Example:

Non-interacting particles

Previous lecture

- $G_0(\alpha\tau, \alpha'\tau') = 0$ for $\alpha \neq \alpha'$

Take $\alpha = \alpha'$.

$\boxed{\text{Notation } \phi \Longleftrightarrow \zeta}$ below $\zeta \longrightarrow \phi$

$$G_0(\alpha, \tau_{k_1} - \tau_{k_2}) = -\lim_{M\to\infty} \frac{1}{Z_0} \int \prod_{k=1}^{M} \left(\prod_{\alpha'} \frac{1}{N} d\phi^*_{\alpha' k} d\phi_{\alpha' k}\right) \phi_{k_1\alpha} \phi^*_{k_2\alpha}\, e^{-\sum_{kk'=1}^{M} \sum_{\alpha} \phi^*_{k\alpha} S^{(\alpha)}_{kk'} \phi_{k'\alpha}} =$$

$\rightarrow$ Moments of the Gaussian distribution (see the exercise class)

$$= -\lim_{M\to\infty} \left[\left[S^{(\alpha)}\right]^{-1}\right]_{k_1 k_2}$$

Inversion of $S^{(\alpha)} \longrightarrow$

$$\left[S^{(\alpha)}\right]^{-1} = \begin{pmatrix} 1 & \zeta a^{M-1} & \zeta a^{M-2} & \cdots & \zeta a \\ a & 1 & \zeta a^{M-1} & & \vdots \\ a^2 & a & 1 & \ddots & \vdots \\ \vdots & \ddots & a & \ddots & \zeta^{M-1} \\ a^{M-1} & \cdots & \cdots & a & 1 \end{pmatrix} \frac{1}{1 - \zeta a^M} \quad (5)$$

$$G_0(\alpha, \tau_{k_1} - \tau_{k_2}) = -\lim_{M\to\infty} \begin{cases} \frac{1}{1-\zeta a^M}\, \zeta\, a^{M + k_1 - k_2} & \tau_{k_1} - \tau_{k_2} < 0 \quad (\text{row \#} < \text{column \#}) \\ \frac{1}{1-\zeta a^M}\, a^{k_1 - k_2} & \tau_{k_1} - \tau_{k_2} > 0 \end{cases}$$

Finish the previous lecture's calculation:

Perturbation theory

$$H = H_0 + V$$

H_0 $\hookrightarrow$ one-body operator

V $\hookrightarrow$ interaction

Here: develop an expansion in V (for Z in the first place)

$$H_0 = \sum_\alpha (\epsilon_\alpha - \mu)\, a^\dagger_\alpha a_\alpha$$

$$V(a^\dagger_\alpha a^\dagger_\beta \dots a_\gamma a_\delta \dots) \quad (\text{normal-ordered})$$

$$Z = \int_{\psi(\beta)=\zeta\psi(0)} \mathcal{D}[\psi^*\psi]\, e^{-\int_0^\beta d\tau \left[\sum_\alpha \psi^*_\alpha(\tau)(\partial_\tau + \epsilon_\alpha - \mu)\psi_\alpha(\tau) + V(\psi^*_\alpha(\tau)\psi^*_\beta(\tau)\dots\psi_\gamma(\tau)\psi_\delta(\tau))\right]}$$

$$= Z_0 \left\langle e^{-\int_0^\beta d\tau\, V(\psi^*_\alpha(\tau)\psi^*_\beta(\tau)\dots\psi_\gamma(\tau)\psi_\delta(\tau)\dots)} \right\rangle_0$$

$\hookrightarrow$ thermal averages wrt H_0.

$$\left\{ \begin{aligned} &\left\langle F(\psi^*_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\dots\psi_{\alpha_{n-1}}(\tau_{n-1})\psi_{\alpha_n}(\tau_n))\right\rangle_0 = \\ &= \frac{1}{Z_0}\int_{\psi(\beta)=\zeta\psi(0)} \mathcal{D}[\psi^*\psi]\, e^{-\sum_\alpha \int_0^\beta d\tau\, \psi^*_\alpha(\partial_\tau + \epsilon_\alpha - \mu)\psi_\alpha} F(\dots) \end{aligned} \right.$$

Perturbation series in V follows from expanding the exponential:

$$\frac{Z}{Z_0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \dots \int_0^\beta d\tau_n \left\langle V(\psi_\alpha^*(\tau_1) \dots \psi_\gamma(\tau_1) \dots) \dots V(\psi_\alpha^*(\tau_n) \dots \psi_\gamma(\tau_n) \dots) \right\rangle_0$$

Wick's theorem (see also the first exercise classes)

$$\frac{\int \mathcal{D}[\psi^*\psi]\, \psi_{i_1}\psi_{i_2}\dots\psi_{i_n}\psi^*_{j_n}\dots\psi^*_{j_1}\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j}}{\int \mathcal{D}[\psi^*\psi]\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j}} = \sum_P \zeta^P M^{-1}_{i_{Pn} j_n} \dots M^{-1}_{i_{P1} j_1}$$

i, j encompass the state and time labels.

Proof: Generating function $G(J^*, J) = \dfrac{\int \mathcal{D}[\psi^*\psi]\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j + \sum_i (J_i^*\psi_i + \psi_i^* J_i)}}{\int \mathcal{D}[\psi^*\psi]\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j}} = e^{\sum_{ij} J_i^* M^{-1}_{ij} J_j}$

Differentiate the LHS wrt sources (J, J^*):

$$\left.\frac{\delta^{2n} G}{\delta J^*_{i_1}\dots\delta J^*_{i_n}\delta J_{j_n}\dots\delta J_{j_1}}\right|_{J=J^*=0} = \zeta^n \frac{\int \mathcal{D}[\psi^*\psi]\, \psi_{i_1}\dots\psi_{i_n}\psi^*_{j_n}\dots\psi^*_{j_1}\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j}}{\int \mathcal{D}[\psi^*\psi]\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j}}$$

Differentiate the RHS wrt (J, J^*):

$$\left.\frac{\delta^{2n}}{\delta J^*_{i_1}\dots\delta J^*_{i_n}\delta J_{j_n}\dots\delta J_{j_1}}\right|_{J=J^*=0} e^{\sum_{ij} J_i^* M^{-1}_{ij} J_j} = \zeta^n \frac{\delta^n}{\delta J^*_{i_1}\dots\delta J^*_{i_n}} \left(\sum_{k_n} J^*_{k_n} M^{-1}_{k_n j_n}\right)\dots\left(\sum_{k_1} J^*_{k_1} M^{-1}_{k_1 j_1}\right) e^{\sum_{ij} J_i^* M^{-1}_{ij} J_j}\Bigg|_{J=J^*=0} =$$

$$= \zeta^n \sum_P \zeta^P M^{-1}_{i_{Pn} j_n} \dots M^{-1}_{i_{P1} j_1} \quad \blacksquare$$

$\psi_j \rightarrow \psi_{\alpha, k}$ (k - time point)

$M^{-1}_{ij} \Leftrightarrow$ 1-particle Green's function

$$= \left(\partial_\tau + \varepsilon_\alpha - \mu\right)^{-1}_{\alpha_1\tau_1, \alpha_2\tau_2} = \delta_{\alpha_1\alpha_2}\, g_{\alpha_1}(\tau_1 - \tau_2 - \eta)$$

→ n-particle Green's function of a noninteracting system = sum of permutations of products of 1-particle Green's functions (contractions).

$$\overbrace{\tilde{a}_\alpha(\tau)\tilde{a}_{\alpha'}(\tau')} = \langle T\tilde{a}_\alpha(\tau)\tilde{a}_{\alpha'}(\tau')\rangle_0$$

$$\overbrace{\tilde{\psi}_\alpha(\tau)\tilde{\psi}_{\alpha'}(\tau')} = \langle \tilde{\psi}_\alpha(\tau)\tilde{\psi}_{\alpha'}(\tau')\rangle_0$$

$$\overbrace{a_\alpha(\tau)a^\dagger_{\alpha'}(\tau')} = \overbrace{\psi_\alpha(\tau)\psi^*_{\alpha'}(\tau')} = \delta_{\alpha\alpha'}\, g_\alpha(\tau-\tau') \qquad \overbrace{a^\dagger_\alpha(\tau)a^\dagger_{\alpha'}(\tau')} = \overbrace{\psi^*_\alpha(\tau)\psi^*_{\alpha'}(\tau')} = 0$$

$$\overbrace{a^\dagger_{\alpha'}(\tau')a_\alpha(\tau)} = \overbrace{\psi^*_{\alpha'}(\tau')\psi_\alpha(\tau)} = \zeta\delta_{\alpha\alpha'}\, g_\alpha(\tau-\tau') \qquad \overbrace{a_{\alpha'}(\tau')a_\alpha(\tau)} = \overbrace{\psi_{\alpha'}(\tau')\psi_\alpha(\tau)} = 0$$

$$\langle T \tilde{a}_{\alpha_1}(\tau_1)\dots\tilde{a}_{\alpha_n}(\tau_n)\rangle_0 = \langle \tilde{\psi}_{\alpha_1}(\tau_1)\dots\tilde{\psi}_{\alpha_n}(\tau_n)\rangle_0 = \sum\left(\begin{array}{c}\text{all complete}\\ \text{contractions}\end{array}\right)$$

E.g. $$\langle T a_{\alpha_1}(\tau_1) a_{\alpha_2}(\tau_2) a^+_{\alpha_2'}(\tau_2') a^+_{\alpha_1'}(\tau_1')\rangle_0 = a_{\alpha_1}(\tau_1) a_{\alpha_2}(\tau_2) a^+_{\alpha_2'}(\tau_2') a^+_{\alpha_1'}(\tau_1')$$

$$+\zeta\, a_{\alpha_1}(\tau_1) a_{\alpha_2}(\tau_2) a^+_{\alpha_2'}(\tau_2') a^+_{\alpha_1'}(\tau_1') =$$

$$= \delta_{\alpha_1\alpha_1'}\delta_{\alpha_2\alpha_2'}\, g_{\alpha_1}(\tau_1-\tau_1')\, g_{\alpha_2}(\tau_2-\tau_2') + \zeta\,\delta_{\alpha_1\alpha_2'}\delta_{\alpha_2\alpha_1'}\, g_{\alpha_1}(\tau_1-\tau_2')\, g_{\alpha_2}(\tau_2-\tau_1')$$

Recall:

$$\frac{Z}{Z_0} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\dots\int_0^\beta d\tau_n \langle V(\psi^*_\alpha(\tau_1)\dots\psi_\gamma(\tau_1)\dots)\dots V(\psi^*_\alpha(\tau_n)\dots\psi_\gamma(\tau_n)\dots)\rangle_0$$

may be represented as sum of contractions, where the propagators $g_\alpha(\tau-\tau')$ connect products of matrix elements of V in all possible ways.

– convenient representation via diagrams

$$\frac{Z}{Z_0} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\dots\int_0^\beta d\tau_n \langle V(\psi^*_\alpha(\tau_1)\dots\psi_\gamma(\tau_1)\dots)\dots V(\psi^*_\alpha(\tau_n)\dots\psi_\gamma(\tau_n)\dots)\rangle_0 \qquad (*)$$

Take V to be a 2-body interaction: $V(\psi^*(\tau),\psi(\tau)) = \frac{1}{2}\sum_{\alpha\beta\gamma\delta}(\alpha\beta|v|\gamma\delta)\,\psi^*_\alpha(\tau)\psi^*_\beta(\tau)\psi_\delta(\tau)\psi_\gamma(\tau)$

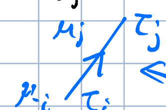
$\rightarrow$ Term of order n in $(*)$: $$\frac{(-1)^n}{n!\,2^n}\sum_{\substack{\alpha_1\beta_1\\ \gamma_1\delta_1}}\dots\sum_{\substack{\alpha_n\beta_n\\ \gamma_n\delta_n}}(\alpha_1\beta_1|v|\gamma_1\delta_1)\dots(\alpha_n\beta_n|v|\gamma_n\delta_n)\int_0^\beta d\tau_1\dots\int_0^\beta d\tau_n \times$$

$$\times \langle \psi^*_{\alpha_1}(\tau_1)\psi^*_{\beta_1}(\tau_1)\psi_{\delta_1}(\tau_1)\psi_{\gamma_1}(\tau_1)\dots\psi^*_{\alpha_n}(\tau_n)\psi^*_{\beta_n}(\tau_n)\psi_{\delta_n}(\tau_n)\psi_{\gamma_n}(\tau_n)\rangle_0 \qquad (**)$$

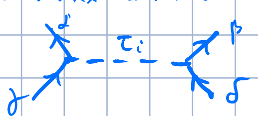
(Labeled) Feynman diagrams

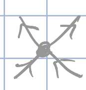
Consider $(**)$, think about a general contribution at order n:

- each contraction will join some $\psi^*_{\mu_i}(\tau_i)$ with some $\psi_{\nu_j}(\tau_j)$ yielding a propagator $\delta_{\mu_i\nu_j}\, g_{\mu_i}(\tau_i-\tau_j)$

  $\rightarrow$ directed line from $\psi^*_{\mu_i}(\tau_i)$ to $\psi_{\nu_j}(\tau_j)$ 

$\Leftrightarrow$ $\delta_{\mu_i\nu_i}\, g_{\mu_i}(\tau_j-\tau_i)$. (for diagonal basis $\mu_i=\mu_j$)

- each interaction $\longrightarrow$ a vertex with 2 ingoing lines corresponding to ψ's and 2 outgoing lines corresponding to ψ^*'s. 

$\Leftrightarrow$ $(\alpha\beta|v|\gamma\delta)$ (or )

Set of possible ways of connecting interactions with propagators $\longleftrightarrow$ set of contractions arising from Wick's th.

$$n=1 \quad \langle \psi^*_{\alpha_1}(\tau_1)\psi^*_{\beta_1}(\tau_1)\psi_{\delta_1}(\tau_1)\psi_{\gamma_1}(\tau_1)\rangle_0$$

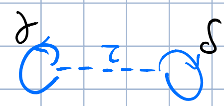
(two contractions)

represented by two diagrams.

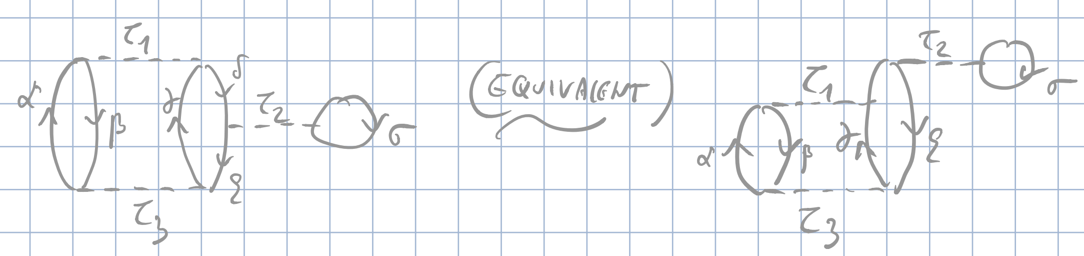
- $$-\frac{1}{2}\sum_{\substack{\alpha_1\beta_1\\ \gamma_1\delta_1}}\int_0^\beta d\tau\, \psi^*_{\alpha_1}(\tau)\psi^*_{\beta_1}(\tau)\psi_{\delta_1}(\tau)\psi_{\gamma_1}(\tau)\,(\alpha_1\beta_1|v|\gamma_1\delta_1) = -\frac{1}{2}\int_0^\beta d\tau_1 \sum_{\gamma\delta} g_\gamma(0)g_\delta(0)\,(\gamma\delta|v|\gamma\delta)$$
- $$-\frac{1}{2}\sum_{\substack{\alpha_1\beta_1\\ \gamma_1\delta_1}}\int_0^\beta d\tau\, \psi^*_{\alpha_1}(\tau)\psi^*_{\beta_1}(\tau)\psi_{\delta_1}(\tau)\psi_{\gamma_1}(\tau)\,(\alpha_1\beta_1|v|\gamma_1\delta_1) = -\frac{\zeta}{2}\int_0^\beta d\tau_1 \sum_{\gamma\delta} g_\gamma(0)g_\delta(0)\,(\delta\gamma|v|\gamma\delta)$$

Propagator at equal times $=\zeta n_\alpha$

$$\rightarrow \frac{Z}{Z_0} = 1-\frac{\beta}{2}\sum_{\gamma\delta} n_\gamma n_\delta\left[(\gamma\delta|v|\gamma\delta)+\zeta(\delta\gamma|v|\gamma\delta)\right]$$



$$\Omega = -T\ln Z \;\rightarrow\; \Omega = \Omega_0 + \frac{1}{2}\sum_{\gamma\delta} n_\gamma n_\delta\left[(\gamma\delta|v|\gamma\delta)+\zeta(\delta\gamma|v|\gamma\delta)\right] \quad (\text{up to order } v^1)$$



Diagrams are distinct when they cannot be continuously deformed one into the other.

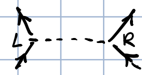
- state labels summed over – not necessary to include them
- for fermions the factor related to $\zeta \longleftrightarrow (-1)^{\#\text{of closed loops}}$

(see e.g. Negele-Orland)

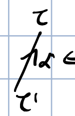
- further simplifications e.g. • exploiting symmetries
  • using symmetrized vertices

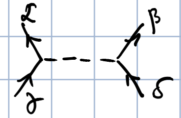
...

$\Longrightarrow$ Rules for constructing (labeled) diagrams providing a representation of contractions contributing to n^{th} order term of perturbative expansion for $\frac{Z}{Z_0}$

1. Draw distinct (labeled) diagrams composed of n vertices  connected by directed lines

   For each distinct diagram, evaluate the corresponding contribution as follows:

2. Assign a single-particle index to each directed line, include the corresponding factor:  $\Leftrightarrow g_\alpha(\tau-\tau')$

3. For each vertex, include the factor  $\Leftrightarrow (\alpha\beta|V|\gamma\delta)$

4. Sum over all single-particle indices and integrate all times over the interval $[0,\beta]$

5. Multiply the result by $\frac{(-1)^n}{n!\,2^n}\zeta^{n_L}$ n_L – number of closed loops of propagators in the diagram.

{ More on this (unlabeled diagrams, symmetrized vertices ...) – e.g. Negele-Orland. }

Linked cluster theorem

We are not so much interested in Z as in $\ln Z$ $(\Omega=-T\ln Z)$.

Linked cluster theorem – only connected diagrams contribute to $\ln Z$.

"Proof" by replica method

- Evaluate Z^n for $n\in\mathbb{N}$ by replicating the system n times.

Consider $$Z^n = e^{n\ln Z} = 1 + n\ln Z + \sum_{m=2}^{\infty}\frac{(n\ln Z)^m}{m!}$$

Assume we evaluate Z^n (in perturbation theory) and expand in n.

Then $\ln Z$ will appear as the factor in front of n^1.

Or (equivalently) continue Z^n to $n\to 0$ and evaluate $\ln Z$ as
$$\lim_{n\to 0^+}\frac{d}{dn}Z^n = \lim_{n\to 0^+}\frac{d}{dn}\left(e^{n\ln Z}\right) = \ln Z.$$

$$\left(\frac{Z}{Z_0}\right)^n = \frac{1}{Z_0^n}\int_{\psi^\sigma_\alpha(\beta)=\zeta\psi^\sigma_\alpha(0)}\prod_{\sigma=1}^{n}\mathcal{D}[\psi^{\sigma*}_\alpha(\tau)\,\psi^\sigma_\alpha(\tau)]\, e^{-\int_0^\beta d\tau\sum_{\sigma=1}^{n}\left(\sum_\alpha \psi^{\sigma*}_\alpha(\partial_\tau+\epsilon_\alpha-\mu)\psi^\sigma_\alpha + V(\psi^{\sigma*}_\alpha,\psi^\sigma_\alpha)\right)}$$