Finish the previous week's calculation:

EXAMPLE Non-interacting particles

$$\hat{H} = \sum_\alpha (\epsilon_\alpha - \mu) a_\alpha^\dagger a_\alpha \qquad \{ \Leftrightarrow \phi$$

$$Z_0 = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_\alpha \frac{1}{N} d\phi_{\alpha k}^* d\phi_{\alpha k} \exp\left\{ -\epsilon \sum_{k=2}^{M} \left[\sum_\alpha \phi_{\alpha k}^* \frac{\phi_{\alpha k} - \phi_{\alpha, k-1}}{\epsilon} + \sum_\alpha (\epsilon_\alpha - \mu) \phi_{\alpha k}^* \phi_{\alpha k-1} \right] + \right.$$

$$\left. - \epsilon \sum_\alpha \phi_{\alpha 1}^* \frac{\phi_{\alpha 1} - \zeta \phi_{\alpha M}}{\epsilon} + \sum_\alpha (\epsilon_\alpha - \mu) \phi_{\alpha 1}^* \phi_{\alpha M} \right\} =$$

$$\left\{ \epsilon = \frac{\beta}{M} \qquad a := 1 - \frac{\beta}{M}(\epsilon_\alpha - \mu) \right\}$$

$$Z_0 = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_\alpha \frac{1}{N} d\phi_{\alpha k}^* d\phi_{\alpha k} \exp\left\{ -\sum_\alpha \left[\phi_{\alpha 1}^* \cdots \phi_{\alpha M}^* \right] \underbrace{\begin{bmatrix} 1 & 0 & \cdots & -\zeta a \\ -a & 1 & & \vdots \\ 0 & -a & \ddots & \\ \vdots & \vdots & \ddots & \\ 0 & & -a & 1 \end{bmatrix}}_{S^{(\alpha)}} \begin{bmatrix} \phi_{\alpha 1} \\ \vdots \\ \phi_{\alpha M} \end{bmatrix} \right\}$$

Gaussian integrations

$$Z_0 = \lim_{M\to\infty} \prod_\alpha \left(\det S^{(\alpha)} \right)^{-\zeta}$$

$$\det S^{(\alpha)} = \left(\begin{array}{c} \text{LAPLACE EXPANSION} \\ \text{with respect to} \\ \text{1st row} \end{array} \right) = \{ \text{to be finished} \}$$

$$\det S^{(\alpha)} = \det \begin{pmatrix} 1 & 0 & & \\ -a & 1 & & \\ 0 & -a & 1 & \\ \vdots & 0 & -a & \\ \vdots & \vdots & \vdots & \\ & & 0 & \end{pmatrix} + (-\zeta a)(-1)^{M+1} \det \begin{pmatrix} -a & 1 & & \\ 0 & -a & 1 & \\ 0 & 0 & -a & 1 \\ & & \ddots & \ddots \quad 1 \\ & 0 & & -a \end{pmatrix}$$

$$= 1 - \zeta a (-1)^{M+1} (-a)^{M-1} = 1 - \zeta a^M =$$

$$= 1 - \zeta \left[1 - \frac{\beta}{M}(\epsilon_\alpha - \mu) \right]^M$$

$$\xrightarrow{M\to\infty} 1 - \zeta e^{-\beta(\epsilon_\alpha - \mu)} \qquad \left\{ \left(1 + \frac{x}{n}\right)^n \xrightarrow{n\to\infty} e^x \right.$$

$$Z_0 = \prod_\alpha \left(1 - \zeta e^{-\beta(\epsilon_\alpha - \mu)} \right)^{-\zeta} \longrightarrow$$ Familiar expression for the GC partition function

Green's functions

(intro) QM: Recall the propagator: $\langle \bar{r} | e^{-i\hat{H}t} | \bar{r}' \rangle$ — $\hat{H}$ - time independent.

(Retarded) One-particle Green's function: $G^R(\bar{r},\bar{r}',t) := -i\theta(t)\langle \bar{r}| e^{-i\hat{H}t} |\bar{r}'\rangle = -i\theta(t)\sum_n e^{-i\epsilon_n t}\varphi_n(\bar{r})\varphi_n^*(\bar{r}')$

(retarded prop.) (spin ignored)

Fourier transform: $\tilde{G}^R(\bar{r},\bar{r}',\omega) = \int_{-\infty}^{\infty} dt\, e^{i(\omega+i\eta)t} G^R(\bar{r},\bar{r}',t) = \sum_n \frac{\varphi_n(\bar{r})\varphi_n^*(\bar{r}')}{\omega + i\eta - \epsilon_n}$ $(\eta = 0^+)$

\\ notation: $G^R(\bar{r},\bar{r}',\omega)$

{Recall the calculation: $\theta(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dx \frac{e^{-ixt}}{x + i0^+}$} (•)

Eigenenergies $(\epsilon_n) \longleftrightarrow$ Poles of $G^R(\bar{r},\bar{r}',\omega)$

Many-body systems

{ $G^R(\vec{r}t,\vec{r}'t') = -i\theta(t-t')\langle 0|\hat{\Psi}(\bar{r},t)\hat{\Psi}^+(\bar{r}',t')|0\rangle$ } — $t'=0$

$$G^R(\bar{r},\bar{r}',t) = -i\theta(t)\langle 0|[\hat{\Psi}(\bar{r},t)\hat{\Psi}^+(\bar{r}',0)]_{-\zeta}|0\rangle \qquad (T=0)$$

Heisenberg op. (position representation)

$$G^R(\bar{r},\bar{r}',\epsilon) = -i\delta(\bar{r}-\bar{r}')$$

{ $A(t) = e^{iHt} A e^{-iHt}$ $(\hbar = 1)$ }

$$G^R(\alpha,\alpha',t) = -i\theta(t)\langle [a_\alpha(t), a^+_{\alpha'}(t)]_{-\zeta}\rangle = -i\theta(t)\frac{1}{Z}\mathrm{Tr}\, e^{-\beta\hat{H}}[a_\alpha(t), a_{\alpha'}(t)]_{-\zeta}$$

→ retarded one-particle Green's function at any T, for any 1-particle basis $\{|\alpha\rangle\}$.

$\hat{H} \longleftrightarrow \hat{H} - \mu\hat{N}$ (GCE).

Non-interacting systems: $\hat{H} = \sum_\alpha (\epsilon_\alpha - \mu) a^+_\alpha a_\alpha$ (diagonal basis)

Solving the e.o.m. ⟶ $G_0^R(\alpha,t) = -i\theta(t) e^{-i(\epsilon_\alpha - \mu)t}$ $(\times\delta_{\alpha\alpha'})$ {Another point of view - see later today}

$$G_0^R(\alpha,\omega) = \int_{-\infty}^{\infty} dt\, e^{i(\omega+i\eta)t} G_0^R(\alpha,t) = \frac{1}{\omega-(\epsilon_\alpha-\mu)+i\eta} \qquad (\eta = 0^+)$$

[Poles $\longleftrightarrow$ single-particle excitation energies]

Matsubara Green's function (1-particle) ("imaginary time", "thermal" G.f.)

(sign conventional (absent in Negele-Orland))

(G) $$G(\alpha,\alpha',\tau) = -\langle T_\tau\, a_\alpha(\tau)\, a^+_{\alpha'}\rangle = -\theta(\tau)\langle a_\alpha(\tau)\, a^+_{\alpha'}\rangle - \zeta\theta(-\tau)\langle a^+_{\alpha'}\, a_\alpha(\tau)\rangle$$

(or $G(\alpha\tau,\alpha'\tau')$) $\tau' = 0$

T_τ – time-ordering operator.

the operators are evolved in imaginary time.

$$\begin{cases} a_\alpha(\tau) = e^{\tau\hat H} a_\alpha e^{-\tau\hat H} \\ a^+_\alpha(\tau) = e^{\tau\hat H} a^+_\alpha e^{-\tau\hat H} \end{cases} \qquad \{a^+_\alpha(\tau) \neq [a_\alpha(\tau)]^+\}$$

For $\tau = it$ recover the standard Heisenberg op.

{No specification of the meaning of $G(\alpha,\alpha',0)$ for now.}

{
- often easier calculable
- recall the connection between real-time dynamics and stat. phys.
}

Consider only $\tau \in [-\beta,\beta]$

- $G(\alpha,\alpha',\tau)$ is (anti)periodic with period β.

- Demonstrate for $\tau \in]-\beta\, 0[$: $G(\alpha,\alpha',\tau) = -\zeta\langle a^+_{\alpha'}\, a_\alpha(\tau)\rangle = \frac{-\zeta}{Z}\mathrm{Tr}\left(e^{-\beta\hat H} a^+_{\alpha'}\, e^{\tau\hat H} a_\alpha e^{-\tau\hat H}\right)$

$$= -\frac{\zeta}{Z}\mathrm{Tr}\left(e^{\tau\hat H} a_\alpha\, e^{-(\tau+\beta)\hat H} a^+_{\alpha'}\right) =$$

$$= -\frac{\zeta}{Z}\mathrm{Tr}\Big(\underbrace{e^{-\beta\hat H} e^{\beta\hat H}}_{1}\, \underbrace{e^{\tau\hat H} a_\alpha e^{-(\tau+\beta)\hat H}}_{a_\alpha(\tau+\beta)}\, a^+_{\alpha'}\Big) = -\frac{\zeta}{Z}\mathrm{Tr}\left(e^{-\beta\hat H} a_\alpha(\underbrace{\tau+\beta}_{>0})\, a^+_{\alpha'}\right) =$$

$$= -\zeta\langle a_\alpha(\tau+\beta)\, a^+_{\alpha'}\rangle = \zeta G(\alpha,\alpha',\tau+\beta)$$

- Analogous calculation for $\tau \in]0,\beta[\longrightarrow G(\alpha,\alpha',\tau) = \zeta G(\alpha,\alpha',\tau-\beta)$ (•)

Expansion in Fourier series:

$$\begin{cases} G(\alpha,\alpha',\tau) = \frac{1}{\beta}\sum_{\omega_n} e^{-i\omega_n\tau} G(\alpha,\alpha',i\omega_n) \\ G(\alpha,\alpha',i\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n\tau} G(\alpha,\alpha',\tau) \end{cases}$$ (•)

with $$\omega_n = \begin{cases} \frac{2\pi}{\beta} n & \text{(bosons)} \\ \frac{2\pi}{\beta}\left(n+\frac{1}{2}\right) & \text{(fermions)} \end{cases}$$ – Matsubara frequencies

$$\left(\sum_{\omega_n} = \sum_{n=-\infty}^{\infty}\right) \qquad \left(\sum_{\omega_n} \xrightarrow{T\to 0^+} \int \frac{d\omega}{2\pi}\right)$$

Non-interacting systems: $\hat{\mathcal{H}} = \sum_\alpha (\epsilon_\alpha - \mu) a_\alpha^\dagger a_\alpha$, evolution of $a_\alpha, a_\alpha^\dagger$ calculated before: $a_\alpha(\tau) = e^{-(\epsilon_\alpha-\mu)\tau} a_\alpha$

$a_\alpha^\dagger(\tau) = e^{-(\epsilon_\alpha-\mu)\tau} a_\alpha^\dagger$

(from def, or solving the e.o.m.)

(*) $$G_0(\alpha,\tau) = -e^{-(\epsilon_\alpha-\mu)\tau}\left[\theta(\tau)(1+\zeta n_\alpha) + \theta(-\tau)\zeta n_\alpha\right] \; ; \quad n_\alpha = \langle a_\alpha^\dagger a_\alpha \rangle$$

$$G_0(\alpha, i\omega_n) = \frac{1}{i\omega_n - (\epsilon_\alpha - \mu)}$$

{another approach – later today}

$$G_0(\alpha,\omega) = G_0(\alpha, i\omega_n \longrightarrow \omega + i\eta)$$

(true also for interacting systems)

Many-particle Green's functions:

- $$G^{(2n)}(\alpha_1\tau_1 \ldots \alpha_n\tau_n ; \alpha_n'\tau_n' \ldots \alpha_1'\tau_1') = (-1)^n \langle T_\tau\, a_{\alpha_1}(\tau_1) \ldots a_{\alpha_n}(\tau_n)\, a_{\alpha_n'}^\dagger(\tau_n') \ldots a_{\alpha_1'}^\dagger(\tau_1') \rangle$$
  (imaginary time) $\quad T_\tau \hat{A}_{\alpha_1}(\tau_1) \ldots \hat{A}_{\alpha_n}(\tau_n) = \zeta^P A_{\alpha_{P(1)}}(\tau_{P(1)}) \ldots \hat{A}_{\alpha_{P(n)}}(\tau_{P(n)})$

  such that $\tau_{P(1)} > \tau_{P(2)} > \ldots > \tau_{P(n)}$
- $$G^{(2n)}(\alpha_1 t_1 \ldots \alpha_n t_n ; \alpha_n' t_n' \ldots \alpha_1' t_1') = (-i)^n \langle T_t\, a_{\alpha_1}(t_1) \ldots a_{\alpha_n}(t_n)\, a_{\alpha_n'}^\dagger(t_n') \ldots a_{\alpha_1'}^\dagger(t_1') \rangle$$

One-body Green's function – path integral

Consider $$G(\alpha\tau, \alpha'\tau') = -\langle T_\tau\, a_\alpha(\tau)\, a_{\alpha'}^\dagger(\tau') \rangle$$

Take $\tau > \tau'$

$$\hookrightarrow G(\alpha\tau,\alpha'\tau') = -\frac{1}{Z}\mathrm{Tr}\left\{ e^{-(\beta-\tau)\hat{\mathcal{H}}} a_\alpha e^{-(\tau-\tau')\hat{\mathcal{H}}} a_{\alpha'}^\dagger e^{-\tau'\hat{\mathcal{H}}} \right\} \qquad (\hat{\mathcal{H}} = \hat{H} - \mu\hat{N})$$

Split the time intervals into M steps. Take $\tau = k_1\epsilon$, $\tau' = k_2\epsilon$. $\epsilon = \frac{\beta}{M}$

Alike for the partition function → use the coherent-state representation of the trace
→ plug the coherent-state unity between each two neighboring time slices.

$$G(\alpha\tau,\alpha'\tau') = -\frac{1}{Z}\int \prod_{k=1}^{M} d\mu_k(\zeta_k)\, e^{-\sum_{k=1}^{M}\sum_\alpha \zeta_{k\alpha}^* \zeta_{k\alpha}} \langle \zeta_M | e^{-\epsilon\hat{\mathcal{H}}} | \zeta_{M-1}\rangle \langle \zeta_{M-1} | e^{-\epsilon\hat{\mathcal{H}}} | \zeta_{M-2}\rangle \ldots \langle \zeta_{k_1+1} | e^{-\epsilon\hat{\mathcal{H}}} a_\alpha | \zeta_{k_1}\rangle \cdot$$
$$\cdot \langle \zeta_{k_1} | e^{-\epsilon\hat{\mathcal{H}}} | \zeta_{k_1-1}\rangle \ldots \langle \zeta_{k_2} | a_{\alpha'}^\dagger e^{-\epsilon\hat{\mathcal{H}}} | \zeta_{k_2-1}\rangle \langle \zeta_{k_2-1} | \ldots \langle \zeta_1 | e^{-\epsilon\hat{\mathcal{H}}} | \zeta_0 \rangle$$

With the b.c. $\xi_M = \zeta \xi_0$, $\xi_M^* = \zeta \xi_0$ $\quad \begin{pmatrix} M \to \infty \\ \epsilon \to 0 \end{pmatrix}$

$$G(\alpha\tau, \alpha'\tau') = -\frac{1}{Z}\int \mathcal{D}[\xi^* \xi]\, \xi_\alpha(\tau)\xi^*_{\alpha'}(\tau')\, e^{-S[\xi^*, \xi]}$$

- Take $\tau < \tau'$

$$\hookrightarrow \quad G(\alpha\tau, \alpha'\tau') = -\frac{\zeta}{Z}\operatorname{Tr}\left[e^{-(\beta-\tau')\hat{H}}\, a^+_{\alpha'}\, e^{-(\tau'-\tau)\hat{H}}\, a_\alpha\, e^{-\tau\hat{H}}\right] = \dots \text{(the same steps)}$$

$$= -\frac{\zeta}{Z}\int \mathcal{D}[\xi^* \xi]\, \xi^*_{\alpha'}(\tau')\xi_\alpha(\tau)\, e^{-S[\xi^* \xi]} =$$

$$= -\frac{1}{Z}\int \mathcal{D}[\xi^* \xi]\, \xi_\alpha(\tau)\xi^*_{\alpha'}(\tau')\, e^{-S[\xi^* \xi]}$$

$\tau = \tau'$ → chronological ordering equivalent to normal ordering
→ creation op. evaluated one step ($+\eta$) later.
→ use the expression for $\tau < \tau'$.

$$G(\alpha\tau, \alpha'\tau') = -\frac{1}{Z}\int \mathcal{D}[\xi^* \xi]\, \xi_\alpha(\tau)\xi^*_{\alpha'}(\tau')\, e^{-S[\xi^*, \xi]}$$

(no T operator!).

This generalizes to statistical averages of time-ordered products of arbitrary operators, n-particle Green's functions in particular.

$$\langle T_\tau\, \mathcal{O}_1(a^+_\alpha(\tau_1), a_\alpha(\tau_1)) \dots \mathcal{O}_n(a^+_\alpha(\tau_n), a_\alpha(\tau_n))\rangle = (\bullet)$$

Chronological ordering taken care of automatically!

$$= \frac{1}{Z}\int \mathcal{D}[\xi^* \xi]\, \mathcal{O}_1(\xi^*_\alpha(\tau_1), \xi_\alpha(\tau_1)) \dots \mathcal{O}_n(\xi^*_\alpha(\tau_n), \xi_\alpha(\tau_n))\, e^{-S[\xi^* \xi]}$$

Example:

Non-interacting particles

Notation below: $\phi \leftrightarrow \xi$, $\xi \to \phi$

$$\hat{H} = \sum_\alpha (\epsilon_\alpha - \mu)\, a^+_\alpha a_\alpha \qquad \xi \Leftrightarrow \phi$$

$$Z_0 = \lim_{M\to\infty}\int \prod_{k=1}^M \prod_{\alpha} \frac{1}{N}\, d\phi^*_{\alpha k}\, d\phi_{\alpha k}\, \exp\Big\{-\epsilon\sum_{k=2}^M\Big[\sum_\alpha \phi^*_{\alpha k}\frac{\phi_{\alpha k}-\phi_{\alpha, k-1}}{\epsilon} + \sum_\alpha(\epsilon_\alpha-\mu)\phi^*_{\alpha k}\phi_{\alpha k-1}\Big] + $$
$$- \epsilon\sum_\alpha \phi^*_{\alpha 1}\frac{\phi_{\alpha 1}-\zeta\phi_{\alpha M}}{\epsilon} + \sum_\alpha(\epsilon_\alpha-\mu)\phi^*_{\alpha 1}\phi_{\alpha M}\Big\} =$$

$$\Big\{\epsilon = \frac{\beta}{M} \qquad a := 1 - \frac{\beta}{M}(\epsilon_\alpha - \mu)\Big\}$$

$$Z_0 = \lim_{M\to\infty}\int \prod_{k=1}^M \prod_{\alpha} \frac{1}{N}\, d\phi^*_{\alpha k}\, d\phi_{\alpha k}\, \exp\Big\{-\sum_\alpha [\phi^*_{\alpha 1} \dots \phi^*_{\alpha M}] \underbrace{\begin{bmatrix} 1 & 0 & \dots & -\zeta a \\ -a & 1 & & \vdots \\ 0 & -a & \ddots & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & & -a & 1 \end{bmatrix}}_{S^{(\alpha)}} \begin{bmatrix}\phi_{\alpha 1} \\ \vdots \\ \phi_{\alpha M}\end{bmatrix}$$

- $G_0(\alpha\tau, \alpha'\tau') = 0$ for $\alpha \neq \alpha'$

Take $\alpha = \alpha'$.

$$G_0(\alpha, \tau_{k_1} - \tau_{k_2}) = -\lim_{M\to\infty}\frac{1}{Z_0}\int \prod_{k=1}^M \Big(\prod_{\alpha'} \frac{1}{N}\, d\phi^*_{\alpha' k}\, d\phi_{\alpha' k}\Big)\, \phi_{k_1\alpha}\phi^*_{k_2\alpha}\, e^{-\sum_{kk'=1}^M \sum_\alpha \phi^*_{k\alpha} S^{(\alpha)}_{kk'} \phi_{k'\alpha}} =$$

$$= -\lim_{M\to\infty}\Big[\big[S^{(\alpha)}\big]^{-1}\Big]_{k_1 k_2}$$

→ Moments of the Gaussian distribution (see the exercise class)

Inversion of $S^{(\alpha)} \longrightarrow$

$$\left[S^{(\alpha)}\right]^{-1} = \begin{pmatrix} 1 & \zeta a^{M-1} & \zeta a^{M-2} & \cdots & \zeta a \\ a & 1 & \zeta a^{M-1} & & \vdots \\ a^2 & a & 1 & \ddots & \\ \vdots & \ddots & a & \ddots & \zeta^{M-1} \\ a^{M-1} & \cdots & & a & 1 \end{pmatrix} \frac{1}{1-\zeta a^M}$$

$$G_0(\alpha, \tau_{k_1}-\tau_{k_2}) = -\lim_{M\to\infty} \begin{cases} \frac{1}{1-\zeta a^M}\, \zeta\, a^{M+k_1-k_2} & \tau_{k_1}-\tau_{k_2} \leq 0 \quad (\text{row \# < column \#}) \\ \frac{1}{1-\zeta a^M}\, a^{k_1-k_2} & \tau_{k_1}-\tau_{k_2} > 0 \end{cases}$$

$\tau_1-\tau_2 = (k_1-k_2)\epsilon$, $\epsilon = \frac{\beta}{M}$, $k_1-k_2 = M\frac{\tau_1-\tau_2}{\beta}$

Remember to use the expression valid for $\tau_{k_1}-\tau_{k_2} < 0$ in the case $k_1 = k_2$! (see p. (4))

- $a^M = \left[1-\frac{\beta}{M}(\epsilon_\alpha-\mu)\right]^M \xrightarrow{M\to\infty} e^{-\beta(\epsilon_\alpha-\mu)}$
- $a^{k_1-k_2} = \left[1-\frac{\beta}{M}(\epsilon_\alpha-\mu)\right]^{\frac{M}{\beta}(\tau_1-\tau_2)} \xrightarrow{M\to\infty} e^{-(\epsilon_\alpha-\mu)(\tau_1-\tau_2)}$
- $\frac{1}{1-\zeta a^M} = \frac{1}{1-\zeta e^{-\beta(\epsilon_\alpha-\mu)}} = \frac{e^{\beta(\epsilon_\alpha-\mu)}}{e^{\beta(\epsilon_\alpha-\mu)}-\zeta} = n_\alpha e^{\beta(\epsilon_\alpha-\mu)}$ ↳ F-D or B-E distribution
- $\frac{1}{1-\zeta a^M}\zeta a^M a^{k_1-k_2} \Rightarrow n_\alpha e^{\beta(\epsilon_\alpha-\mu)} \zeta e^{-\beta(\epsilon_\alpha-\mu)} e^{-(\epsilon_\alpha-\mu)(\tau_1-\tau_2)} = n_\alpha \zeta e^{-(\epsilon_\alpha-\mu)(\tau_1-\tau_2)}$
- $\frac{1}{1-\zeta a^M} a^{k_1-k_2} \longrightarrow n_\alpha e^{\beta(\epsilon_\alpha-\mu)} e^{-(\epsilon_\alpha-\mu)(\tau_1-\tau_2)} = \frac{e^{\beta(\,)}-\zeta+\zeta}{e^{\beta(\,)}-\zeta} e^{-(\,)(\,)} = (1+\zeta n_\alpha)e^{-(\epsilon_\alpha-\mu)(\tau_1-\tau_2)}$

$$G_0(\alpha, \tau_1-\tau_2) = \begin{cases} -\zeta n_\alpha e^{-(\tau_1-\tau_2)(\epsilon_\alpha-\mu)} & \tau_1 \leq \tau_2 \\ -(1+\zeta n_\alpha) e^{-(\tau_1-\tau_2)(\epsilon_\alpha-\mu)} & \tau_1 > \tau_2 \end{cases}$$

(in agreement with the result from solving the e.o.m.)

Another way (simpler) to calculate the path integral for G_0 (and $\mathcal{Z}_0$ alike)

$$G(\alpha\tau, \alpha'\tau') = -\frac{1}{\mathcal{Z}}\int \mathcal{D}[\zeta^*\zeta]\, \zeta_\alpha(\tau)\zeta^*_{\alpha'}(\tau')\, e^{-S[\zeta^*\zeta]} \longrightarrow \text{diagonalize } S \text{ by Fourier transform.}$$

$$\hookrightarrow G_0(\alpha, \tau_1-\tau_2) = -\langle T a_\alpha(\tau_1) a^\dagger_{\alpha'}(\tau_2)\rangle_0 = -\delta_{\alpha\alpha'} e^{-(\epsilon_\alpha-\mu)(\tau_1-\tau_2)}\left\{\Theta(\tau_1-\tau_2-\eta)(1+\zeta n_\alpha) + \zeta n_\alpha \Theta(\tau_2-\tau_1+\eta)\right\} =$$

$$= -\delta_{\alpha\alpha'}\, g_\alpha(\tau_1-\tau_2-\eta) \qquad \eta = 0^+$$

→ Key quantity for perturbation theory.