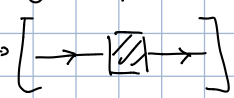
- Review previous lecture, continue with spectral representation and analytical continuation.

- Self energy, Dyson's equation

Perturbation expansion for g ($\mathcal{G}$) → every (connected) diagram contains two external lines

→ In momentum/frequency representation: (momentum/frequency conserved at each vertex)

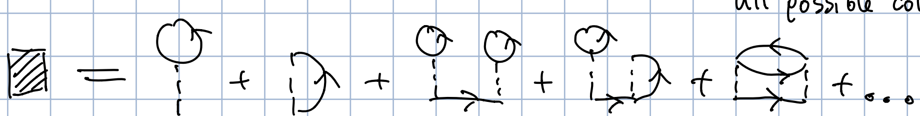
any diagram → 

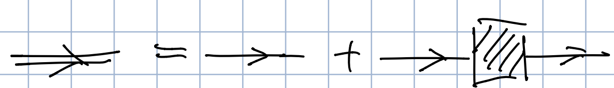
→ the corresponding algebraic expression:

$$g_0(\vec{k}\sigma, \omega_n)\underbrace{A(\vec{k}\sigma, \omega_n)}_{\text{determined by the structure of the diagram.}} g_0(\vec{k}\sigma, \omega_n)$$

→ We may write:

$$g(\vec{k}\sigma, \omega_n) = g_0(\vec{k}\sigma, \omega_n) + g_0(\vec{k}\sigma, \omega_n)\underbrace{\tilde{\Sigma}(\vec{k}\sigma, \omega_n)}_{\text{from summing } A \text{ over all possible connected diagrams}} g_0(\vec{k}\sigma, \omega_n)$$

$\tilde{\Sigma}(\vec{k}\sigma, \omega_n) =$ 

$g(\vec{k}\sigma, \omega_n) =$ 

Two classes of diagrams in the expansion of $\tilde{\Sigma}$:

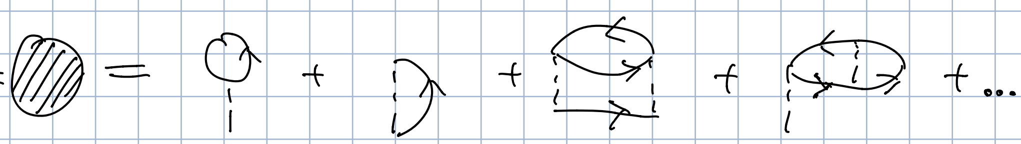
- reducible (can be separated into two pieces by cutting one propagator line)
- irreducible (cannot ...)

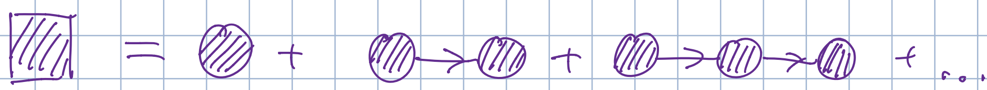
$\Sigma(\vec{k}\sigma, \omega_n) :=$ sum over all the contributions represented by irreducible diagrams

↓

self energy ("proper", "irreducible").

$$\tilde{\Sigma}(\bar{k}\sigma,\omega_n)=\Sigma(\bar{k}\sigma,\omega_n)+\Sigma(\bar{k}\sigma,\omega_n)\,g_0(\bar{k}\sigma,\omega_n)\,\Sigma(\bar{k}\sigma,\omega_n)+\Sigma(\bar{k}\sigma,\omega_n)\,g_0(\bar{k}\sigma,\omega_n)\,\Sigma(\bar{k}\sigma,\omega_n)\,g_0(\bar{k}\sigma,\omega_n)\,\Sigma(\bar{k}\sigma,\omega_n)+\dots$$

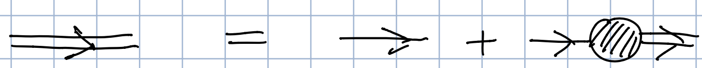
$\Sigma(\bar{k}\sigma,\omega) =$  (irreducible diagrams)



$\rightarrow$ insert into $g(\bar{k}\sigma,\omega_n)$ $\qquad k=(\bar{k}\sigma,\omega_n)$

$$g(\bar{k}\sigma,\omega_n)\overset{\text{notation}}{=}g(k)=g_0(k)+g_0(k)\Sigma(k)g_0(k)+g_0(k)\Sigma(k)g_0(k)\Sigma(k)g_0(k)+\dots$$

$$=g_0(k)+g_0(k)\Sigma(k)g(k)$$



$$\hookrightarrow g(k)=\frac{g_0(k)}{1-g_0(k)\Sigma(k)}=$$

$$\boxed{g^{-1}(k)=g_0^{-1}(k)-\Sigma(k)} \qquad =\frac{1}{g_0^{-1}(k)-\Sigma(k)}$$

(Dyson's eq.)

$\cancel{\dfrac{1}{i\omega_n-(\epsilon_{\bar{k}}-\mu)}}$

$\Sigma(k)$ – represented by 1PI (irreducible) diagrams.

$$\rightarrow g(\bar{k}\sigma,\omega_n)=\frac{1}{i\omega_n-(\epsilon_{\bar{k}}-\mu)-\Sigma(\bar{k}\sigma,\omega_n)}$$

Retarded (real time) G.f.:

$$i\omega_n\longrightarrow\omega+i\eta \qquad \eta=0^+$$

$$G^R(\bar{k}\sigma,\omega)=\frac{1}{\omega-(\epsilon_{\bar{k}}-\mu)-\Sigma^R(\bar{k}\sigma,\omega)+i\eta}$$

May be dropped if $\mathrm{Im}\,\Sigma^R\neq 0$

$$\Sigma^R(\bar{k}\sigma,\omega)=\Sigma(\bar{k}\sigma,\,i\omega_n\rightarrow\omega+i\eta)$$ (Retarded self-energy)

Energy and lifetime of excitations

$$\Sigma^R(\bar{k}\sigma,\omega) = \mathrm{Re}\,\Sigma^R(\bar{k}\sigma,\omega) + i\,\mathrm{Im}\,\Sigma^R(\bar{k}\sigma,\omega)$$

$$S(\bar{k}\sigma,\omega) = -\frac{1}{\pi}\,\mathrm{Im}\,G^R(\bar{k}\sigma,\omega)$$ (•) – Review (e.g. previous sem.)

$$\rightarrow S(\bar{k}\sigma,\omega) = -\frac{1}{\pi}\,\frac{\mathrm{Im}\,\Sigma^R}{[\omega-(\epsilon_{\bar{k}}-\mu)-\mathrm{Re}\,\Sigma^R]^2+[\mathrm{Im}\,\Sigma^R]^2}$$

(Becomes Dirac delta for $\mathrm{Im}\,\Sigma^R \rightarrow 0$)

→ delta peaks at excitation energies located at $\omega = \epsilon_{\bar{k}} - \mu + \mathrm{Re}\,\Sigma^R$ (resonant energies)

$$\mathrm{Im}\,\Sigma^R \neq 0 \Rightarrow \text{peak broadening}$$

Considering G^R in the time domain:

$$G^R(\bar{k}\sigma,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, G^R(\bar{k}\sigma,\omega) \underset{\uparrow}{\simeq} \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega t}}{\omega-\epsilon'_{\bar{k}\sigma}+i\gamma}$$

"typical situation"
→ dominant contributions from the vicinity of the resonances. → expand.

[Review (•)]

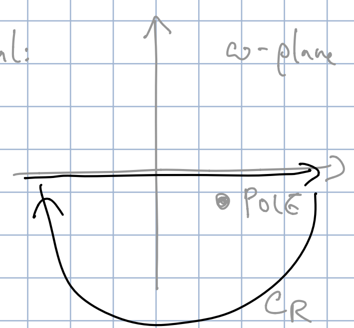
$$\epsilon'_{\bar{k}\sigma} = \epsilon_{\bar{k}} - \mu + \mathrm{Re}\,\Sigma^R(\bar{k}\sigma,\epsilon'_{\bar{k}\sigma})$$

$$\gamma = -\mathrm{Im}\,\Sigma^R(\bar{k}\sigma,\epsilon'_{\bar{k}\sigma})$$

$$\{\, S(\bar{k}\sigma,\omega) \geq 0 \Rightarrow \mathrm{Im}\,\Sigma^R \leq 0 \Rightarrow \gamma > 0 \,\}$$

Doing the integral:

$t > 0$



$$\int_{C_R} \frac{e^{-i\omega t}}{\omega-\epsilon'+i\gamma}\, d\omega \xrightarrow{R\to\infty} 0$$

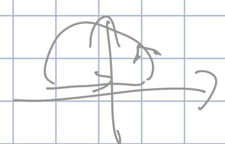
(Jordan's lemma)

$$\Rightarrow G^R(\bar{k}\sigma,t) = \frac{1}{2\pi}\oint_{C} d\omega\, \frac{e^{-i\omega t}}{\omega-\epsilon'+i\gamma}$$

C direction

Residue th. $\rightarrow G^R(\bar{k}\sigma,t) = -2\pi i\,\frac{1}{2\pi}\,\underset{\omega=\epsilon'-i\gamma}{\mathrm{Res}}\,\frac{e^{-i\omega t}}{\omega-\epsilon'+i\gamma} =$

$$= -i\,e^{-i\epsilon' t}\,e^{-\gamma t}$$

$t < 0 \rightarrow$  $\rightarrow G^R = 0.$

$$G^R(\bar{h}\sigma, t) = -i\,\Theta(t)\, e^{-\epsilon'_{\bar{h}\sigma} t}\, e^{-\gamma t}$$

Effects of interactions:
- energy shift ($\mathrm{Re}\,\Sigma^R$)
- damping of excitations ($\mathrm{Im}\,\Sigma^R$)

Excitations' lifetime: $\tau \sim \dfrac{-1}{\mathrm{Im}\,\Sigma^R(\bar{h}\sigma, \epsilon'_{\bar{h}\sigma})}$