2, \mathbb{C}) ; \ det(g) = 1 \}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \quad a \cdot d - b \cdot c = 1$$

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{array}{c} U^{\dagger} \sigma^{\mu} U = \Lambda^{\mu}{}_{\nu} \sigma^{\nu} \\ \text{maps } SL(2, \mathbb{C}) \ni U \longrightarrow \Lambda \in SO^{+}(3,1) \end{array}}$$

Lie algebra:

one condition $det(g) = 1$

$$g = e^{iT} \Rightarrow tr(T) = 0$$

$\underline{SL(2, \mathbb{C})}$ is spanned by $\left\{ \frac{\sigma^a}{2}, i\frac{\sigma^b}{2} \right\}$

Clearly $\underline{SL(2, \mathbb{C})}$ contain $\underline{SU(2)} = \underline{SO(3)}$

other:

$$\left[\frac{\sigma^a}{2}, i\frac{\sigma^b}{2}\right] = i\epsilon^{abc}\left(i\sigma^c/2\right)$$

$$\left[i\frac{\sigma^a}{2}, i\frac{\sigma^b}{2}\right] = -i\epsilon^{abc}\,\sigma^c/2$$

FAKT: $\underline{SO(1,3)} \sim \underline{SL(2,\mathbb{C})}$

$$M^{0i} \to i\frac{\sigma^a}{2} \qquad M^{ij} \to \sigma^a$$

commutation relations of $\left\{\frac{\sigma^a}{2}, i\frac{\sigma^b}{2}\right\}$ are that of $\underline{SO(1,3)}$ thus both Lie algebras are isomorphic.