Perturbation theory for the Green's function

$$\left\{ Z = \int_{\psi(\beta)=\zeta\psi(0)} \mathcal{D}[\psi^*\psi]\, e^{-\int_0^\beta d\tau \left[\sum_\alpha \psi^*_\alpha(\tau)(\partial_\tau + \epsilon_\alpha - \mu)\psi_\alpha(\tau) + V(\psi^*_\alpha(\tau)\psi^*_\beta(\tau)\ldots\psi_\gamma(\tau)\psi_\delta(\tau))\right]} \right.$$

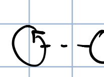
$$S_{int}[\psi^*,\psi] = \frac{1}{2}\int_0^\beta d\tau \sum_{\substack{\alpha_1\alpha_2\\ \alpha_1'\alpha_2'}} (\alpha_1\alpha_2|V|\alpha_1'\alpha_2')\,\psi^*_{\alpha_1}(\tau)\psi^*_{\alpha_2}(\tau)\psi_{\alpha_2'}(\tau)\psi_{\alpha_1'}(\tau)$$

$$G(\alpha_1\tau_1,\alpha_2\tau_2) = -\langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\rangle =$$

$$= -\frac{Z_0}{Z}\langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\, e^{-S_{int}}\rangle_0$$

To compute G to a given order in V expand both Z and $\langle\ldots\rangle_0$

Work it out at 1st order to begin with:

- Z - calculated before ( + )
- $-\langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\, e^{-S_{int}}\rangle_0 \simeq G_0(\alpha_1\tau_1,\alpha_2\tau_2) + \langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\, S_{int}\rangle_0 =$

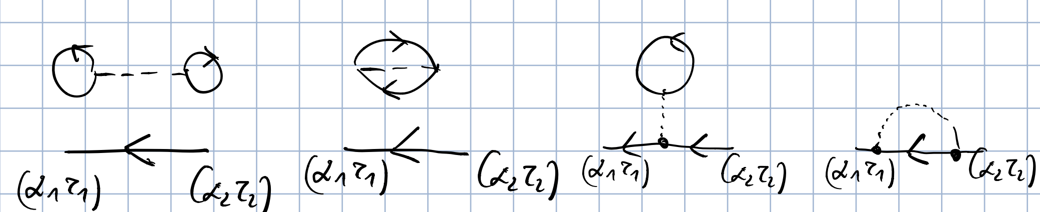
$$= G_0(\alpha_1\tau_1,\alpha_2\tau_2) + \frac{1}{2}\sum_{\substack{\beta_1\beta_2\\ \beta_1'\beta_2'}} (\beta_1\beta_2|V|\beta_1'\beta_2')\int_0^\beta d\tau \langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\,\psi^*_{\beta_1}(\tau)\psi^*_{\beta_2}(\tau)\psi_{\beta_2'}(\tau)\psi_{\beta_1'}(\tau)\rangle_0$$

$$= \left\{\begin{matrix}\text{Wick}\\ \text{th}\end{matrix}\right\} = G_0(\alpha_1\tau_1,\alpha_2\tau_2) - \sum_{\substack{\beta_1\beta_2\\ \beta_1'\beta_2'}} (\beta_1\beta_2|V|\beta_1'\beta_2')\int_0^\beta d\tau \Big\{\frac{1}{2}G_0(\alpha_1\tau_1,\alpha_2\tau_2)\big[G_0(\beta_1'\tau,\beta_1\tau)G_0(\beta_2'\tau,\beta_2\tau) +$$

$$+\zeta G_0(\beta_1'\tau,\beta_2\tau)G_0(\beta_2'\tau,\beta_1\tau)\big] +$$

$$+ G_0(\alpha_1\tau_1,\beta_1\tau)\big[G_0(\beta_2'\tau,\alpha_2\tau_2)G_0(\beta_1'\tau,\beta_2\tau) +$$

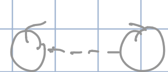
$$+\zeta G_0(\beta_1'\tau,\alpha_2\tau_2)G_0(\beta_2'\tau,\beta_2\tau)\big]\Big\}$$

$\Longrightarrow$ 

Only indices carried by the internal lines are summed over.

Combine this with the expression for $\frac{Z}{Z_0}$ (at order $\mathcal{O}(V)$)

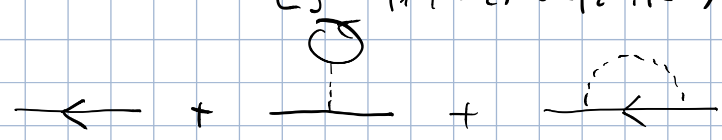
$$\frac{Z}{Z_0} = 1 - \left(\frac{1}{2}\right)$$  $$-\left(\frac{\zeta}{2}\right)$$  $$+\mathcal{O}(V^2)$$

$$\Rightarrow \frac{Z_0}{Z} = 1 + \left(\frac{1}{2}\right)$$  $$+\left(\frac{\zeta}{2}\right)$$  $$+\mathcal{O}(V^2)$$

simply because $\frac{1}{1-x+\mathcal{O}(x^2)} = 1+x+\mathcal{O}(x^2)$

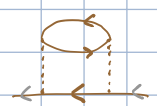
②

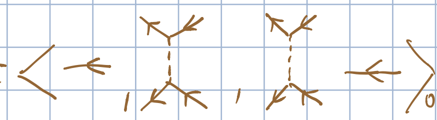
$$\longrightarrow G(\alpha_1\tau_1, \alpha_2\tau_2) = G_0(\alpha_1\tau_1, \alpha_2\tau_2) - \sum_{\substack{\beta_1\beta_2\\ \beta_1'\beta_2'}} (\beta_1, \beta_2|v|\beta_1'\beta_2') \int_0^\beta d\tau\, G_0(\alpha_1\tau_1, \beta_1\tau) \times$$
$$\times \left[\zeta G_0(\beta_1'\tau, \alpha_2\tau_2) G_0(\beta_2'\tau, \beta_2\tau) + G_0(\beta_2'\tau, \alpha_2\tau_2) G_0(\beta_1'\tau, \beta_2\tau) \right]$$

$\longrightarrow$ 

Only connected diagrams (general result, valid at any order in v → proof by replica method)

Combinatorial factors → direct analysis of the number of contractions leading to a diagram.

e.g. 

Obtained from: $\frac{1}{2!\,2^2}$ 

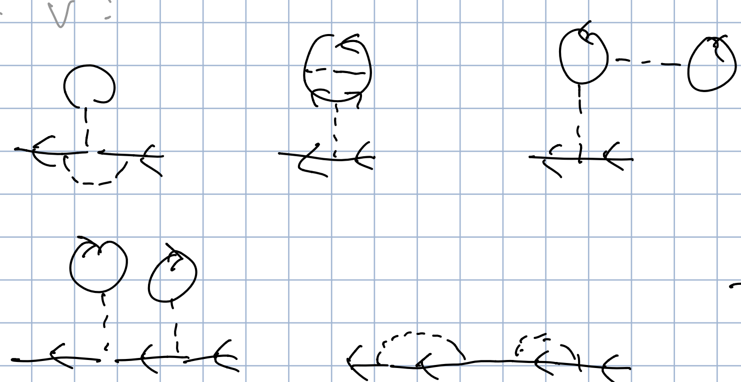
- interchange of vertices → factor of 2
- connection of outgoing line to vertex → $\times 2$
- connection of ingoing line to vertex → $\times 2$.

$\frac{1}{2!\,2^2} \cdot 2 \cdot 2 \cdot 2 = 1.$ (this turns out to be a general result proof → e.g. N-O.) and for any $G^{(2n)}$

- Sign of a diagram ⇔ $(-1)^n \zeta^{n_c}$

Diagram rules — analogous to those for Z, but 2 external legs (1 ingoing and 1 outgoing).

At order v^2:

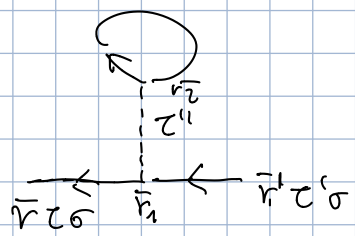


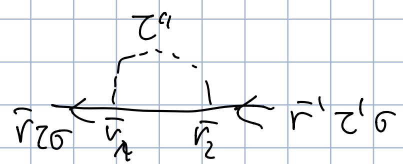
+ 5 MORE

(IN TOTAL 10).

Example, 1st order, position basis

$$(\bar r_1 \sigma_1, \bar r_2 \sigma_2 | \hat V | \bar r_3 \sigma_3\ \bar r_4 \sigma_4) = V(\bar r_1 - \bar r_2)\,\delta(\bar r_1 - \bar r_3)\,\delta(\bar r_2 - \bar r_4)\cdot \delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4}.$$





$$-\zeta \int_0^\beta d\tau'' \int d^3 r_1 \int d^3 \bar r_2 \sum_{\sigma'} V(\bar r_1 - \bar r_2)\, G_{0\sigma}(\bar r, \tau, \bar r_1, \tau'')\, G_{0\sigma}(\bar r_1, \tau'', \bar r' \tau') \times G_{0\sigma'}(\bar r_2 \tau'', \bar r_2\ \tau''^{+}) +$$

$$-\int_0^\beta d\tau'' \int d^3 r_1 \int d^3 r_2\ v(\bar r_1 - \bar r_2)\, G_{0\sigma}(\bar r \tau, \bar r_1 \tau'')\, G_{0\sigma}(\bar r_1 \tau'', \bar r_2 \tau''^{+}) \times G_{0\sigma}(\bar r_2 \tau'', \bar r' \tau').$$

Useful when translation invariant:

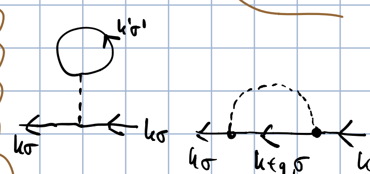
$(\vec k, \omega_n) \to G_0$ diagonal but V not.

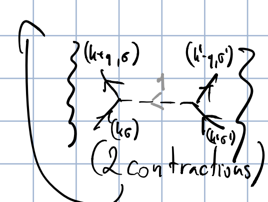
Action $\to S = S_0 + S_{int}$

$$S_0 = \sum_{k\sigma} \psi^*_\sigma(k)[-G^{-1}_{0\sigma}(k)]\psi_\sigma(k)$$

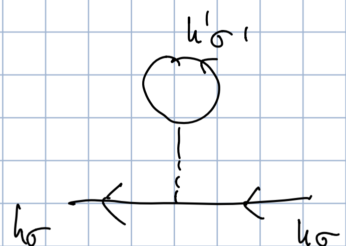
$$S_{int} = \frac{1}{2\beta V} \sum_{kk'q} \sum_{\sigma\sigma'} V(q)\, \psi^*_\sigma(k+q)\, \psi^*_{\sigma'}(k'-q)\, \psi_{\sigma'}(k')\, \psi_\sigma(k)$$

$$(\ |\hat V|\ ) \qquad V_{\bar q}\ \delta_{\bar k_1 + \bar k_2, \bar k_3 + \bar k_4}$$

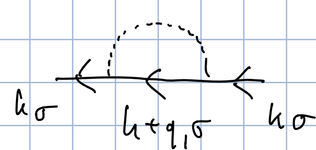




$$\sum_{k'} = \sum_{\omega_n'} \int \frac{d^3 k'}{(2\pi)^3}$$



$k = (\vec k, \omega_n)$



$$(\ \cdot\ )\ \frac{-\zeta}{\beta V} \sum_{k'\sigma'} v(0)\, G_{0\sigma}(k)^2\, G_{0\sigma'}(k')\, e^{i\omega_n' \eta} \qquad \eta = 0^+$$

$$-\frac{1}{\beta V} \sum_q v(\bar q)\, G_{0\sigma}(k)^2\, G_{0\sigma}(k+q)\, e^{i\omega_{n+m}\eta}$$

CHECK

Real-time Green's functions

(in particular) describe system's response to external perturbations

Previous calculations for τ imaginary (perturbation theory in particular)

Can we learn anything about the real-time G. function from this?

Terminology review:

$\hat{H}$ - time indep. $\qquad \mathcal{H} := \hat{H} - \mu \hat{N} \qquad \hbar = 1$

Heisenberg op. $A(t) = e^{i\mathcal{H}t} A e^{-i\mathcal{H}t}$ (A has no explicit time dependence)

- Retarded G.f. $G^R_{AB}(t,t') = -i\theta(t-t')\langle [A(t), B(t')]_{-\zeta} \rangle = G^R_{AB}(t-t')$
  (important - occurs in the Kubo formula)
- Advanced G.f. $G^A_{AB}(t,t') = +i\theta(t'-t)\langle [A(t), B(t')]_{-\zeta} \rangle = G^A_{AB}(t-t')$
- Causal G.f. $G^c_{AB}(t,t') = -i\langle T A(t) B(t') \rangle$
- Correlation f. $C_{AB}(t,t') = \langle A(t) B(t') \rangle$

Defined for any A, B.

Single-particle G.f. $A \to a_\alpha$, $B \to a^+_{\alpha'}$. E.g. $A = \Psi_\sigma(\vec{r})$, $B = \Psi^+_{\sigma'}(\vec{r}')$

or $A = a_{\vec{k}\sigma}$, $B = a^+_{\vec{k}'\sigma'}$

$\to G^R(\vec{r}\sigma t, \vec{r}'\sigma' t') = -i\theta(t-t')\langle [\Psi_\sigma(\vec{r}t), \Psi^+_{\sigma'}(\vec{r}'t')]_{-\zeta} \rangle$ and so on

Physical meaning - consider the causal G.f., take $t > t'$, use the eigenbasis of $\mathcal{H}$:

$\mathcal{H}|n\rangle = E|n\rangle$

$$iG^c(\vec{r}\sigma t, \vec{r}'\sigma' t') = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta\mathcal{H}}\Psi_\sigma(\vec{r}t)\Psi^+_{\sigma'}(\vec{r}'t')\right] =$$

$$= \frac{1}{Z}\sum_n e^{-\beta E_n} \underbrace{\langle n| e^{i\mathcal{H}t}\Psi_\sigma(\vec{r})}_{\langle\beta|} \underbrace{e^{-i\mathcal{H}(t-t')}\Psi^+_{\sigma'}(\vec{r}')e^{-i\mathcal{H}t'}|n\rangle}_{|\alpha\rangle}$$

$\langle n| \ldots |n\rangle$ - prob. amplitude of finding the system with an extra particle $(\vec{r},\sigma)$ at time t if a particle $(\vec{r}'\sigma')$ was added at an earlier time t'.

$iG^c(\vec{r}\sigma t, \vec{r}'\sigma' t')$ - ensemble average of this probability amplitude.

For $t<t'$ $iG(\vec{r}\sigma t, \vec{r}'\sigma' t')$ – ensemble average of an analogous amplitude of a hole 'propagating' from $(\vec{r}\sigma t)$ to $(\vec{r}'\sigma' t')$

Consider now the translationally invariant system (also spin-independent $\mathcal{H}$ for simplicity)

$$G^R(\vec{r}\sigma t, \vec{r}'\sigma' t') = \delta_{\sigma\sigma'} G^R(\vec{r}\sigma t, \vec{r}'\sigma t')$$ • depends only on $t-t'$ and $\vec{r}-\vec{r}'$
↳ take $t'=0$

$$\rightarrow G^R(\vec{r}-\vec{r}', \sigma, t) = -i\theta(t)\langle [\Psi_\sigma(\vec{r},t), \Psi^\dagger_\sigma(\vec{r}',0)]_{-\zeta}\rangle$$ (analogously for the other G.f.)

$$\text{F.t.}: \quad \Psi_\sigma(\vec{r}t) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\vec{r}} a_{\vec{k}\sigma}(t), \quad \Psi^\dagger_\sigma(\vec{r}t) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{-i\vec{k}\vec{r}} a^\dagger_{\vec{k}\sigma}(t)$$

$$(\bullet) \quad G^R(\vec{r}-\vec{r}', \sigma, t) = -i\theta(t)\frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}(\vec{r}-\vec{r}')}\langle [a_{\vec{k}\sigma}(t), a^\dagger_{\vec{k}\sigma}(0)]_{-\zeta}\rangle = \frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}(\vec{r}-\vec{r}')} G^R(\vec{k}, \sigma, t)$$

$$G^R(\vec{k}\sigma t) = -i\theta(t)\langle [a_{\vec{k}\sigma}(t), a^\dagger_{\vec{k}\sigma}(0)]_{-\zeta}\rangle$$

other G.f. – analogously.

Spectral representation

$$\mathcal{H} = \sum_n E_n |n\rangle\langle n|$$ (spectral rep. of $\mathcal{H}$)

Express the retarded G.f in terms of the (exact) energy spectrum and eigenstates.
(unknown)

Spectral rep. → simple way of obtaining G^R from the Matsubara G.f.

$$G^R(\vec{k}\sigma t) = \underbrace{-i\theta(t)\langle a_{\vec{k}\sigma}(t) a^\dagger_{\vec{k}\sigma}(0)\rangle}_{:=\mathcal{A}} + \zeta\underbrace{i\theta(t)\langle a^\dagger_{\vec{k}\sigma}(0) a_{\vec{k}\sigma}(t)\rangle}_{:=\mathcal{B}} = \mathcal{A} - \zeta\mathcal{B}$$

$\{a_{\vec{k}\sigma} = a_{\vec{k}\sigma}(0)\}$

$$\langle a_{\vec{k}\sigma}(t) a^\dagger_{\vec{k}\sigma}(0)\rangle \overset{(*)}{=} \frac{1}{Z}\sum_n \langle n| e^{-\beta\mathcal{H}} e^{i\mathcal{H}t} a_{\vec{k}\sigma} \underset{\uparrow}{e^{-i\mathcal{H}t}} a^\dagger_{\vec{k}\sigma}|n\rangle = \frac{1}{Z}\sum_{nm} e^{-\beta E_n} e^{-i(E_m-E_n)t}\langle n|a_{\vec{k}\sigma}|m\rangle\langle m|a^\dagger_{\vec{k}\sigma}|n\rangle =$$

$$\mathbb{1} = \sum_m |m\rangle\langle m|$$

$$= -\int_{-\infty}^{\infty} P(\vec{k}\sigma\omega') e^{-i\omega' t} d\omega' \; ; \quad P(\vec{k}\sigma\omega') := -\frac{1}{Z}\sum_{nm} e^{-\beta E_n} |\langle m|a^\dagger_{\vec{k}\sigma}|n\rangle|^2 \delta(\omega' - (E_m - E_n))$$

$$\Rightarrow \mathcal{A} = i\Theta(t)\int_{-\infty}^{\infty} P(\bar{k}\sigma\omega')e^{-i\omega' t}d\omega'$$

$$\langle a^+_{\bar{k}\sigma}(0)a_{\bar{k}\sigma}(t)\rangle = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}a^+_{\bar{k}\sigma}e^{iHt}a_{\bar{k}\sigma}e^{-iHt}\right] = \frac{1}{Z}\mathrm{Tr}\left[e^{iHt}a_{\bar{k}\sigma}e^{-i(Ht-i\beta H)}a^+_{\bar{k}\sigma}\right] = \{1 = e^{-\beta H}e^{\beta H}\}$$

$$= \frac{1}{Z}\mathrm{Tr}\Big[\underbrace{e^{-\beta H}e^{iH(t-i\beta)}}_{=e^{iHt}}a_{\bar{k}\sigma}e^{-iH(t-i\beta)}a^+_{\bar{k}\sigma}\Big] \to \text{the same as } \circledast \text{ but } t \to t-i\beta$$

$$= -\int_{-\infty}^{\infty} P(\bar{k}\sigma\omega')e^{-i\omega'(t-i\beta)}d\omega'$$

$$\Rightarrow -\zeta\mathcal{B} = -i\zeta\Theta(t)\int_{-\infty}^{\infty} P(\bar{k}\sigma\omega')e^{-i\omega'(t-i\beta)}d\omega'$$

F.t of G^R: $G^R(\bar{k}\sigma\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\underbrace{G^R(\bar{k}\sigma t)}_{\mathcal{A}-\zeta\mathcal{B}} = \int_{-\infty}^{\infty} dt\, e^{i\omega t}(\mathcal{A}-\zeta\mathcal{B})$

→ vanishes for $t<0$

(in particular: $\Theta(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty}dx\,\frac{e^{-ixt}}{x+i0^+}$)

$= (\;)$

Review lect XI-XIII from previ. sem.

$$G^R(\bar{k}\sigma\omega) = \lim_{\eta\to 0^+}\int_{-\infty}^{\infty}\frac{S(\bar{k}\sigma\omega')}{\omega-\omega'+i\eta}d\omega' \quad (\text{spectral rep. of } G^R(\bar{k}\sigma\omega))$$

Here $S(\bar{k}\sigma\omega) = -P(\bar{k}\sigma\omega)(1-\zeta e^{-\beta\omega})$

Go now to the imaginary time Green's function: (1-particle, position rep.)

$$\mathcal{G}(\bar{r}-\bar{r}',\sigma,\tau) = -\langle T\Psi_\sigma(\bar{r}\tau)\Psi^+_\sigma(\bar{r}'0)\rangle$$

$$\mathcal{G}(\bar{r}-\bar{r}',\sigma,\tau) = \frac{1}{V}\sum_{k}\mathcal{G}(\bar{k}\sigma\tau)e^{i\bar{k}(\bar{r}-\bar{r}')}$$

$$\mathcal{G}(\bar{k}\sigma\tau) = -\langle T a_{\bar{k}\sigma}(\tau)a^+_{\bar{k}\sigma}(0)\rangle$$

Spectral rep. of $\mathcal{G}(\bar{k}\sigma\omega_n) = \int_0^\beta \mathcal{G}(\bar{k}\sigma\tau)e^{i\omega_n\tau}d\tau \quad (\circledast)$

$$\mathcal{G}(\bar{k}\sigma\tau) = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}e^{H\tau}a_{\bar{k}\sigma}e^{-H\tau}a^+_{\bar{k}\sigma}\right] = -\frac{1}{Z}\sum_{mn}\langle n|e^{-\beta H}e^{H\tau}a_{\bar{k}\sigma}|m\rangle\langle m|e^{-H\tau}a^+_{\bar{k}\sigma}|n\rangle =$$

$$= -\frac{1}{Z}\sum_{mn}e^{-\beta E_n}e^{-(E_m-E_n)\tau}\langle n|a_{\bar{k}\sigma}|m\rangle\langle m|a^+_{\bar{k}\sigma}|n\rangle = \int_{-\infty}^{\infty}P(\bar{k}\sigma\omega')e^{-\omega'\tau}d\omega'$$

(Recall from the real-time calculation:

$$P(\bar{k}\sigma\omega') := -\frac{1}{Z}\sum_{mn}e^{-\beta E_n}|\langle m|a^+_{\bar{k}\sigma}|n\rangle|^2\delta(\omega'-(E_m-E_n))$$

$$S(\bar{k}\sigma\omega) = -P(\bar{k}\sigma\omega)(1-\zeta e^{-\beta\omega})$$

Back to (*):

$$\mathcal{G}(\bar{k}\sigma\omega_n) = \int_{-\infty}^{\infty} d\omega' \int_0^{\beta} d\tau\, P(\bar{k}\sigma\omega')\, e^{(i\omega_n-\omega')\tau} = \int_{-\infty}^{\infty} d\omega' P(\bar{k}\sigma\omega') \frac{e^{(i\omega_n-\omega')\beta}-1}{i\omega_n-\omega'} =$$

$$= \left\{ e^{i\omega_n\beta} = \zeta \right\} = -\int_{-\infty}^{\infty} d\omega' P(\bar{k}\sigma\omega') \frac{1-\zeta e^{-\beta\omega'}}{i\omega_n-\omega'} =$$

$$= \int_{-\infty}^{\infty} d\omega' \frac{S(\bar{k}\sigma\omega')}{i\omega_n-\omega'}$$

In summary:

$$\boxed{\begin{aligned} G^R(\bar{k}\sigma\omega) &= \lim_{\eta\to 0^+} \int_{-\infty}^{\infty} \frac{S(\bar{k}\sigma\omega')}{\omega-\omega'+i\eta} d\omega' \\ \mathcal{G}(\bar{k},\sigma,\omega_n) &= \int_{-\infty}^{\infty} \frac{S(\bar{k}\sigma\omega')}{i\omega_n-\omega'} \end{aligned}}$$

$$F(\bar{k}\sigma z) = \int_{-\infty}^{\infty} \frac{S(\bar{k}\sigma\omega')}{z-\omega'} d\omega' \quad (\text{analytic everywhere except } \mathbb{R})$$

$$G^R(\bar{k}\sigma\omega) = F(\bar{k}\sigma, z=\omega+i0^+)$$

$$\mathcal{G}(\bar{k}\sigma\omega_n) = F(\bar{k}\sigma, z=i\omega_n)$$

$$G^R(\bar{k},\sigma,\omega) = \mathcal{G}(\bar{k}\sigma, i\omega_n)\Big|_{i\omega_n \to \omega+i0^+}$$