Instability growth rate for small permeability contrasts

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We begin with a dimensionless form of Eqs. (1)-(3), using the downstream scaling of length:

$$\xi = \frac{x'}{l_d}, \qquad \eta = \frac{y}{l_d}, \qquad \tau = \frac{\gamma_a v_0 t}{l_d}, \tag{1}$$

and dimensionless velocity, concentration, and porosity fields:

$$\hat{\boldsymbol{v}} = \boldsymbol{v}/v_0, \qquad \hat{c} = c/c_{in}, \qquad \hat{\phi} = (\phi - \phi_0)/(\phi_{max} - \phi_0). \tag{2}$$

The pressure can be eliminated by replacing Darcy's equation with the compatibility equation (6):

$$\hat{v}_\xi \partial_\xi \hat{c} + \hat{v}_\eta \partial_\eta \hat{c} - Pe^{-1}(\partial_\xi^2 \hat{c} + \partial_\eta^2 \hat{c}) = -(1 + Pe^{-1})\hat{c}, \tag{3}$$

$$\partial_\tau \hat{\phi} - \partial_\xi \hat{\phi} = (1 + Pe^{-1})\hat{c}, \tag{4}$$

$$\partial_\xi \hat{v}_\xi + \partial_\eta \hat{v}_\eta = 0, \tag{5}$$

$$\partial_\eta \hat{v}_\xi - \alpha \hat{v}_\xi \partial_\eta \hat{\phi} = \partial_\xi \hat{v}_\eta - \alpha \hat{v}_\eta \partial_\xi \hat{\phi}, \tag{6}$$

where the (time-dependent) terms in γ_a have been dropped, corresponding to the typical limiting case of small acid capacity.

Substituting perturbations of the form of Eq. (11) leads to coupled equations for the one-dimensional fields $\delta\hat{\phi}(\xi)$, $\delta\hat{c}(\xi)$, and $\delta\hat{v}(\xi)$:

$$(\partial_\xi \hat{c}_b)\delta\hat{v} = (\partial_\xi - \hat{\omega})\delta\hat{\phi} + \left[Pe^{-1}(\partial_\xi^2 - \hat{u}^2) - \partial_\xi\right]\delta\hat{c}, \tag{7}$$

$$(1 + Pe^{-1})\delta\hat{c} = (-\partial_\xi + \hat{\omega})\delta\hat{\phi}, \tag{8}$$

$$\alpha \hat{u}^2 \delta\hat{\phi} = (-\partial_\xi^2 + \alpha(\partial_\xi \hat{\phi}_b)\partial_\xi + \hat{u}^2)\delta\hat{v}. \tag{9}$$

For the purposes of numerical solution it is convenient to combine these equations into a single fifth-order equation for $\delta\hat{\phi}$,

$$\left\{\left[\partial_\xi^2 + \alpha e^{-\xi}\partial_\xi - \hat{u}^2\right] e^{\xi} \left[Pe^{-1}(\partial_\xi^2 - \hat{u}^2 - 1) - \partial_\xi - 1\right]\left[\partial_\xi - \hat{\omega}\right] + \alpha\hat{u}^2\right\}\delta\hat{\phi} = 0. \tag{10}$$

The equations for the upstream perturbations (without the reaction terms and with constant porosity) are:

$$(\partial_\xi \hat{c}_b)\delta\hat{v} = \left[Pe^{-1}(\partial_\xi^2 - \hat{u}^2) - \partial_\xi\right]\delta\hat{c}, \tag{11}$$

$$0 = (-\partial_\xi^2 + \hat{u}^2)\delta\hat{v}. \tag{12}$$

The upstream and downstream perturbations are connected by continuity conditions in $\hat{\phi}$, $\hat{v}_\xi$, $\hat{v}_\eta$, $\hat{c}$, and $\partial_\xi \hat{c}$ at the front $\xi = \xi_f$, where perturbation in the front position also grows exponentially $\xi_f = \xi_0 \cos(\hat{u}\eta) e^{\hat{\omega}\tau}$.

In the limit that the permeability contrast α is small, we can make a regular perturbation expansion around $\alpha = 0$. Expanding $\hat{\omega}$, $\delta\hat{\phi}$, $\delta\hat{c}$, and $\delta\hat{v}$ in powers of α,

$$\hat{\omega} = \hat{\omega}_0 + \alpha\hat{\omega}_1 + \ldots, \tag{13}$$

the zeroth order solution for $\delta\hat{\phi}$ is

$$(\delta\hat{\phi})_0 = A_0 e^{-(\hat{u}+1)\xi} + B_0 e^{\lambda\xi}, \tag{14}$$

with

$$\lambda = \frac{1}{2}\left(Pe - \sqrt{(Pe+2)^2 + 4\hat{u}^2}\right). \tag{15}$$

Applying the continuity conditions at the front we find $(\delta\hat{\phi})_0 = \xi_0 e^{\lambda\xi}$ and

$$w_0 = \beta = \frac{1}{2}\left(Pe - \sqrt{Pe^2 + 4\hat{u}^2}\right). \tag{16}$$

The first-order (in α) solution can be found in a similar fashion; after applying the continuity conditions at the front, we obtain an explicit expression for the first-order growth rate, which is always positive:

$$\hat{\omega}_1 = \frac{c_1 + c_2\beta - c_3\lambda - c_4\beta\lambda}{2(1+Pe)[1+Pe(1+\hat{u})][(1+2\hat{u}+Pe(1+\hat{u})]}, \tag{17}$$

where the coefficients are polynomials in $\hat{u}$ and Pe:

$$c_1 = \hat{u}\{1 + 2\hat{u} + Pe(1+\hat{u})[3 + \hat{u} + Pe(3 + \hat{u} + Pe)]\}, \tag{18}$$

$$c_2 = 1 + 2\hat{u} + Pe\{3 + 4\hat{u} + \hat{u}^2 + Pe[2 + 3\hat{u} + Pe(1+\hat{u})]\}, \tag{19}$$

$$c_3 = Pe\,\hat{u}^2, \tag{20}$$

$$c_4 = Pe\,[\hat{u} + Pe(1+\hat{u})]. \tag{21}$$