Contents

1 General

Group G is a set G and a map $\circ$, such that: (1)$\circ$ is associative; (2)$g_1, g_2 \in G \Rightarrow g_1 \circ g_2 \in G$; (3)there exists a unity element e, such that $e \circ g = g \circ e = g$; (4) for every g there exists a unique g^{-1} such that $g \circ g^{-1} = g^{-1} \circ g = e$

Action of a group on a vector space V**.** To any group element g we associate an operator $T_g : V \to V$. If T_g is linear and such that $T_g(T_h) = T_{g \circ h}$ then it is called a **representation** of the group G. $dim(V)$ is called the dimension of the representation. If $dim(V) < \infty$ than in most cases (!) T_g can be represented as a matrix.

Discrete groups are groups having finite number of elements e.g.
$Z_n = (0, 1, 2, ..., n-1, + \bmod(n))$.

1.1 Lie groups and algebras

are (1) differentiable manifolds and (2) $\circ$ and $g \to g^{-1}$ are smooth (infinitely differentiable) maps. This allows to choose coordinates on G.

Lie algebras. Lie group around unity element can be represented as

$$g = e^{i\omega_a T^a}, \quad a = 1, ..dim(G). \tag{1.1}$$

T^a are constant matrices which form the basis of the Lie algebra of G. Lie a. of G will be denoted by $\underline{G}$. Thus $\underline{G}$ is a vector space over real numbers R with the basis $\{T^a\}$. Moreover $\underline{G}$ must be closed under commutator i.e.

$$[T^a, T^b] = i f^{ab}{}_c T^c \tag{1.2}$$

The matrices T^a are also called **generators** of the Lie group. **Metric** on Lie algebra: $g^{ab} = \mathrm{tr}\{T^a T^b\}$. For compact Lie groups the metric can be taken to be unity $g^{ab} = \delta^{ab}$, thus we shall not distinguish position of group indices in this (most common in this lecture) case.

Carten subalgebra is maximal albelian subalgebra of $\underline{G}$.

1.2 Some classical groups

1.2.1 O(N)

leaves invariant scalar the product in R^n i.e. let $v \in R^n$, $O \in O(n)$ and $v' = Ov$ than $v'^T v' = v^T v$. This yields

$$O^T O = 1 \tag{1.3}$$

From above we get that Lie a. is set of imaginary, antisymmetric matrices:

$$(T^a)^T = -T^a \tag{1.4}$$

O(n) has two disconnected components characterized by $\det(O) = \pm 1$. **SO(n)** is subset of O(n) characterized by $\det(O) = 1$. It is connected. O(n) leaves invariant scalar the product in R^n i.e. let $v \in R^n$, $O \in O(n)$ and $v' = Ov$ than $v'^T v' = v^T v$.

1.2.2 U(N)

$$U^\dagger U = 1 \tag{1.5}$$

From above we get that Lie a. is set of hermitian matrices:

$$(T^a)^\dagger = T^a \tag{1.6}$$

SU(n) is subset of U(n) characterized by $\det(U) = 1$. It is connected. U(n) leaves invariant the scalar product in C^n i.e. let $v \in C^n$, $U \in U(n)$ and $v' = Uv$ than $v'^\dagger v' = v^\dagger v$.

1.2.3 SO(3) vs SU(2)

The group SO(3) has the same algebra as SU(2) but on the group level $SO(3) = SU(2)/Z_2$. The identified element is

$$\begin{aligned} \exp\{i2\pi\sigma^3/2\} &= -1, \quad SU(2) \\ \exp\{i2\pi L_3\} &= 1, \quad SO(3) \end{aligned} \tag{1.7}$$

Thus reps for which (1.7) acts nontrivially are ot reps of SU(2) but they are reps of SO(3).

1.3 Lorentz - $O(3,1)$

Def. throught representation: (Lorentz group) leaves invariant the scalar product in R^4 with metric $\eta_{\mu\nu}$ (Minkowski space) . Let $v \in R^4$, $\Lambda \in O(3,1)$ and $v' = Ov$ than $v'^T \eta v' = v^T \eta v$. Thus[1]

$$\Lambda^T \eta \Lambda = \eta \quad \text{i.e.} \quad \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \tag{1.8}$$

O(3,1) has four disconnected components characterized by $\det(O) = \pm 1$, $\text{sign}(\Lambda^0{}_0) = \pm 1$. The **proper Lorentz group** is characterized by $\det(O) = 1$ and $\text{sign}(\Lambda^0{}_0) = 1$. It is connected.

Algebra

From the above we get that Lie a. is a set of imaginary matrices such that:

$$(\eta T^a)^T = -\eta T^a \tag{1.9}$$

i.e. ηT^a form $o(n+1)$ lie algebra.

We define (see below)

$$L^i_\pm = i(M^{0i} \pm \frac{i}{2} \epsilon^{ijk} M^{jk}) \leftarrow (?) \tag{1.10}$$

these form $so(3) \times so(3) \sim sl(2) \times sl(2)$ algebras. We can take its 2d reps and exponentate it and get 4d general Lorentz transformation.

1. scalar: $\phi' = \phi$.
2. vector: $v'^\mu = \Lambda^\mu{}_\nu v^\nu$, $v'_\mu = v_\nu (\Lambda^{-1})^\nu{}_\mu$
3. second rank tensor: $t'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma t^{\rho\sigma}$

WARNING. Position x^μ is not a tensor. It transforms under full Poincare group as follows: $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$. According to (1.8) the metric η does not transform under Lorentz group i.e. it is a scalar.

With the help of the metric η we can define quantities with the lower indices e.g. co-vectors: $v_\mu \equiv \eta_{\mu\nu} v^\nu$. Transformation properties of these quantities are according to the above definition. So: $v'_\mu = \eta_{\mu\nu} v'^\nu = \eta_{\mu\nu} \Lambda^\nu{}_\rho \eta^{\rho\sigma} v_\sigma = v_\nu (\Lambda^{-1})^\nu{}_\mu$.

From spinors

$$\begin{aligned} \psi'(x') &= U\psi(x), \quad \overline{\psi}' = \overline{\psi} U^{-1}, \quad U^{-1}\gamma^\mu U = \Lambda^\mu{}_\nu \gamma^\nu \\ \overline{\psi}'\psi' &= \overline{\psi}\psi, \ \overline{\psi}'\gamma_5\psi' = \overline{\psi}\gamma_5\psi & (1.11) \\ \overline{\psi}'\gamma^\mu\psi' &= \Lambda^\mu{}_\nu \overline{\psi}\gamma^\nu\psi, \ \overline{\psi}'\gamma_5\gamma^\mu\psi' = \Lambda^\mu{}_\nu \overline{\psi}\gamma_5\gamma^\nu\psi & (1.12) \end{aligned}$$

[1] Notes on indices

1.3.1 Poincare group

is semidirect product of the Lorentz group and the translation group in R^4. Elements (Λ, a). Multiplication $(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a)$.

2 Representations

A rep R of group G is a linear map $R : G \to \; End(V)$ where V is a vector space, respecting

$$R(g_1)R(g_2) = R(g_1 g_2), g_1, g_2 \in G \tag{2.1}$$

For Lie algebra we have

$$[R(t_1), R(t_2)] = R([t_1, t_2]) \tag{2.2}$$

$\dim(V)$ is called dimension of rep.

Because $R(t)$ is a linear map it can be represented as matrix (for $\dim(V) < \infty$).

Cartan subalgebra of Lie algebra: maximal set of commuting T's denoted by H_i.

2.1 Finite dim. reps.

HWIR Highest weight irreducible representations.

2.1.1 Adjoint irreps

Adjoint action of group $V \to \; g\, V g^{-1} \in V$ so $V = v_a T^a, \; v_a \in \mathbb{R}$
Adjoint action of algebra

$$\delta_a V \to \; i[T^a, V] = -v_b f^{ab}{}_c T^c \equiv v_b \, i(T^a_{adj})_c{}^b \, T^c \longrightarrow \; (T^a_{adj})_c{}^b = i f^{ab}{}_c \tag{2.3}$$

Do $[T^a_{adj}, T^b_{adj}] = i f^{ab}{}_j \, T^j_{adj}$ hold ? i.e $\; i f^{aj}{}_d (i f^{be}{}_j) - (i f^{bj}{}_d)(i f^{ae}{}_j) = i f^{ab}{}_j (i f^{je}{}_d)$

From the Jacobi identity

$$[[T^a, T^b], T^e] + cycl = 0 \quad \text{i.e. } 0 = f^{abj} f^{jed} + f^{bej} f^{jad} + f^{eaj} f^{jbd} \tag{2.4}$$

i.e. Adjoint actions define adjoint irreps (with dim=dim of group).

2.1.2 CONJUGATE IRREP

If T^a is an irrep then $-(T^a)^T$ and $-(T^a)^*$ also. For unitary irrep $-(T^a)^T = -(T^a)^*$.

These reps can be equivalent to the original one i.e.

$$-(T^a)^T = ST^aS^{-1} \tag{2.5}$$

For unitary reps they are called real if T^a can be choose to be antisymmetric (also purely imaginary) so $S = 1$ or pseudo-real if it is impossible so $S \neq 1$. (see Weiberg: QFT II, p.384)