P.T. CONTINUED

$$\langle T\tilde{a}_{\alpha_1}(\tau_1)\dots\tilde{a}_{\alpha_n}(\tau_n)\rangle_0 = \langle \tilde{\psi}_{\alpha_1}(\tau_1)\dots\tilde{\psi}_{\alpha_n}(\tau_n)\rangle_0 = \sum \begin{pmatrix}\text{all complete}\\ \text{contractions}\end{pmatrix}$$

$$\text{E.g. } \langle T a_{\alpha_1}(\tau_1) a_{\alpha_2}(\tau_2) a^\dagger_{\alpha_2'}(\tau_2') a^\dagger_{\alpha_1'}(\tau_1')\rangle_0 = a_{\alpha_1}(\tau_1) a_{\alpha_2}(\tau_2) a^\dagger_{\alpha_2'}(\tau_2') a^\dagger_{\alpha_1'}(\tau_1')$$

$$+\zeta\, a_{\alpha_1}(\tau_1) a_{\alpha_2}(\tau_2) a^\dagger_{\alpha_2'}(\tau_2') a^\dagger_{\alpha_1'}(\tau_1') =$$

$$= \delta_{\alpha_1\alpha_1'}\delta_{\alpha_2\alpha_2'} g_{\alpha_1}(\tau_1-\tau_1') g_{\alpha_2}(\tau_2-\tau_2') + \zeta\,\delta_{\alpha_1\alpha_2'}\delta_{\alpha_2\alpha_1'} g_{\alpha_1}(\tau_1-\tau_2') g_{\alpha_2}(\tau_2-\tau_1')$$

Recall:

$$\frac{Z}{Z_0} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\dots\int_0^\beta d\tau_n \langle V(\psi^*_\alpha(\tau_1)\dots\psi_\gamma(\tau_1)\dots)\dots V(\psi^*_\alpha(\tau_n)\dots\psi_\gamma(\tau_n)\dots)\rangle_0$$

may be represented as sum of contractions, where the propagators $g_\alpha(\tau-\tau')$ connect products of matrix elements of V in all possible ways.

– convenient representation via diagrams

$$\frac{Z}{Z_0} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\dots\int_0^\beta d\tau_n \langle V(\psi^*_\alpha(\tau_1)\dots\psi_\gamma(\tau_1)\dots)\dots V(\psi^*_\alpha(\tau_n)\dots\psi_\gamma(\tau_n)\dots)\rangle_0 \qquad (*)$$

Take V to be a 2-body interaction: $V(\psi^*(\tau),\psi(\tau)) = \frac{1}{2}\sum_{\alpha\beta\gamma\delta}(\alpha\beta|v|\gamma\delta)\psi^*_\alpha(\tau)\psi^*_\beta(\tau)\psi_\delta(\tau)\psi_\gamma(\tau)$

→ Term of order n in $(*)$: $\dfrac{(-1)^n}{n!\,2^n}\sum_{\substack{\alpha_1\beta_1\\\gamma_1\delta_1}}\dots\sum_{\substack{\alpha_n\beta_n\\\gamma_n\delta_n}}(\alpha_1\beta_1|v|\gamma_1\delta_1)\dots(\alpha_n\beta_n|v|\gamma_n\delta_n)\int_0^\beta d\tau_1\dots\int_0^\beta d\tau_n \,\times$

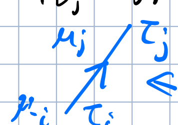
$$\times\langle\psi^*_{\alpha_1}(\tau_1)\psi^*_{\beta_1}(\tau_1)\psi_{\delta_1}(\tau_1)\psi_{\gamma_1}(\tau_1)\dots\psi^*_{\alpha_n}(\tau_n)\psi^*_{\beta_n}(\tau_n)\psi_{\delta_n}(\tau_n)\psi_{\gamma_n}(\tau_n)\rangle_0 \qquad (**)$$

(Labeled) Feynman diagrams

Consider $(**)$, think about a general contribution at order n:

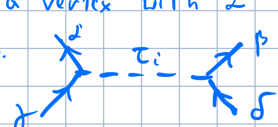
– each contraction will join some $\psi^*_{\mu_i}(\tau_i)$ with some $\psi_{\nu_j}(\tau_j)$ yielding a propagator $\delta_{\mu_i\nu_j}g_{\mu_i}(\tau_i-\tau_j)$

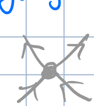
→ directed line from $\psi^*_{\mu_i}(\tau_i)$ to $\psi_{\nu_j}(\tau_j)$



$\Longleftrightarrow \delta_{\mu_i\nu_i}g_{\mu_i}(\tau_j-\tau_i)$. (for diagonal basis $\mu_i=\mu_j$)

– each interaction ⟶ a vertex with 2 ingoing lines corresponding to ψ's and 2 outgoing lines corresponding to ψ^*'s.



$\Longleftrightarrow (\alpha\beta|v|\gamma\delta)$ (or )

Set of possible ways of connecting interactions with propagators $\longleftrightarrow$ set of contractions arising from Wick's th.

$$n=1 \quad \langle \psi^*_{\alpha_1}(\tau_1)\psi^*_{\beta_1}(\tau_1)\psi_{\delta_1}(\tau_1)\psi_{\gamma_1}(\tau_1)\rangle_0$$

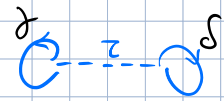
(two contractions)

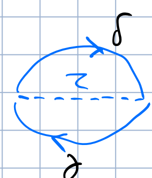
represented by two diagrams.

- $$-\frac{1}{2}\sum_{\substack{\alpha_1\beta_1\\ \gamma_1\delta_1}}\int_0^\beta d\tau\, \psi^*_{\alpha_1}(\tau)\psi^*_{\beta_1}(\tau)\psi_{\delta_1}(\tau)\psi_{\gamma_1}(\tau)\,(\alpha_1\beta_1|v|\gamma_1\delta_1) = -\frac{1}{2}\int_0^\beta d\tau_1 \sum_{\gamma\delta} g_\gamma(0)g_\delta(0)\,(\gamma\delta|v|\gamma\delta)$$
- $$-\frac{1}{2}\sum_{\substack{\alpha_1\beta_1\\ \gamma_1\delta_1}}\int_0^\beta d\tau\, \psi^*_{\alpha_1}(\tau)\psi^*_{\beta_1}(\tau)\psi_{\delta_1}(\tau)\psi_{\gamma_1}(\tau)\,(\alpha_1\beta_1|v|\gamma_1\delta_1) = -\frac{\zeta}{2}\int_0^\beta d\tau_1 \sum_{\gamma\delta} g_\gamma(0)g_\delta(0)\,(\delta\gamma|v|\gamma\delta)$$

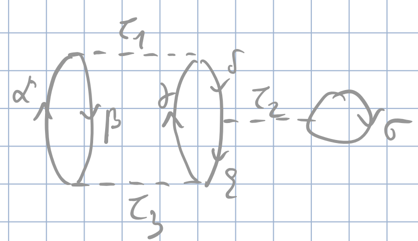
Propagator at equal times $=\zeta n_\alpha$

$$\rightarrow \frac{Z}{Z_0} = 1-\frac{\beta}{2}\sum_{\gamma\delta} n_\gamma n_\delta\left[(\gamma\delta|v|\gamma\delta)+\zeta(\delta\gamma|v|\gamma\delta)\right]$$

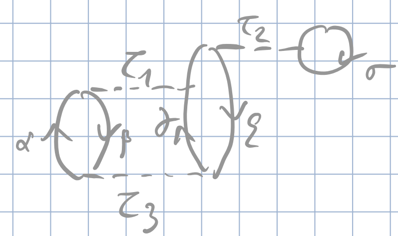




$$\Omega=-T\ln Z \;\longrightarrow\; \Omega=\Omega_0+\frac{1}{2}\sum_{\gamma\delta} n_\gamma n_\delta\left[(\gamma\delta|v|\gamma\delta)+\zeta(\delta\gamma|v|\gamma\delta)\right] \quad (\text{up to order } v^1)$$



(EQUIVALENT)

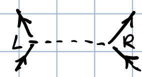


Diagrams are distinct when they cannot be continuously deformed one into the other.

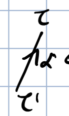
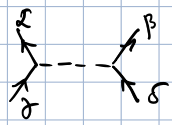
- state labels summed over – not necessary to include them
- for fermions the factor related to $\zeta \longleftrightarrow (-1)^{\#\text{of closed loops}}$
- further simplifications e.g. • exploiting symmetries
  • using symmetrized vertices
  ...

(See e.g. Negele-Orland)

$\Longrightarrow$ Rules for constructing (labeled) diagrams providing a representation of contractions contributing to n^{th} order term of perturbative expansion for $\frac{Z}{Z_0}$

1. Draw distinct (labeled) diagrams composed of n vertices  connected by directed lines

   For each distinct diagram, evaluate the corresponding contribution as follows:

2. Assign a single-particle index to each directed line, include the corresponding factor:  $\leftrightarrow g_\alpha(\tau-\tau')$
3. For each vertex, include the factor  $\leftrightarrow (\alpha\beta|v|\gamma\delta)$
4. Sum over all single-particle indices and integrate all times over the interval $[0,\beta]$
5. Multiply the result by $\frac{(-1)^n}{n!\,2^n}\zeta^{n_L}$ $\quad n_L$ – number of closed loops of propagators in the diagram.

{ More on this (unlabeled diagrams, symmetrized vertices ...) – e.g. Negele-Orland }

Linked cluster theorem

We are not so much interested in Z as in $\ln Z$ $(\Omega = -T\ln Z)$.

Linked cluster theorem – only connected diagrams contribute to $\ln Z$.

Proof by replica method

- Evaluate Z^n for $n\in\mathbb{N}$ by replicating the system n times.

Consider $Z^n = e^{n\ln Z} = 1 + n\ln Z + \sum_{m=2}^{\infty}\frac{(n\ln Z)^m}{m!}$

Assume we evaluate Z^n (in perturbation theory) and expand in n.

Then $\ln Z$ will appear as the factor in front of n^1.

Or (equivalently) continue Z^n to $n\to 0$ and evaluate $\ln Z$ as

$$\lim_{n\to 0^+}\frac{d}{dn}Z^n = \lim_{n\to 0^+}\frac{d}{dn}\left(e^{n\ln Z}\right) = \ln Z.$$

Recall: $$\frac{Z}{Z_0} = \frac{1}{Z_0}\int_{\psi_\alpha(\beta)=\zeta\psi_\alpha(0)} \mathcal{D}[\psi_\alpha^*(\tau)\psi_\alpha(\tau)]\, e^{-\int_0^\beta d\tau\left[\sum_\alpha \psi_\alpha^*(\partial_\tau-\mu+\epsilon_\alpha)\psi_\alpha + V(\psi_\alpha^*,\psi_\alpha)\right]}$$

$$\left(\frac{Z}{Z_0}\right)^n = \frac{1}{Z_0^n}\int_{\psi_\alpha^\sigma(\beta)=\zeta\psi_\alpha^\sigma(0)}\prod_{\sigma=1}^{n}\mathcal{D}[\psi_\alpha^{\sigma *}(\tau)\,\psi_\alpha^\sigma(\tau)]\, e^{-\int_0^\beta d\tau\sum_{\sigma=1}^{n}\left(\sum_\alpha \psi_\alpha^{\sigma *}(\partial_\tau+\epsilon_\alpha-\mu)\psi_\alpha^\sigma + V(\psi_\alpha^{\sigma *},\psi_\alpha^\sigma)\right)}$$

Feynman rules for $\left(\frac{Z}{Z_0}\right)^n$ the same as for $\frac{Z}{Z_0}$, but each propagator carries an extra σ index. Propagators entering or leaving a given vertex have the same σ index. All σ-s are summed over from 1 to n. $\Rightarrow$ Each connected part of a diagram carries a single σ index; $\sum_\sigma$ gives a factor of n. A diagram with n_c connected parts is $\sim n^{n_c}$, and the diagrams $\sim n$ are only the connected ones.

$\ln \frac{Z}{Z_0} = \sum(\text{connected diagrams})$ $\quad \Omega = -\beta^{-1}\ln Z \quad \Omega_0 = -\beta^{-1}\ln Z_0$

$\to \Omega = \Omega_0 - \beta^{-1}\sum(\text{connected diagrams})$

Example: Imagine we are interested in calculating Z, Ω up to 2^{nd} order in V.

Recall:

$$\frac{Z}{Z_0} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1 \ldots \int_0^\beta d\tau_n \langle V(\psi^*_\alpha(\tau_1)\ldots\psi_\gamma(\tau_1)\ldots)\ldots V(\psi^*_\alpha(\tau_n)\ldots\psi_\gamma(\tau_n)\ldots)\rangle_0 \qquad (*)$$

Take V to be a 2-body interaction: $V(\psi^*(\tau),\psi(\tau)) = \frac{1}{2}\sum_{\alpha\beta\gamma\delta}(\alpha\beta|v|\gamma\delta)\psi^*_\alpha(\tau)\psi^*_\beta(\tau)\psi_\delta(\tau)\psi_\gamma(\tau)$

$\to$ Term of order n in $(*)$: $\frac{(-1)^n}{n!2^n}\sum_{\substack{\alpha_1\beta_1\\\gamma_1\delta_1}}\ldots\sum_{\substack{\alpha_n\beta_n\\\gamma_n\delta_n}}(\alpha_1\beta_1|v|\gamma_1\delta_1)\ldots(\alpha_n\beta_n|v|\gamma_n\delta_n)\int_0^\beta d\tau_1\ldots\int_0^\beta d\tau_n \,\times$

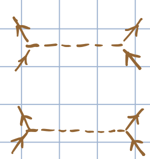
$$\times\langle\psi^*_{\alpha_1}(\tau_1)\psi^*_{\beta_1}(\tau_1)\psi_{\delta_1}(\tau_1)\psi_{\gamma_1}(\tau_1)\ldots\psi^*_{\alpha_n}(\tau_n)\psi^*_{\beta_n}(\tau_n)\psi_{\delta_n}(\tau_n)\psi_{\gamma_n}(\tau_n)\rangle_0 \qquad (**)$$

Follow the diagrammatic rules deriving from Wick's theorem:

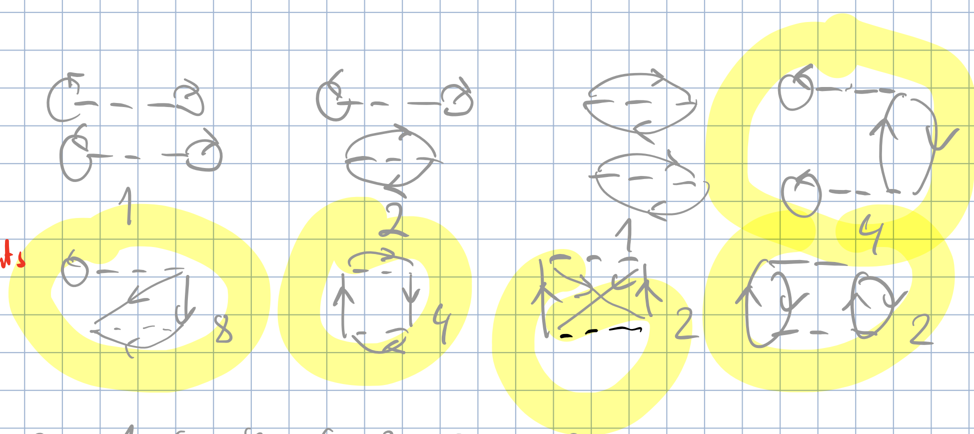
1^{st} order  – two possible contractions leading to distinct diagrams

$\{\text{sign} \leftrightarrow (-1)^n\zeta^{n_c}\}$

$\frac{1}{2}$ (2 closed loops) $\qquad \frac{1}{2}$ (1 closed loop)

2^{nd} order  there will be $(2n)! = 4! = 24$ contractions $\left(\begin{array}{l}\text{order 3:}\\ (2\cdot 3)! = 720\end{array}\right)$

$\frac{1}{n!2^n} = \frac{1}{8}$



(o) coefficients

- only 8 distinct diagrams
- only 5 connected

• check the sum rule $\to$ $1+2+1+4+8+4+2+2 = 24$ ok!

In practice: • calculations beyond 2^{nd} order become rapidly cumbersome
• resummations up to infinite order (of subclasses of diagrams) often necessary.