

Introduction to the theory of fundamental interactions. Problems.

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (1, \vec{\sigma}) \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

1 Lorentz group

Zad. 1.1 Show that

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} = \int \frac{d^4k}{(2\pi)^4} 2\pi \delta(k^2 - m^2) \theta(k^0),$$

where $\theta(x) = 1$ dla $x \geq 0$ i $\theta(x) = 0$ dla $x < 0$. Moreover, the below may be useful:

$$\delta[y(x)] = \sum_i \frac{\delta(x - x_i)}{|y'(x_i)|}, \quad y' = \frac{dy}{dx}, \quad x_i - \text{poles of } y(x).$$

Is the above Lorentz invariant. ?

Zad. 1.2 ψ is a spinor transforming under $U \in SL(2, C)$ according to

$$U = e^{-1/8 \omega_{\mu\nu} \sigma^{\mu\nu}}, \quad \sigma^{\mu\nu} = (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu),$$

in the following way: $\psi \rightarrow U\psi$. Show that $\tilde{\psi} = i\sigma^2 \psi^*$ transforms as $\tilde{\psi} \rightarrow \bar{U}\tilde{\psi}$, where

$$\bar{U} = e^{-1/8 \omega_{\mu\nu} \bar{\sigma}^{\mu\nu}}, \quad \bar{\sigma}^{\mu\nu} = (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)$$

Zad. 1.3 Show that bispinor

$$\psi_c = C \bar{\psi}^T$$

transforms under Lorentz group the same way as ψ if $\psi' = U\psi$ then $\psi'_c = U\psi_c$

Hints: $C\gamma^\mu C^{-1} = -\gamma^{\mu T}$, $\gamma^0(\gamma^\mu)^+ \gamma^0 = \gamma^\mu$ and

$$U = \exp\{i\frac{1}{2} \omega_{\mu\nu} S^{\mu\nu}\}, \quad S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu].$$

2 Classical fields

Zad. 2.1 Let ϕ be complex scalar field which dynamics is given by the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

Show conservation of the current: $j^\mu = -ig(\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi)$ (i.e. $\partial_\mu j^\mu = 0$) if the equations of motions (e.o.m.) are respected.

Zad. 2.2 U(1) gauge field interacts with two charged scalar fields of the charge 2 and -1. Construct the most general relativistic and locally U(1) invariant Lagrangian for this theory, having at most two derivatives and containing at most quartic non-derivative terms in fields.

FERMIONIC FIELDS

Zad. 2.3 Show that if ψ respects Dirac equation then $\bar{\psi} \equiv \psi^\dagger \gamma^0$ respects:

$$(i\partial_\mu \bar{\psi} \gamma^\mu + \bar{\psi} m) = 0$$

Hint: $\gamma^0 \gamma^\mu \gamma^0 = \gamma^\mu$

Zad. 2.4

- Solve free Dirac equation for a particle and anti-particle of mass m at rest. Linearly independent solutions denote as: $u_s(0)e^{-imt}$ for particle and as $v_s(0)e^{imt}$ for antiparticle ($s = 1, 2$). Normalize u, v so that $\bar{u}_s u_r = \delta_{rs}$, $\bar{v}_s v_r = -\delta_{rs}$.
- Show that solutions to free Dirac equation for a particle and anti-particle of mass m and the momentum p are

$$\begin{aligned} \psi(x) &= u_s(p)e^{-ipx}, & u_s(p) &= (\not{p} + m)u_s(0) \\ \psi(x) &= v_s(p)e^{ipx}, & v_s(p) &= (\not{p} - m)v_s(0) \end{aligned}$$

Display solutions in an explicit way.

- * Show one must rescale the above solutions by $N = 1/(2(\omega_p + m))$ in order to $\sum_s u_s(p)\bar{u}_s(p) = (\not{p} + m)$ hold. Hint: $(\not{p} + m)\gamma^0(\not{p} + m) = 2\omega_p(\not{p} + m)$.

Zad. 2.5 Prove the Gordon's equality

$$\bar{u}(p)\gamma^\mu u(k) = \frac{1}{2m} \bar{u}(p)[p^\mu + k^\mu + i\sigma^{\mu\nu}(p - k)_\nu]u(k),$$

where $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $u(p)$ respects free Dirac equation $(i\gamma^\mu p_\mu - m)u(p) = 0$.

Zad. 2.6 Show that fermionic U(1) current

$$j^\mu = -\bar{\psi} \gamma^\mu \psi$$

is conserved $\partial_\mu j^\mu = 0$ if ψ respects free equations of motions (e.o.m.) $(i\gamma^\mu \partial_\mu - m)\psi = 0$.

Zad. 2.7 Formula

$$\psi(x) \rightarrow \psi'(x) = e^{i\alpha\gamma_5} \psi(x),$$

where $\alpha \in \mathbb{R}$ defines the chiral transformation. Show that $\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$ is invariant under the chiral transformation iff $m \rightarrow 0$.

GAUGE FIELDS

Zad. 2.8 Show that Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \Phi_\nu \partial^\mu \Phi^\nu + \frac{1}{2} (\partial_\mu \Phi^\mu)^2 + \frac{m^2}{2} \Phi_\mu \Phi^\mu$$

for real vector field Φ_α leads to the following e.o.m.

$$(\partial^2 + m^2)\Phi_\mu - \partial_\mu \partial_\nu \Phi^\nu = 0$$

and that classical Φ_μ respects Lorentz condition $\partial_\alpha \Phi^\alpha = 0$.

Zad. 2.9 SU(2) invariant Lagrangian $\phi \in \underline{2}$

$$\mathcal{L} = (D_\mu \phi)^\dagger D_\mu \phi, \quad D_\mu = \partial_\mu - ig \frac{\sigma^a}{2} A_\mu^a, \quad a = 1, 2, 3$$

display in an explicit form with use of the variables

$$\phi = \begin{pmatrix} f^+ \\ f^0 \end{pmatrix}, \quad A_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \pm iA_\mu^2)$$

3 Quanta

Zad. 3.1 Using canonical commutation relations

$$[a(k), a^+(k')] = \tilde{\delta}(k - k'), \quad [b(k), b^+(k')] = \tilde{\delta}(k - k').$$

- Show that for arbitrary x_0, y_0 the following holds

$$[\phi(x), \phi^+(y)] = i \int \tilde{d}k (e^{-ik(x-y)} - e^{ik(x-y)})|_{k_0=\omega_k}.$$

- Calculate $[\phi(x), \phi(y)]$.

Zad. 3.2 Express Hamiltonian of the complex scalar field ϕ

$$H = \int d^3x ((\partial_0\phi^+)(\partial_0\phi) + \nabla\phi^+ \cdot \nabla\phi + m^2\phi^+\phi)$$

through creation and annihilation operators.

Zad. 3.3 Express Hamiltonian for the free Dirac field Ψ

$$H = \int d^3x \mathcal{H} =: \int d^3x \Psi^+ i\partial_0\Psi :$$

through creation and annihilation operators.

Zad. 3.4 Check that in the interaction picture in which the time evolution of an operator $A_I(t)$ is given by free Hamiltonian H_0 :

$$A_I(t) = e^{iH_0t} A e^{-iH_0t},$$

scalar field operators $\phi_I(x)$, $\pi_I(x)$ respects the same commutation relations as in Schrödinger picture.

$$[\phi_I(\vec{x}, t), \pi_I(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y}).$$

Zad. 3.5 Lagangian of the complex scalar field is

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \lambda(\phi^* \phi)^2$$

- (a). express (time independent) charge operator $Q = -ig \int d^3y (\phi^* \partial_t \phi - \partial_t \phi^* \phi)(y)$ through creation and anihilation operators.

Show that in quantum theory

- (b). the field ϕ has charge g i.e. $[Q, \phi(x)] = g\phi(x)$. (Hint:one does not need expression of Q through creation and anihilation operators.)

- (c). $Q = \int d^3x j^0$ is time independent i.e. $[H, Q] = 0$.

Zad. 3.6 Show that canonical quatization leads to the formula

$$\langle 0|T\phi(x)\phi(x')|0 \rangle = \int \tilde{d}p [\theta(t-t')\phi_p^{(+)}(x)\phi_p^{(+)*}(x') + \theta(t'-t)\phi_p^{(-)}(x)\phi_p^{(-)*}(x')]$$

$(\phi_p^{(\pm)}(x) = e^{\mp i p x})$ and then differentiate it to obtain

$$(\partial^2 + m^2) \langle 0|T\phi(x)\phi(x')|0 \rangle = -i\delta^{(4)}(x-x')$$

Hint: $\partial_t \theta(t-t') = \delta(t-t')$, $\tilde{d}p = \frac{d^3p}{(2\pi)^3 2\omega_p}$

Zad. 3.7 Directly differentiating $\langle 0|T\phi(x)\phi(x')|0 \rangle$ and using canonical commutation relation between fields and their momenta show that

$$(\partial^2 + m^2) \langle 0|T\phi(x)\phi(x')|0 \rangle = -i\delta^{(4)}(x-x')$$

Hint: $\partial_t \theta(t) = \delta(t)$.

FERMIONS

Zad. 3.8 (a) Express the charge operator for the complex fermion field ψ $Q = -g \int d^3x \psi^+ \psi$ through creation and anihilation operators.

(b) Show that $[Q, \psi(x)] = g\psi(x)$

4 Feynman diagrams

Zad. 4.1 Two complex scalars ϕ_i , ($i = 1, 2$) of masses m_i interact by

$$\mathcal{L}_I = -\lambda|\phi_1|^2|\phi_2|^2,$$

Using canonical quantization calculate the transition amplitude : particle 1 + antiparticle 1 to particle 2 + antiparticle 2 i.e.

$$1 + \bar{1} \rightarrow 2 + \bar{2}$$

Zad. 4.2 Two complex scalars ψ of the masses m interacts with the scalar ϕ of the mass $M > 2m$

$$\mathcal{L}_I = -\lambda\phi|\psi|^2,$$

Calculate:

(a). decay amplitude : ϕ to particle ψ + antiparticle $\bar{\psi}$ i.e.

$$\phi \rightarrow \psi + \bar{\psi}$$

Why does the amplitude vanishes for $M < 2m$?

and scattering amplitudes

(b). $2\psi \rightarrow 2\psi$

(c). $\psi + \phi \rightarrow \psi + \phi$

(d). $2\phi \rightarrow \psi + \bar{\psi}$

(e). Show that the amplitude below vanishes

$$\phi \rightarrow \psi + \psi, \quad 2\psi \rightarrow \psi + \bar{\psi}$$

5 Standard Model (SM)

Quark constituents of mezoons: $K^- = \bar{u}s$, $\pi^- = \bar{u}d$, $\pi^0 = (\bar{u}u - \bar{d}d)$.

Masses: $m_{\pi^-} = 140 \text{ MeV}$, $m_{\pi^0} = 135 \text{ MeV}$, $m_{K^-} = 494 \text{ MeV}$, $m_{K^0} = 498 \text{ MeV}$, $m_e = 0.5 \text{ MeV}$, $m_\mu = 100 \text{ MeV}$, $m_\tau = 1.8 \text{ GeV}$.

Zad. 5.1 Calculate the mass of the Higgs field H in SM knowing that the potential for the scalar doublet is $V(\phi) = \lambda(\phi^+\phi - \frac{v^2}{2})^2$ and the Higgs field is defined by

$$\phi = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + H \end{pmatrix}$$

Zad. 5.2 Draw the simplest Feynman diagrams leading to decays:

(a). $K^- \rightarrow l + \bar{\nu}_l$

(b). $K^- \rightarrow l + \bar{\nu}_l + \pi^0$

(c). $K^- \rightarrow \pi^- + \pi^0$

where l is one of the leptons.

Zad. 5.3 Draw the simplest Feynman diagrams leading to decays:

(a). $\pi^- \rightarrow l + \bar{\nu}_l$

(b). $\pi^- \rightarrow l + \bar{\nu}_l + \pi^0$

where l is one of the leptons. Which leptons can be produced in such processes.

Zad. 5.4 Calculate coupling of fields in the interaction Lagrangian of the SM

(a) $\bar{e}eZ$, (b) $\bar{\nu}_e\nu_eZ$, (c) $\bar{e}\nu_eW$, (d) W^+W^-Z , (e) W^+W^-A

(f) $W^+\bar{u}d$, (g) $W^+\bar{u}s$.

Zad. 5.5 A. Draw the simplest Feynman diagrams leading to the following processes:

(a). $e + \nu_\mu \rightarrow e + \nu_\mu$

(b). $e + \nu_e \rightarrow e + \nu_e$

(c). $e + \nu_\mu \rightarrow \mu + \nu_e$

Justify the answer enumerating appropriate couplings in the interaction Lagrangian of the SM

B. Calculate transition amplitudes for the processes (a), (c) from the part A.