Coherent states: summary

(Now consider both fermions and bosons)

$$\zeta = \begin{cases} +1 & \text{bosons} \\ -1 & \text{fermions} \end{cases} \qquad [a_\alpha, a_\beta^+]_{-\zeta} = \delta_{\alpha\beta}$$

$$|\xi\rangle = e^{\zeta \sum_\alpha \xi_\alpha a_\alpha^+}|0\rangle \qquad a_\alpha|\xi\rangle = \xi_\alpha|\xi\rangle \qquad \langle\xi|a_\alpha^+ = \langle\xi|\xi_\alpha^*$$

ξ, ξ^* – $\begin{cases}\text{complex numbers for boson} \\ \text{Grassmann numbers for fermions}\end{cases}$

$$a_\alpha^+|\xi\rangle = \zeta\frac{\partial}{\partial\xi_\alpha}|\xi\rangle$$

$$\langle\xi|A(a_\alpha^+, a_\alpha)|\xi'\rangle = e^{\sum_\alpha \xi_\alpha^*\xi_\alpha'} A(\xi_\alpha^*, \xi_\alpha') \quad (A \text{ normal ordered})$$

$$d\mu(\xi) = \prod_\alpha \left(\frac{d\xi_\alpha^* d\xi_\alpha}{\mathcal{N}}\right) \qquad 1 = \int d\mu(\xi)\, e^{-\sum_\alpha \xi_\alpha^*\xi_\alpha}|\xi\rangle\langle\xi|$$

$$\mathcal{N} = \begin{cases} 2\pi i & \text{bosons} \\ 1 & \text{fermions}\end{cases} \qquad \mathrm{Tr}A = \int d\mu(\xi)\, e^{-\sum_\alpha \xi_\alpha^*\xi_\alpha}\langle\zeta\xi|A|\xi\rangle$$

Propagator in QM vs partition function (case of a single particle)

$$|\psi(t)\rangle = \mathcal{U}(t)|\psi(t=0)\rangle$$

$\mathcal{U}(t)$ → evolution operator (Schrödinger picture) $\quad \mathcal{U}(t) = e^{-\frac{i\hat{H}}{\hbar}t}$ (for time-independent $\hat{H}$)

Probability amplitude to find a particle at position q_f at time t_f:

$$\psi(q_f, t_f) = \langle q_f|\psi(t_f)\rangle = \langle q_f|\hat{\mathcal{U}}(t_f - t_i)|\psi(t_i)\rangle = \int dq_i\, \underbrace{\mathcal{U}(q_f, q_i, t_f - t_i)}_{=\langle q_f|\hat{\mathcal{U}}(t_f-t_i)|q_i\rangle}\psi(q_i, t_i)$$

– propagator (probability amplitude for a particle to propagate from q_i to q_f in a time $t_f - t_i$)

$$\hat{H}|n\rangle = \epsilon_n|n\rangle$$

$$\mathcal{U}(q_f, q_i, t_f - t_i) = \langle q_f|e^{-\frac{i\hat{H}}{\hbar}(t_f - t_i)}|q_i\rangle = \sum_n \langle q_f|n\rangle e^{-\frac{i\epsilon_n}{\hbar}(t_f - t_i)}\langle n|q_i\rangle$$

$$= \sum_n e^{-\frac{i}{\hbar}\epsilon_n(t_f - t_i)}\varphi_n(q_f)\varphi_n^*(q_i)$$

if we know $\{|n\rangle, \epsilon_n\}$, we can calculate $\mathcal{U}$

$\mathcal{U}(q_f, q_i, t)$ can also be represented as a path integral $\rightarrow$ see the exercise class

- Consider first an infinitesimal time ϵ and recall the Baker-Campbell-Hausdorff formula $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]-[Y,[X,Y]])+\dots}$

To first order in ϵ: $\mathcal{U}(q_f, q_i, \epsilon) = \langle q_f | e^{-i\frac{\hat{H}}{\hbar}\epsilon} | q_i\rangle \simeq \langle q_f | e^{-i\epsilon\frac{\hat{p}^2}{2m}\frac{1}{\hbar}} e^{-i\frac{\epsilon}{\hbar}V(\hat{q})} | q_i\rangle$

$\left(\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})\right)$

$1 = \sum_p |p\rangle\langle p|$ allows for evaluating $\mathcal{U}(q_f, q_i, \epsilon)$

$\mathcal{U}(q_f, q_i, \epsilon) = \left(\frac{m}{2\pi i \epsilon}\right)^{\frac{1}{2}} e^{iS(q_f, q_i, \epsilon)}$

$\left\{ e^{\epsilon\hat{X}+\epsilon\hat{Y}} = 1 + \epsilon(\hat{X}+\hat{Y}) + \mathcal{O}(\epsilon^2) = e^{\epsilon\hat{X}} e^{\epsilon\hat{Y}} e^{\mathcal{O}(\epsilon^2)} \right\}$

- $\mathcal{U}(q_f, q_i, t_f - t_i) = \langle q_f | \underbrace{e^{-i\hat{H}\frac{\epsilon}{\hbar}} e^{-i\hat{H}\frac{\epsilon}{\hbar}} \dots e^{-i\hat{H}\frac{\epsilon}{\hbar}}}_{N \text{ times}} | q_i\rangle$, $N\cdot\epsilon = t_f - t_i$.

At the end send $N\rightarrow\infty$ $(\epsilon\rightarrow 0)$

At each time step insert $\int_q |q\rangle\langle q|$

$$\mathcal{U}(q_f, q_i, t_f - t_i) = \int\left(\prod_{k=1}^{N-1} dq_k\right) \langle q_f | e^{-i\frac{\hat{H}}{\hbar}\epsilon} | q_{N-1}\rangle\langle q_{N-1} | e^{-i\frac{\hat{H}}{\hbar}\epsilon} | q_{N-2}\rangle \dots \langle q_1 | e^{-i\frac{\hat{H}}{\hbar}\epsilon} | q_i\rangle =$$

$\begin{pmatrix} q_0 = q_i \\ q_N = q_f \end{pmatrix}$ $= \int\prod_{k=1}^{N-1} dq_k \prod_{k=1}^{N} \mathcal{U}(q_k, q_{k-1}, \epsilon)$ $\dots \rightarrow$ see the exercise class.

<u>Partition function</u> $\rightarrow$ $Z = \mathrm{Tr}\, e^{-\beta\hat{H}} = \int dq \underbrace{\langle q | e^{-\beta\hat{H}} | q\rangle}_{\text{Matrix element of the evolution operator in imaginary time!}}$ (single particle as above)

$\left(e^{-i\frac{\hat{H}}{\hbar}t} \xrightarrow{t\rightarrow -i\beta\hbar} e^{-\beta\hat{H}} \right)$

Real-time dynamics connected to (equilibrium) statistical mechanics by $t\rightarrow -i\tau$ (so called Wick rotation)

$\{q_0 = q_i,\ q_N = q_f\}$

$$\mathcal{U}(q_f, q_i, -i\tau) = \langle q_f | e^{-\frac{\hat{H}\tau}{\hbar}} | q_i\rangle \overset{(?)}{=} \lim_{N\to\infty}\left(\frac{mN}{2\pi\tau\hbar}\right)^{\frac{N}{2}} \int\prod_{k=1}^{N-1} dq_k\, e^{-\frac{\epsilon}{\hbar}\sum_{k=1}^{N}\left[\frac{m}{2}\frac{(q_k - q_{k-1})^2}{\epsilon^2} + V(q_{k-1})\right]} =$$

$$= \int_{q(0)=q_i}^{q(\tau)=q_f} \mathcal{D}[q]\, e^{-S_E[q]\frac{1}{\hbar}} \qquad S_E(q) = \int_0^\tau d\tau' \left[\frac{m}{2}\dot{q}^2 + V(q)\right]$$

("symbolic" notation — the paths are not differentiable)

{ paths which are not continuous are exponentially suppressed since $\frac{q_k - q_{k-1}}{\epsilon} \to \infty$. }

$$Z = \int dq\, \mathcal{U}(q, q, -i\beta\hbar) = \int_{q(\beta\hbar)=q(0)} \mathcal{D}[q]\, e^{-S_E[q]\frac{1}{\hbar}} \qquad \text{(no operators...)}$$

Classical limit: $\beta\hbar \to 0^+ \to$ the time interval becomes "infinitesimal" and can be calculated with a single time step ($N=1$)

$$Z_{cl} = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{\frac{1}{2}} \int dq\, e^{-\beta\hbar\left[\frac{m}{2}\frac{(q-q)^2}{(\beta\tau)^2} + V(q)\right]/\hbar} = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{\frac{1}{2}} \int dq\, e^{-\beta V(q)} =$$

$$= \int dq\, e^{-\beta V(q)} \int dp\, e^{-\beta\frac{p^2}{2m}} \underbrace{\sqrt{\frac{\beta/2m}{\pi}} \sqrt{\frac{m}{2\pi\beta\hbar^2}}}_{\frac{1}{2\pi\hbar}} = \int \frac{dp}{h} \int dq\, e^{-\beta\left(\frac{p^2}{2m} + V(q)\right)}$$

$$\frac{\sqrt{\pi}}{\sqrt{\beta/2m}}$$

(Further on $\hbar = 1$).

Path integral for the many-body systems

Recall:

$$d\mu(\xi) = \prod_\alpha \left(\frac{d\xi_\alpha^* d\xi_\alpha}{\mathcal{N}}\right) \qquad \mathcal{N} = \begin{cases} 2\pi i & \text{bosons} \\ 1 & \text{fermions} \end{cases}$$

$$1 = \int d\mu(\xi)\, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} |\xi\rangle\langle\xi|$$

$$\mathrm{Tr}\, A = \int d\mu(\xi)\, e^{-\sum_\alpha \xi_\alpha^*\xi_\alpha} \langle \zeta\xi | A | \xi\rangle$$

$$Z = \mathrm{Tr}\, e^{-\beta\hat{H}} = \int d\mu(\xi)\, e^{-\sum_\alpha \xi_\alpha^*\xi_\alpha} \langle \zeta\xi | e^{-\beta\hat{H}} | \xi\rangle$$

$\hat{H}$ is normal-ordered

$(\hat{H} = :\hat{H}:)$ (normal-ordered exponential)

$$\to e^{-\epsilon\hat{H}} = :e^{-\epsilon\hat{H}}: + \mathcal{O}(\epsilon^2)$$

Here GCA $(-\beta\hat{H} \leftrightarrow -\beta(\hat{H} - \mu\hat{N}))$!!!

$$Z = \int d\mu(\xi)\, e^{-\sum_\alpha \xi_\alpha^*\xi_\alpha} \langle \zeta\xi | \underbrace{e^{-\epsilon\hat{H}}\, e^{-\epsilon\hat{H}} \dots e^{-\epsilon\hat{H}}}_{M \gg 1 \text{ times}} | \xi \rangle \qquad \left(M = \frac{\beta}{\epsilon}\right)$$

insert the coherent-state unity between each $e^{-\beta\epsilon\hat{H}}\; e^{-\beta\epsilon\hat{H}}$ (M-1 times)

$0 \quad \epsilon \quad \beta$

$$\langle \xi_k | e^{-\epsilon\hat{H}(a_\alpha^+, a_\alpha)} | \xi_{k-1} \rangle = \langle \xi_k | \xi_{k-1} \rangle\, e^{-\epsilon H(\xi_{k\alpha}^*, \xi_{k-1\alpha})} + \mathcal{O}(\epsilon^2).$$

Introduce $\xi_0 = \zeta \xi_M \qquad \xi \to \xi_0$

$\xi_0^* = \zeta \xi_M^*$

No operators!

$$Z = \int \prod_{k=1}^{M} d\mu_{(k)}(\xi_{(k)})\, e^{-\epsilon \sum_{k=1}^{M}\left[\sum_\alpha \xi^*_{(k)\alpha} \frac{(\xi_{(k)\alpha} - \xi_{(k-1)\alpha})}{\epsilon} + H(\xi^*_{(k)\alpha}, \xi_{(k-1)\alpha})\right]}$$

for $\epsilon \to 0$, $M \to \infty$, $\epsilon M = \beta$

$\{\xi_{(0)\alpha}, \xi_{(1)\alpha} \dots \xi_{(M)\alpha}\} \longrightarrow \xi_\alpha(\tau)$ (continuous trajectory)

$\tau \in]0, \beta] \to$ imaginary time

boundary conditions: $\xi_\alpha(\beta) = \zeta \xi_\alpha(0)$

$\xi^*_\alpha(\beta) = \zeta \xi^*_\alpha(0)$

Symbolic notation: $\epsilon \sum_{k=1}^{M} \longrightarrow \int_0^\beta d\tau$

$$\xi^*_{(k)\alpha} \frac{\xi_{(k)\alpha} - \xi_{(k-1)\alpha}}{\epsilon} \longrightarrow \xi^*_\alpha(\tau)\partial_\tau \xi_\alpha(\tau)$$

$$H(\xi^*_{(k)\alpha}, \xi_{(k-1)\alpha}) \longrightarrow H(\xi^*_\alpha(\tau^+), \xi_\alpha(\tau))$$

$$Z = \int_{\substack{\xi(\beta) = \zeta\xi(0) \\ \xi^*(\beta) = \zeta \xi^*(0)}} \mathcal{D}[\xi^*, \xi]\, e^{-S[\xi^*, \xi]} \quad (*)$$

$$S[\xi^*, \xi] = \int_0^\beta d\tau \left[\sum_\alpha \xi^*_\alpha(\tau)\partial_\tau \xi_\alpha(\tau) + H(\xi^*_\alpha(\tau^+), \xi_\alpha(\tau))\right]$$

$$\mathcal{D}[\xi^*, \xi] = \lim_{M\to\infty} \prod_{k=0}^{M} d\mu_{(k)}(\xi_{(k)})$$

(supplemented with the b.c (*))

EXAMPLE Non-interacting particles

$$\hat{H} = \sum_\alpha (\epsilon_\alpha - \mu) a^\dagger_\alpha a_\alpha \qquad \xi \Leftrightarrow \phi$$

$$Z_0 = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_\alpha \frac{1}{N} d\phi^*_{\alpha k}\, d\phi_{\alpha k}\, \exp\left\{-\epsilon \sum_{k=2}^{M}\left[\sum_\alpha \phi^*_{\alpha k} \frac{\phi_{\alpha k} - \phi_{\alpha, k-1}}{\epsilon} + \sum_\alpha (\epsilon_\alpha - \mu)\phi^*_{\alpha k}\phi_{\alpha k-1}\right] + \right.$$

$$\left. - \epsilon \sum_\alpha \phi^*_{\alpha, 1} \frac{\phi_{\alpha 1} - \zeta \phi_{\alpha M}}{\epsilon} + \sum_\alpha (\epsilon_\alpha - \mu)\phi^*_{\alpha 1}\phi_{\alpha M}\right\} =$$

$$\left\{\epsilon = \frac{\beta}{M} \qquad a := 1 - \frac{\beta}{M}(\epsilon_\alpha - \mu)\right\}$$

$$Z_0 = \lim_{n\to\infty} \int \prod_{k=1}^{n} \prod_{\alpha} \frac{1}{N} d\phi^*_{\alpha k} d\phi_{\alpha k} \exp\left\{ -\sum_{\alpha} \left[\phi^*_{\alpha 1} \cdots \phi^*_{\alpha n}\right] \underbrace{\begin{bmatrix} 1 & 0 & \cdots & & -\zeta a \\ -a & 1 & & & \vdots \\ 0 & -a & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \\ 0 & & & -a & 1 \end{bmatrix}}_{S^{(\alpha)}} \begin{bmatrix} \phi_{\alpha 1} \\ \vdots \\ \vdots \\ \phi_{\alpha n} \end{bmatrix} \right\}$$

Gaussian integrations.

$$Z_0 = \lim_{n\to\infty} \prod_{\alpha} \left[\det S^{(\alpha)}\right]^{-\zeta}$$

$$\det S^{(\alpha)} = \begin{pmatrix} \text{LAPLACE EXPANSION} \\ \text{with respect to} \\ \text{1st row} \end{pmatrix} = \left\{ \text{to be finished} \right\}$$