Coherent states for the many-body problem (continue)

Previous lecture: coherent states for bosons: $|\phi\rangle = e^{\sum_\alpha \phi_\alpha a_\alpha^+}|0\rangle$ $\quad a_\alpha|\phi_\alpha\rangle = \phi_\alpha|\phi_\alpha\rangle$

α - labels single-particle basis states

$\phi_\alpha \in \mathbb{C}$

$$|\phi\rangle = \sum_{n_{\alpha_1}, n_{\alpha_2}, \dots, n_{\alpha_p} \dots} \phi_{n_{\alpha_1} n_{\alpha_2} \dots n_{\alpha_p} \dots} |n_{\alpha_1} n_{\alpha_2} \dots n_{\alpha_p} \dots\rangle \quad (*)$$

$$\phi_{n_{\alpha_1} n_{\alpha_2} \dots n_{\alpha_i} \dots} = \frac{\phi_{\alpha_1}^{n_{\alpha_1}}}{\sqrt{n_1!}} \frac{\phi_{\alpha_2}^{n_{\alpha_2}}}{\sqrt{n_{\alpha_2}!}} \cdots \frac{\phi_{\alpha_i}^{n_{\alpha_i}}}{\sqrt{n_{\alpha_i}!}} \cdots$$

- Overlap: $\langle\phi|\phi'\rangle = e^{\sum_\alpha \phi_\alpha^* \phi'_\alpha}$
- $\mathrm{Tr} A = \int \prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi_\alpha^* \phi_\alpha} \langle\phi|A|\phi\rangle$
- $\langle\phi|A(a_\alpha^+, a_\alpha)|\phi'\rangle = A(\phi_\alpha^*, \phi'_\alpha) e^{\sum_\alpha \phi_\alpha^* \phi'_\alpha}$ (normal-ordered op.)

  $a_\alpha^+ a_{\alpha'}^+ a_\beta a_{\beta'} \quad a a^+ a^+ a$

- Occupation number n_α (for each α) is Poisson distributed with mean value $|\phi_\alpha|^2$:

$$|\langle n_{\alpha_1} n_{\alpha_2} \dots|\phi\rangle|^2 = \prod_\alpha \frac{|\phi_\alpha|^{2n_\alpha}}{n_\alpha!}$$

$P(n) = e^{-\lambda}\frac{\lambda^n}{n!}$

$\lambda \to |\phi_\alpha|^2$

- $\langle\hat N\rangle = \frac{\langle\phi|\hat N|\phi\rangle}{\langle\phi|\phi\rangle} = \frac{\sum_\alpha \langle\phi|a_\alpha^+ a_\alpha|\phi\rangle}{\langle\phi|\phi\rangle} = \sum_\alpha \phi_\alpha^* \phi_\alpha$

- $\langle\hat N^2\rangle - \langle\hat N\rangle^2 = (\cdot) = \langle\hat N\rangle$

In the thermodynamic limit $\langle\hat N\rangle \to \infty$ and $\frac{\sigma}{\langle\hat N\rangle} = \frac{1}{\sqrt{\langle\hat N\rangle}} \to$ sharply peaked around $\langle\hat N\rangle$.

Coherent states for fermions

Analogous construction for fermions? $\longrightarrow$ eigenvalues must anticommute!!!

Algebra of anticommuting "numbers" – Grassmann algebra

vector space with additional multiplication operation (associative, anticommutative)

Grassmann algebra $\longrightarrow$ set of generators $\{\xi_\alpha\}$. Consider $\alpha \in \{1,2,\dots,n\}$ now.

- generators anticommute $\xi_\alpha \xi_\beta + \xi_\beta \xi_\alpha = 0$ $\quad$ ($\xi_\alpha^2 = 0$ in particular)
- basis of the Grassmann algebra $\rightarrow$ distinct products of generators
- a "number" in the Grassmann algebra $\rightarrow$ linear combination (with complex coefficients) of the "numbers" $\{1, \xi_{\alpha_1}, \xi_{\alpha_1}\xi_{\alpha_2}, \dots \xi_{\alpha_1}\xi_{\alpha_2}\cdots\xi_{\alpha_n}\}$ $\quad$ ($\alpha_1 < \alpha_2 < \dots < \alpha_n$ - convention)
- dimension of the Grassmann algebra $= 2^n$. (•)

• Addition defined as usual in vector spaces (commutative and associative)

- consider n even ($n = 2p$)
- define a conjugation operation $\rightarrow$ pick p generators ξ_α, to each of them associate the conjugate generator ξ_α^*

$$\left.\begin{aligned}
(\xi_\alpha)^* &= \xi_\alpha^* \\
(\xi_\alpha^*)^* &= \xi_\alpha \\
(\lambda \xi_\alpha)^* &= \lambda^* \xi_\alpha^* \quad (\text{for } \lambda \in \mathbb{C}) \\
(\xi_{\alpha_1} \cdots \xi_{\alpha_n})^* &= \xi_{\alpha_n}^* \xi_{\alpha_{n-1}}^* \cdots \xi_{\alpha_1}^*
\end{aligned}\right\} \text{ Conjugation in Grassmann algebra}$$

Integration and differentiation

For simplicity – take the case with only 2 generators: ξ, ξ^* $\rightarrow$ algebra spanned by 4 elements:

$$\{1, \xi, \xi^*, \xi^* \xi\}$$

Consider a Grassmann-valued function $A(\xi^*,\xi) = a_0 + a_1\xi + \bar{a}_1\xi^* + a_{12}\xi^*\xi$

Derivative – defined to be "identical" to the usual derivative, but in order to act with $\frac{\partial}{\partial\xi}$ on ξ, ξ must be adjacent to $\frac{\partial}{\partial\xi}$ $\leadsto$ e.g. $\frac{\partial}{\partial\xi}(\xi^*\xi) = \frac{\partial}{\partial\xi}(-\xi\xi^*) = -\xi^*$

"operational" definition (no sense of talking about "infinitesimals")

$$\frac{\partial}{\partial\xi}A(\xi^*,\xi) = a_1 - a_{12}\xi^*$$

$\frac{\partial}{\partial\xi}$ is nilpotent $\left(\left(\frac{\partial}{\partial\xi}\right)^2 = 0\right)$

$$\frac{\partial}{\partial\xi^*}A(\xi^*,\xi) = \bar{a}_1 + a_{12}\xi$$

$$\frac{\partial}{\partial\xi^*}\frac{\partial}{\partial\xi}A(\xi^*,\xi) = -a_{12} = -\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi^*}A(\xi^*,\xi)$$

$\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi^*}\right.$ anticommute$\left.\right)$

(generalization to cases with more generators straightforward)

Integral – linear mapping with the property that the integral of an "exact differential form" is zero $\leadsto$

"operational" definition (no analog to Riemann or Lebegue integral)

$$\int d\xi\, 1 = 0 \qquad \int d\xi^*\, 1 = 0$$

$$\int d\xi\, \xi = 1 \quad \text{(definition)} \qquad \int d\xi^*\, \xi^* = 1$$

Mnemonic: Grassmann integration is identical to Grassmann differentiation...

$$\int d\xi\, A(\xi^*,\xi) = \int d\xi\,(a_0 + a_1\xi + \bar{a}_1\xi^* + a_{12}\xi^*\xi) = a_1 - a_{12}\xi^*$$

$$\int d\xi^* A(\xi^*,\xi) = \bar{a}_1 + a_{12}\xi$$

$$\int d\xi^* d\xi\, A(\xi^*,\xi) = -a_{12} = -\int d\xi\, d\xi^*\, A(\xi^*,\xi)$$

Why such definitions ??? → see later

Scalar product of Grassmann functions: $\langle f|g\rangle = \int d\xi^* \int d\xi\, e^{-\xi^*\xi} f^*(\xi)\, g(\xi^*)$

$\longrightarrow$ structure of a Hilbert space.

Back to fermions

To each a_α associate a generator ξ_α (Grassmann algebra $\mathcal{G}$)

$a_\alpha^+ \longrightarrow \xi_\alpha^*$

Generalized Fock space → set of linear combinations of states of the Fock space $\mathcal{F}$ with coefficients in $\mathcal{G}$.

Add a commutation rule between a's and ξ's : $[\tilde{\xi}, \tilde{a}]_+ = 0$ $\quad \tilde{\xi} \in \{\xi_\alpha, \xi_\alpha^*\}$

$(\tilde{\xi}\tilde{a})^+ = \tilde{a}^+ \tilde{\xi}^*$ $\quad \tilde{a} \in \{a_\beta, a_\beta^+\}$

Define a fermion coherent state $|\xi\rangle = e^{-\sum_\alpha \xi_\alpha a_\alpha^+}|0\rangle = \prod_\alpha (1 - \xi_\alpha a_\alpha^+)|0\rangle$

(exact because any $\xi_\alpha a_\alpha^+$ commutes with any $\xi_\beta a_\beta^+$)

- Verify that this is an eigenstate of the annihilation operators:

For a single state α $\quad a_\alpha(1 - \xi_\alpha a_\alpha^+)|0\rangle = -a_\alpha \xi_\alpha |1_\alpha\rangle = \xi_\alpha a_\alpha |1_\alpha\rangle = \xi_\alpha |0\rangle = \xi_\alpha (1 - \xi_\alpha a_\alpha^+)|0\rangle$

$$a_\alpha|\xi\rangle = a_\alpha \prod_\beta (1 - \xi_\beta a_\beta^+)|0\rangle = \prod_{\beta\neq\alpha}(1 - \xi_\beta a_\beta^+)\, a_\alpha (1 - \xi_\alpha a_\alpha^+)|0\rangle = \prod_{\beta\neq\alpha}(1-\xi_\beta a_\beta^+)\,\xi_\alpha \underbrace{(1 - \xi_\alpha a_\alpha^+)}_{|0\rangle} = \xi_\alpha \prod_\beta (1 - \xi_\beta a_\beta^+)|0\rangle = \xi_\alpha |\xi\rangle$$

- The adjoint state : $\langle\xi| = \langle 0| e^{-\sum_\alpha a_\alpha \xi_\alpha^*} \longrightarrow \langle\xi| a_\alpha^+ = \langle\xi|\xi_\alpha^*$

- Action of a_α^+ on $|\xi\rangle$:

$$a_\alpha^+|\xi\rangle = a_\alpha^+(1 - \xi_\alpha a_\alpha^+)\prod_{\beta\neq\alpha}(1 - \xi_\beta a_\beta^+)|0\rangle = a_\alpha^+ \prod_{\beta\neq\alpha}(1 - \xi_\beta a_\beta^+)|0\rangle = -\frac{\partial}{\partial\xi_\alpha}(1 - \xi_\alpha a_\alpha^+)\prod_{\beta\neq\alpha}(1 - \xi_\beta a_\beta^+)|0\rangle = -\frac{\partial}{\partial \xi_\alpha}|\xi\rangle$$

- Overlap of two coherent states :

$$\langle 0|\left(\prod_\beta (1 - a_\beta \xi_\beta^*)\right)\prod_\alpha (1 - \xi'_\alpha a_\alpha^{+})|0\rangle \overset{(\bullet)}{=} \langle 0|\prod_\alpha (1 + \xi_\alpha^* a_\alpha)(1 - \xi'_\alpha a_\alpha^+)|0\rangle =$$

$$= \langle 0|\prod_\alpha (1 + \xi_\alpha^* a_\alpha - \xi'_\alpha a_\alpha^+ + \xi_\alpha^* \xi'_\alpha a_\alpha a_\alpha^+)|0\rangle \overset{(\bullet)}{=} \langle 0|\prod_\alpha (1 + \xi_\alpha^* \xi'_\alpha)|0\rangle$$

$$= \prod_\alpha (1 + \xi_\alpha^* \xi'_\alpha) = e^{\sum_\alpha \xi_\alpha^* \xi'_\alpha}$$

- Closure relation: $\int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} |\xi\rangle\langle\xi| = 1$ (unity in the physical fermionic Fock space!)

Proof:

$$A := \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} |\xi\rangle\langle\xi|$$

$|\alpha_1 \dots \alpha_n\rangle$, $|\beta_1 \dots \beta_m\rangle$ } basis states from the Fock space

$$\langle \alpha_1 \dots \alpha_n | \beta_1 \dots \beta_m \rangle = \langle \alpha_1 \dots \alpha_n | A | \beta_1 \dots \beta_m \rangle \qquad (*)$$

This we show.

$$\langle \alpha_1 \dots \alpha_n | \xi \rangle = \langle 0 | a_{\alpha_n} \dots a_{\alpha_1} | \xi \rangle = \xi_{\alpha_n} \dots \xi_{\alpha_1}$$

$$\langle \alpha_1 \dots \alpha_n | A | \beta_1 \dots \beta_m \rangle = \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \langle \alpha_1 \dots \alpha_n | \xi \rangle \langle \xi | \beta_1 \dots \beta_m \rangle$$

$$= \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \prod_\beta (1 - \xi_\beta^* \xi_\beta) \, \xi_{\alpha_n} \dots \xi_{\alpha_1} \xi_{\beta_1}^* \dots \xi_{\beta_m}^*$$

Consider the integral arising for a particular state γ

$$\int d\xi_\gamma^* d\xi_\gamma (1 - \xi_\gamma^* \xi_\gamma) \begin{Bmatrix} \xi_\gamma \xi_\gamma^* \\ \xi_\gamma^* \\ \xi_\gamma \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

Integral non-zero only if each γ is either occupied in both $|\alpha_1 \dots \alpha_n\rangle$ and $|\beta_1 \dots \beta_m\rangle$ $\rightarrow$ $\beta_1 \dots \beta_m$ is a permutation of $(\alpha_1 \dots \alpha_n)$ $\rightarrow n = m$, $\beta_i = \alpha_{P(i)}$ (PERMUTATION)

($\bullet$) evaluate this integral

Hint: $\xi_{\alpha_n} \dots \xi_{\alpha_1} \xi_{\beta_1}^* \dots \xi_{\beta_n}^* = (-1)^P \xi_{\alpha_n} \dots \xi_{\alpha_1} \xi_{\alpha_1}^* \dots \xi_{\alpha_n}^*$.

$$\int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \prod_\beta (1 - \xi_\beta^* \xi_\beta) \, \xi_{\alpha_n} \dots \xi_{\alpha_1} \xi_{\alpha_{P(1)}}^* \dots \xi_{\alpha_{P(n)}}^* = \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \prod_\beta (1 - \xi_\beta^* \xi_\beta) (-1)^P \xi_{\alpha_n} \dots \xi_{\alpha_1} \xi_{\alpha_1}^* \dots \xi_{\alpha_n}^* \overset{(\bullet)}{=}$$

$$= (-1)^P$$

This demonstrates $(*)$

• Trace of an operator

$$\mathrm{Tr}\,A = \sum_n \langle n|A|n\rangle \qquad \text{e.g. } A = e^{-(\hat{H}-\mu\hat{N})}$$

$$\mathrm{Tr}\,A = \int \prod_\alpha d\xi^*_\alpha d\xi_\alpha\, e^{-\sum_\alpha \xi^*_\alpha \xi_\alpha} \sum_n \langle n|\xi\rangle\langle\xi|A|n\rangle = \circledast$$

Be careful about interchanging the order of $\langle n|\xi\rangle$ and $\langle\xi|A|n\rangle$

Assume A does not change the particle number.

$$A|n\rangle = \sum_m a_m |m\rangle \qquad p - \text{number of particles}$$

$$|n\rangle = a^+_{\alpha_1} \cdots a^+_{\alpha_p}|0\rangle \qquad \langle n| = \langle 0| a_{\alpha_p} \cdots a_{\alpha_1}$$

$$|m\rangle = a^+_{\beta_1} \cdots a^+_{\beta_p}|0\rangle \qquad \langle n|\xi\rangle = \xi_{\alpha_p} \cdots \xi_{\alpha_1}$$

$$\langle n|\xi\rangle\langle\xi|m\rangle = (\xi_{\alpha_p} \cdots \xi_{\alpha_1})(\xi^*_{\beta_1} \cdots \xi^*_{\beta_p}) = \left((-1)^p\right)^p (\xi^*_{\beta_1} \cdots \xi^*_{\beta_p})(\xi_{\alpha_p} \cdots \xi_{\alpha_1})$$

$$= (-1)^{p^2} \langle\xi|m\rangle\langle n|\xi\rangle \qquad (-1)^{p^2} = (-1)^p$$

$$\langle n|\xi\rangle\langle\xi|m\rangle = (-1)^{\overset{\text{particle number}}{p}} \langle\xi|m\rangle\langle n|\xi\rangle$$

$$\langle n|\xi\rangle\langle\xi|m\rangle = (-1)^p \langle\xi|m\rangle\langle n|\xi\rangle = \langle -\xi|m\rangle\langle n|\xi\rangle$$

E.g. $|\xi\rangle = (1-\xi_1 a^+_1)(1-\xi_2 a^+_2)|0\rangle \qquad (n=2)$

$$= |0\rangle - \xi_1|10\rangle - \xi_2|01\rangle - \xi_1\xi_2|11\rangle$$

$$|-\xi\rangle = |0\rangle + \xi_1|10\rangle + \xi_2|01\rangle - \xi_1\xi_2|11\rangle$$

(added a minus to the components with odd particle number)

- $\langle 0|\xi\rangle\langle\xi|0\rangle = 1\cdot 1 = \langle -\xi|0\rangle\langle 0|\xi\rangle$
- $\langle 10|\xi\rangle\langle\xi|10\rangle = (-\xi_1)(-\xi^*_1) = -\xi^*_1\xi_1 = -\langle\xi|10\rangle\langle 10|\xi\rangle = \langle -\xi|10\rangle\langle 10|\xi\rangle$
- $\langle 11|\xi\rangle\langle\xi|11\rangle = (-\xi_1\xi_2)(-\xi^*_2\xi^*_1) = (\xi^*_2\xi^*_1)(\xi_1\xi_2) = \langle\xi|11\rangle\langle 11|\xi\rangle$

$$\langle -\xi|11\rangle\langle 11|\xi\rangle =$$

Going back to ⊗: $\circledast = \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \sum_n \sum_m a_m \langle -\xi | m \rangle \langle n | \xi \rangle$

$$= \left\{ \sum_m a_m |m\rangle = A|n\rangle \right\} = \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \langle -\xi | A \sum_n |n\rangle\langle n | \xi \rangle$$

$$= \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \langle -\xi | A | \xi \rangle$$

$$\mathrm{Tr} A = \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \langle -\xi | A | \xi \rangle$$