Example:

Non-interacting particles

Previous lecture

- $G_0(\alpha\tau, \alpha'\tau') = 0$ for $\alpha \neq \alpha'$

Take $\alpha = \alpha'$.

$\boxed{\text{Notation } \phi \leftrightarrow \zeta}$ below $\zeta \to \phi$

$$G_0(\alpha, \tau_{k_1} - \tau_{k_2}) = -\lim_{M\to\infty} \frac{1}{Z_0} \int \prod_{k=1}^{M} \left(\prod_{\alpha'} \frac{1}{N} d\phi^*_{\alpha' k} d\phi_{\alpha' k} \right) \phi_{k_1\alpha} \phi^*_{k_2\alpha}\, e^{-\sum_{kk'=1}^{M} \sum_{\alpha} \phi^*_{k\alpha} S^{(\alpha)}_{kk'} \phi_{k'\alpha}} =$$

$$= -\lim_{M\to\infty} \left[\left[S^{(\alpha)} \right]^{-1} \right]_{k_1 k_2}$$

→ Moments of the Gaussian distribution (see the exercise class)

Inversion of $S^{(\alpha)} \longrightarrow$

$$\left[S^{(\alpha)} \right]^{-1} = \begin{pmatrix} 1 & \zeta a^{M-1} & \zeta a^{M-2} & \cdots & \zeta a \\ a & 1 & \zeta a^{M-1} & & \vdots \\ a^2 & a & 1 & \ddots & \\ \vdots & \ddots & a & \ddots & \zeta^{M-1} \\ a^{M-1} & \cdots & & a & 1 \end{pmatrix} \frac{1}{1 - \zeta a^M} \qquad (5)$$

$$G_0(\alpha, \tau_{k_1} - \tau_{k_2}) = -\lim_{M\to\infty} \begin{cases} \dfrac{1}{1-\zeta a^M}\, \zeta a^{M+k_1-k_2} & \tau_{k_1} - \tau_{k_2} < 0 \quad (\text{row}\# < \text{column}\#) \\ \dfrac{1}{1-\zeta a^M}\, a^{k_1-k_2} & \tau_{k_1} - \tau_{k_2} > 0 \end{cases}$$

Finish the previous lecture's calculation:

Perturbation theory

$$H = H_0 + V$$

$V \to$ interaction

$H_0 \to$ one-body operator

Here: develop an expansion in V (for Z in the first place)

$$H_0 = \sum_{\alpha} (\epsilon_\alpha - \mu)\, a^\dagger_\alpha a_\alpha$$

$$V(a^\dagger_\alpha a^\dagger_\beta \ldots a_\gamma a_\delta \ldots) \qquad (\text{normal-ordered})$$

$$Z = \int_{\psi(\beta)=\zeta\psi(0)} \mathcal{D}[\psi^* \psi]\, e^{-\int_0^\beta d\tau \left[\sum_\alpha \psi^*_\alpha(\tau)(\partial_\tau + \epsilon_\alpha - \mu)\psi_\alpha(\tau) + V(\psi^*_\alpha(\tau)\psi^*_\beta(\tau)\ldots\psi_\gamma(\tau)\psi_\delta(\tau)) \right]}$$

$$= Z_0 \left\langle e^{-\int_0^\beta d\tau\, V(\psi^*_\alpha(\tau)\psi^*_\beta(\tau)\ldots\psi_\gamma(\tau)\psi_\delta(\tau)\ldots)} \right\rangle_0$$

$\langle \ldots \rangle_0 \to$ thermal averages wrt H_0.

$$\left\langle F(\psi^*_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\ldots\psi_{\alpha_{n-1}}(\tau_{n-1})\psi_{\alpha_n}(\tau_n)) \right\rangle_0 = \frac{1}{Z_0} \int_{\psi(\beta)=\zeta\psi(0)} \mathcal{D}[\psi^*\psi]\, e^{-\sum_\alpha \int_0^\beta d\tau\, \psi^*_\alpha(\partial_\tau + \epsilon_\alpha - \mu)\psi_\alpha} F(\ldots)$$

Perturbation series in V follows from expanding the exponential:

$$\frac{Z}{Z_0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \dots \int_0^\beta d\tau_n \langle V(\psi_\alpha^*(\tau_1) \dots \psi_\gamma(\tau_1) \dots) \dots V(\psi_\alpha^*(\tau_n) \dots \psi_\gamma(\tau_n) \dots) \rangle_0$$

Wick's theorem

$$\frac{\int \mathcal{D}[\psi^*\psi]\, \psi_{i_1}\psi_{i_2}\dots\psi_{i_n}\psi^*_{j_n}\dots\psi^*_{j_1}\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j}}{\int \mathcal{D}[\psi^*\psi]\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j}} = \sum_P \zeta^P M^{-1}_{i_{Pn}j_n} \dots M^{-1}_{i_{P1}j_1}$$

i,j encompass the state and time labels.

Proof: Generating function $G(J^*, J) = \frac{\int \mathcal{D}[\psi^*\psi]\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j + \sum_i (J_i^*\psi_i + \psi_i^* J_i)}}{\int \mathcal{D}[\psi^*\psi]\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j}} = e^{\sum_{ij} J_i^* M^{-1}_{ij} J_j}$

Differentiate the LHS wrt sources (J, J^*):

$$\left.\frac{\delta^{2n} G}{\delta J^*_{i_1}\dots\delta J^*_{i_n}\delta J_{j_n}\dots\delta J_{j_1}}\right|_{J=J^*=0} = \zeta^n \frac{\int \mathcal{D}[\psi^*\psi]\, \psi_{i_1}\dots\psi_{i_n}\psi^*_{j_n}\dots\psi^*_{j_1}\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j}}{\int \mathcal{D}[\psi^*\psi]\, e^{-\sum_{ij}\psi_i^* M_{ij}\psi_j}}$$

Differentiate the RHS wrt (J, J^*):

$$\left.\frac{\delta^{2n}}{\delta J^*_{i_1}\dots\delta J^*_{i_n}\delta J_{j_n}\dots\delta J_{j_1}} e^{\sum_{ij} J_i^* M^{-1}_{ij} J_j}\right|_{J=J^*=0} = \zeta^n \frac{\delta^n}{\delta J^*_{i_1}\dots\delta J^*_{i_n}} \left(\sum_{k_n} J^*_{k_n} M^{-1}_{k_n j_n}\right) \dots \left(\sum_{k_1} J^*_{k_1} M^{-1}_{k_1 j_1}\right) e^{\sum_{ij} J_i^* M^{-1}_{ij} J_j}\Big|_{J=J^*=0} =$$

$$= \zeta^n \sum_P \zeta^P M^{-1}_{i_{Pn}j_n} \dots M^{-1}_{i_{P1}j_1}$$

$$\psi_j \to \psi_{\alpha, k} \qquad (k - \text{time point})$$

$$M^{-1}_{ij} \Leftrightarrow \text{1-particle Green's function}$$

$$= (\partial_\tau + \varepsilon_\alpha - \mu)^{-1}_{\alpha_1\tau_1, \alpha_2\tau_2} = \delta_{\alpha_1\alpha_2}\, g_{\alpha_1}(\tau_1 - \tau_2 - \eta)$$

→ n-particle Green's function of a noninteracting system = sum of permutations of products of 1-particle Green's functions (contractions).

$$\overbrace{\tilde{a}_\alpha(\tau)\tilde{a}_{\alpha'}}(\tau') = \langle T\tilde{a}_\alpha(\tau)\tilde{a}_{\alpha'}(\tau')\rangle_0$$

$$\overbrace{\tilde{\psi}_\alpha(\tau)\tilde{\psi}_{\alpha'}}(\tau') = \langle \tilde{\psi}_\alpha(\tau)\tilde{\psi}_{\alpha'}(\tau')\rangle_0$$

$$\overbrace{a_\alpha(\tau)\, a^+_{\alpha'}}(\tau') = \overbrace{\psi_\alpha(\tau)\psi^*_{\alpha'}}(\tau') = \delta_{\alpha\alpha'}\, g_\alpha(\tau - \tau') \qquad \overbrace{a^+_\alpha(\tau)\, a^+_{\alpha'}}(\tau') = \overbrace{\psi^*_\alpha(\tau)\psi^*_{\alpha'}}(\tau') = 0$$

$$\overbrace{a^+_{\alpha'}(\tau')\, a_\alpha}(\tau) = \overbrace{\psi^*_{\alpha'}(\tau')\psi_\alpha}(\tau) = \zeta\delta_{\alpha\alpha'}\, g_\alpha(\tau - \tau') \qquad \overbrace{a_{\alpha'}(\tau')\, a_\alpha}(\tau) = \overbrace{\psi_{\alpha'}(\tau')\psi_\alpha}(\tau) = 0$$