Effective boundary conditions for creeping flow along a periodic rough surface

B. CICHOCKI[1], P. SZYMCZAK[1] & F. FEUILLEBOIS[2]

[1] *Warsaw University, Institute of Theoretical Physics,,*
ul. Hoża 69, PL-00-681 Warsaw, Poland
cichocki@fuw.edu.pl

[2] *Laboratoire de Physique et Mécanique des Milieux Hétérogènes , ESPCI*
10, rue Vauquelin, 75005 Paris, France

Abstract The creeping flow along a periodic rough surface is calculated as a series in the slope of the roughness grooves. On a scale much larger than the grooves, this flow is equivalent to that over a smooth plane which is shifted from the top of the riblets. The convergence of the series for the shift distance in term of the slope is accelerated by use of Euler transformation and of the existence of a limit for large slope. The case of a flow along the grooves is presented in detail. The result for the shift is typically valid for a slope up to 2. A flow perpendicular to grooves can be treated in a similar way. Asymptotic behaviour for large slope depends on the profile shape.

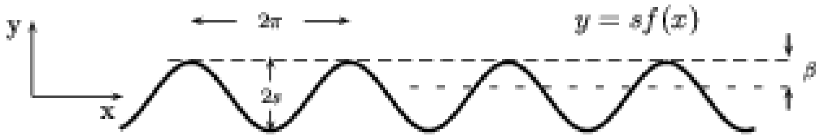
Flow past a wall that is rough
Is a problem to do off-the-cuff;
When the surface is toothed,
It's effectively smoothed,
By a mapping that's complex enough.

1. Introduction

Modelling the flow of a viscous fluid along a rough surface is relevant for various applications and for a better understanding of fundamental problems like modelling the boundary condition of a porous material (see e.g. Taylor 1971), reducing the shear stress of a turbulent boundary layer as compared with that on a smooth surface (Bechert & Bartenwerfer 1989, Luchini, Manzo & Pozzi 1991).

The rough surface considered here has periodic corrugations with wavelength λ along one dimension. The corrugation profile is symmetric, but its shape is otherwise not restricted. Normalising distances with $\lambda/(2\pi)$, and using the notation shown in the figure, the conditions for the profile $f(x)$ are:

(i) $f(x+2\pi) = f(x)$ and (ii) $f(x+\pi) = -f(x)$



A pure shear flow at infinity (viz. on a macroscopic scale) is applied along the surface. The size of the roughness and the fluid velocity on that scale are here assumed to be small compared with those on the macroscopic scale, so that the Reynolds number is small compared with unity and the creeping flow equations apply. A major problem is to ascertain an "equivalent" boundary condition to be applied on the macroscopic scale. That is, our goal is to show that the flow at infinity is equivalent to a pure shear flow along a smooth plane located at a distance $\beta\lambda/(2\pi)$ below the top of the corrugations. Here, β denotes the normalised shift.

2. Earlier results

Two sets of problems may be considered: parallel flows and cross-flows with respect to the grooves. Both are well documented.

For flows parallel to the grooves, the fluid velocity is harmonic, as explained below. Richardson (1971) calculated in particular by conformal mapping and Schwarz-Christoffel transformation an equivalent slip velocity for a flow along a row of parallel and equidistant thick semi-infinite slabs (his formula 4.4). From that formula, we derive the following normalised shift for semi-infinite plates of zero thickness[1]: $\beta = 2\log 2$. Bechert & Bartenwerfer (1989) calculated flows along various profiles (sawtooth, trapezoidal valleys, blade riblets viz. wall attached barriers) by conformal mapping. Luchini, Manzo & Pozzi (1991) used the numerical boundary element technique to solve flows along sinusoidal, scalloped and sawtooth profiles. Wang (1994) calculated in particular the flow along blade riblets by a collocation technique.

[1]Note that there is a misprint in the slip velocity he gives for that case (a factor 2 is missing), but the result follows easily from his equation (4.4) which is correct. There is also a misprint in Hocking's (1976) quotation of Richardson's (1971) result: a factor 4 is missing. The same result as in (Richardson 1971) was obtained independently by Bechert & Bartenwerfer (1989) from the limit of a sawtooth profile and again by Jeong (2001), using the Wiener-Hopf technique. Jeong remarks he recovers Richardson's (1971) result but without pointing out the misprint in that paper. We have redone the calculation by conformal mapping as a check.

For cross-flows, the Stokes equations have to be solved. Richardson (1973) calculated the cases of sinusoidal and scalloped profiles by conformal mapping. Hocking (1976) provided an interesting series solution that he applied to the sinusoidal profile, improving over Richardson (1973). However his solution leads to numerical problems for large slopes. He then calculated the case of an infinite slope, viz. of a row of semi-infinite plates of zero thickness, by the Wiener-Hopf technique, with the result[2] : $\beta = 0.5569$. Luchini, Manzo & Pozzi (1991) applied the numerical boundary element technique to the same profiles as for the parallel flows. Davis (1993) considered in particular the case of blade riblets. He used a distribution of singularities over the riblets and solved a Fredholm integral equation for this distribution. Wang (1994) solved this problem independently by a collocation technique. Tuck & Kouzoubov (1995) treated the sinusoidal profile using (Hocking 1976) type of solution together with a collocation technique.

Our approach is to use a solution in the spirit of (Hocking 1976), to expand as a series in the slope and to accelerate the convergence of the series so as to extend its application range. This approach is in principle valid for any profile with limited slope.

3. The expansion method

We will present here the case of a flow parallel to grooves, since the formulation then is simpler. The velocity is normalised by $\kappa\lambda/2\pi$, where κ denotes the shear gradient at infinity. The velocity field is of the form $v(x, y)\,\vec{e}_z$, where $\vec{e}_z$ denotes the unit vector perpendicular to the (x, y) plane. Stokes equations then reduce to Laplace equation for $v(x, y)$ (this would more generally be true when starting from Navier-Stokes equations). A general form of solution satisfying the condition at infinity is

$$v(x, y) = y + d_0 + \sum_{n=1}^{\infty} d_n e^{-ny} \cos nx. \qquad (1)$$

The equivalent smooth plane is at $y = -d_0$ so that the normalised shift is (cf. Figure):

$$\beta = s + d_0. \qquad (2)$$

[2]Unaware of this paper, other authors redid this calculation independently also with Wiener-Hopf technique: (Luchini, Manzo & Pozzi 1991) with the result $\beta = 0.556475$; and (Jeong 2001) with the result $\beta = 0.5567$.

The boundary condition on the rough surface $y = sf(x)$ reads:

$$0 = sf(x) + d_0 + \sum_{n=1}^{\infty} d_n e^{-nsf(x)} \cos nx. \tag{3}$$

Note that all d_i coefficients in this equation are implicitly functions of s. By changing s to $-s$, f to $-f$ and using assumption (ii) one shows that d_{2m}'s are even in s while d_{2m+1}'s are odd. Next one expands d_i's in powers of s, e.g. d_0 takes the form

$$d_0(s) = \sum_{n=0}^{\infty} a_{0,n} s^{2(n+1)}. \tag{4}$$

After substituting such expansions into (3) as well as expanding the function $e^{-nsf(x)} \cos nx$ into Fourier series, one ends up with a system of linear equations for the coefficients $a_{i,n}$. In particular, solving for $a_{0,n}$ allows us to find the shift from (2) and (4).

Alas, the series (4) is slowly convergent. As an example, in Table 1 we present the first ten $a_{0,n}$'s for the profile $f(x) = \cos x$. However, as coefficients $a_{0,n}$ are of alternating signs, the convergence can be accelerated by use of the Euler transformation (see eg. Knopp 1958)

$$a_{0,k} \to b_k = \sum_{n=0}^{k} \begin{pmatrix} k \\ n \end{pmatrix} a_{0,n-k}. \tag{5}$$

In this way, one obtains:

$$d_0(s) = \sum_{k=0}^{\infty} b_k \left(\frac{s^2}{1+s^2} \right)^{k+1}. \tag{6}$$

Now from the existence of the limit $\beta(s \to \infty) = \beta_\infty$ and relation (2) one infers that asymptotically $d_o(s) \to -s$ as $s \to \infty$. Denoting $z = s^2/(1+s^2)$, one gets:

$$\sum_{k=0}^{\infty} b_k z^{k+1} \sim -\frac{1}{\sqrt{1-z}}, \qquad z \to 1 \tag{7}$$

so that asymptotically the coefficient b_k should obey:

$$b_k \sim \frac{-1}{\sqrt{\pi k}}, \qquad k \to \infty. \tag{8}$$

This means that the asymptotic behaviour of the series under consideration is the same as that of the polylogarithm function $Li_{1/2}$ defined as

(see Lewin 1981)

$$Li_{1/2}(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{\sqrt{k}}. \tag{9}$$

Once the asymptotic behaviour is known, we can make the convergence faster by subtracting the asymptotic terms, i.e.

$$d_0(s) = \sum_{k=0}^{\infty} c_k \left(\frac{s^2}{1+s^2}\right)^{k+1} - \frac{1}{\sqrt{\pi}} \frac{s^2}{1+s^2} Li_{1/2}\left(\frac{s^2}{1+s^2}\right), \tag{10}$$

with $c_0 = b_0$ and

$$c_k = b_k + 1/\sqrt{\pi k} \qquad \text{for} \qquad k = 1, 2, \ldots . \tag{11}$$

The above series is fast convergent for $s \leq 2$. In Table 1 the coefficients $b_n/2^{n+1}$ and $c_n/2^{n+1}$ are given for the profile $f(x) = \cos x$. The factor $1/2^{n+1}$ corresponds to setting $s = 1$ in the series (6) and (10).

The cross-flow problem can be treated in an analogous way. In that case, Stokes equations give the biharmonic equation for the stream function. The number of equations then is doubled.

Table 1. The coefficients $a_{0,n}$, $b_n/2^{n+1}$ and $c_n/2^{n+1}$ for the profile $f(x) = \cos x$ as defined by Eqs. (4), (5) and (11) respectively.

n	$a_{0,n}$	$b_n/2^{n+1}$	$c_n/2^{n+1}$
0	$-5. \cdot 10^{-1}$	$-2.5 \cdot 10^{-1}$	$-2.5 \cdot 10^{-1}$
1	$1.25 \cdot 10^{-2}$	$-9.375 \cdot 10^{-2}$	$4.730 \cdot 10^{-2}$
2	$-5.7292 \cdot 10^{-2}$	$-3.841 \cdot 10^{-2}$	$1.146 \cdot 10^{-2}$
3	$3.1033 \cdot 10^{-2}$	$-1.662 \cdot 10^{-2}$	$3.743 \cdot 10^{-3}$
4	$-1.8363 \cdot 10^{-2}$	$-7.437 \cdot 10^{-3}$	$1.379 \cdot 10^{-3}$
5	$1.1583 \cdot 10^{-3}$	$-3.403 \cdot 10^{-3}$	$5.389 \cdot 10^{-4}$
6	$-7.6990 \cdot 10^{-3}$	$-1.581 \cdot 10^{-3}$	$2.184 \cdot 10^{-4}$
7	$5.3257 \cdot 10^{-3}$	$-7.422 \cdot 10^{-4}$	$9.080 \cdot 10^{-5}$
8	$-3.8684 \cdot 10^{-3}$	$-3.511 \cdot 10^{-4}$	$3.850 \cdot 10^{-5}$
9	$2.8903 \cdot 10^{-3}$	$-1.671 \cdot 10^{-4}$	$1.660 \cdot 10^{-5}$
10	$-2.2217 \cdot 10^{-3}$	$-7.986 \cdot 10^{-5}$	$7.255 \cdot 10^{-6}$

4. Summary

The use of series expansion was until now restricted, since numerical problems arose when the slope was not small (Hocking 1976). This technique of accelerating the convergence of series thus provides an extension of such solutions in a domain of practical interest : results obtained for the normalised shift are typically valid for $s < 2$.

Our numerical results also suggest that the asymptotic behaviour of $\beta(s)$ for $s \to \infty$ may depend on the shape of the corrugation crests. In case of the sawtooth profile one expects that $\beta(s) - \beta_\infty \sim a/s$ whereas for crests with vanishing first derivative the asymptotic behaviour is more like $\beta(s) - \beta_\infty \sim b/\sqrt{s}$.

Finding the constants a and b would provide the $\beta(s)$ dependence on the whole s range, by combining the precise results of the previous section with asymptotic expansions.

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