CONTINUE WITH P.T

- A function $g(\tau)$ (anti)periodic in $[0,\beta] \rightarrow$ $g(\tau) = \frac{1}{\beta}\sum_{\omega_n} \tilde{g}(\omega_n) e^{-i\omega_n \tau}$, $\quad \tilde{g}(\omega_n) = \int_0^\beta d\tau\, g(\tau) e^{i\omega_n \tau}$

  $\omega_n = \frac{2\pi}{\beta} n$ periodic (bosons)

  $\omega_n = \frac{2\pi}{\beta}(n+\frac{1}{2})$ antiperiodic (fermions)

- A function $f(\vec{r})$ periodic in a box of length L (in each direction)

  $V = L^3$

  $\vec{k}_n = \frac{2\pi}{L}\vec{n}$, $\vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$

  $f(\vec{r}) = \frac{1}{L^3}\sum_{\vec{k}_n} \tilde{f}(\vec{k}_n) e^{i\vec{k}_n\vec{r}} \xrightarrow{L\to\infty} \int \frac{d^3k}{(2\pi)^3}\tilde{f}(\vec{k}) e^{i\vec{k}\vec{r}}$

  $\tilde{f}(\vec{k}_n) = \int_V d^3r\, f(\vec{r}) e^{-i\vec{k}_n\vec{r}} \xrightarrow{L\to\infty} \int_{\mathbb{R}^3} d^3r\, f(\vec{r}) e^{-i\vec{k}\vec{r}}$

Frequency/momentum representation

Spatially homogeneous system $\rightarrow H_0$ translationally invariant $\rightarrow$ eigenstates $=$ (plane waves) (+ spin)

$|k_x k_y k_z)$.

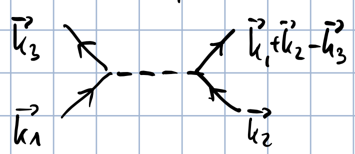
Standard formulation $\rightarrow$ finite box with periodic b.c.

Recall the matrix element of 2-body interaction in momentum basis

$$(\vec{k}_{n_1} \vec{k}_{n_2} | v | \vec{k}_{n_3} \vec{k}_{n_4}) = \int \frac{d^3x_1 d^3x_2}{V^2} v(\vec{x}_1 - \vec{x}_2) e^{i[(\vec{k}_{n_1}-\vec{k}_{n_3})\vec{x}_1 + (\vec{k}_{n_2}-\vec{k}_{n_4})\vec{x}_2]} =$$

$$= \int \frac{d^3r\, d^3R}{V^2} v(\vec{r}) e^{i\vec{R}(\vec{k}_{n_1}+\vec{k}_{n_2}-\vec{k}_{n_3}-\vec{k}_{n_4}) + \frac{i}{2}\vec{r}(\vec{k}_{n_1}-\vec{k}_{n_2}-\vec{k}_{n_3}+\vec{k}_{n_4})} =$$

$$= \frac{1}{V}\delta_{\vec{k}_{n_1}+\vec{k}_{n_2}-\vec{k}_{n_3}-\vec{k}_{n_4}}\, \tilde{v}(\vec{k}_{n_1}-\vec{k}_{n_3})$$



$\left\{ \begin{array}{l} \text{Each interaction conserves momentum,} \\ \text{sum of momenta entering a vertex equal} \\ \text{sum of momenta leaving the vertex} \end{array} \right\}$

Each diagram — product of connected parts

consider a connected diagram with m vertices

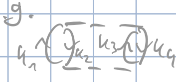
$\rightarrow$ there are $2m$ propagators $g_{k_n}(\tau_i - \tau_j)$, each summed over k_n

$\rightarrow$ m momentum-conserving δ's

$\rightarrow$ a factor of V^{-m}

$\rightarrow$ diagram is connected, momentum is conserved at each vertex

$\Rightarrow$ one of the δ's is redundand. e.g.



$k_1 + k_3 = k_2 + k_4$

at each vertex $\to$ 1 constraint. (2)

$\Rightarrow$ $(m-1)$ constraints on $2m$ momenta $\Rightarrow$ sum over $(m+1)$ independent momenta.

For $V \to \infty$ $\sum_{k_n} \to \frac{V}{(2\pi)^3} \int d^3k$ $\Rightarrow$ Yields a factor V^{m+1}

In total: V^{m+1} from independent integrals; V^{-m} from interaction vertices

$\Rightarrow$ factor of V for each connected contribution

A diagram with n_c connected parts $\sim V^{n_c}$; only fully connected diagrams give extensive contributions.

(Compare the linked cluster theorem).

Time $\to$ frequency $(\tau \to \omega_n)$

$$\tilde{g}_\alpha(\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n \tau} e^{-(\epsilon_\alpha-\mu)\tau}\Big[\theta(\tau)\underbrace{(1+\zeta n_\alpha)} + \theta(-\tau) n_\alpha\Big] = (*)$$

$$1+\frac{\zeta}{e^{\beta(\epsilon_\alpha-\mu)}-\zeta} = \frac{e^{\beta(\epsilon_\alpha-\mu)}}{e^{\beta(\epsilon_\alpha-\mu)}-\zeta}$$

$$(*) = \frac{1}{i\omega_n-(\epsilon_\alpha-\mu)}\Big[e^{i\omega_n\beta}e^{-(\epsilon_\alpha-\mu)\beta}-1\Big]\frac{e^{\beta(\epsilon_\alpha-\mu)}}{e^{\beta(\epsilon_\alpha-\mu)}-\zeta} = (**)$$

$$e^{i\omega_n\beta} = \begin{cases} e^{i\frac{2\pi}{\beta}n\beta} = 1 & (\text{Bosons}) \\ e^{i\frac{2\pi(n+\frac{1}{2})}{\beta}\beta} = -1 & (\text{Fermions}) \end{cases} = \zeta$$

$$(**) = \frac{1}{i\omega_n-(\epsilon_\alpha-\mu)}\left(\frac{\zeta e^{-(\epsilon_\alpha-\mu)\beta}-1}{e^{\beta(\epsilon_\alpha-\mu)}-\zeta}\, e^{\beta(\epsilon_\alpha-\mu)}\right) = \frac{-1}{i\omega_n-(\epsilon_\alpha-\mu)}$$

$$= -1$$

$$\tilde{g}_\alpha(\omega_n) = \frac{-1}{i\omega_n-(\epsilon_\alpha-\mu)}$$

$$g_\alpha(\tau) = \sum_{\omega_n}\frac{-1}{\beta}e^{-i\omega_n\tau}\frac{1}{i\omega_n-(\epsilon_\alpha-\mu)}$$

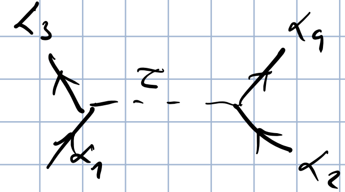
$\tau = 0 \to g_\alpha(\tau) \to g_\alpha(\tau-\eta)$

$$g_\alpha(\tau) = \sum_{\omega_n}\frac{-1}{\beta}e^{-i\omega_n(\tau-\eta)}\frac{1}{i\omega_n-(\epsilon_\alpha-\mu)} =$$

$$= g_\alpha(\tau-\eta)$$

Back to diagrams: $\int_0^\beta d\tau$

A time integration associated with the vertex



$$\int_0^\beta d\tau\, g_{\alpha_1}(\tau-\tau_1)\, g_{\alpha_2}(\tau-\tau_2)\, g_{\alpha_3}(\tau_3-\tau)\, g_4(\tau_4-\tau) \overset{\text{FT}}{=}$$

$$\left(\sum_{\omega_i}\right) \underbrace{\int_0^\beta d\tau\, e^{-i(\omega_{n_1}+\omega_{n_2}-\omega_{n_3}-\omega_{n_4})\tau}}_{\beta\,\delta_{n_1+n_2-n_3-n_4}} \prod_{i=1}^{4} \frac{\tilde{g}(\omega_{n_i})}{\beta}\, e^{i(\omega_{n_1}\tau_1+\omega_{n_2}\tau_2-\omega_{n_3}\tau_3-\omega_{n_4}\tau_4)}$$

- Frequency is conserved at each vertex
- Each vertex gives a factor of β

$\rightarrow$ alike for $\vec{k}$ $m+1$ independent frequency summations

$\rightarrow$ each propagator has a factor of $\frac{1}{\beta}$, each integral yields a β
$\Rightarrow$ overall factor of β^{-m}

Rules for (unlabeled) Feynman diagrams in momentum/frequency representation:

- Draw all distinct diagrams composed of n vertices connected by directed lines.
(distinct $\leftrightarrow$ cannot be continuously deformed so as to coincide)
- Work out the signs/coefficients (e.g. by enumerating contractions).
- Assign momentum/frequency labels to each (directed) line. For each connected part containing m interactions pick $m+1$ propagators to label independently, use momentum/frequency conservation at each vertex to label the remaining propagators. With each directed line associate the factor $\nearrow \vec{k}, \omega_n \leftrightarrow \tilde{g}_{\vec{k}}(\omega_n) = \frac{-1}{i\omega_n - (\epsilon_{\vec{k}}-\mu)}$
For prop. beginning and ending at the same vertex include the factor $e^{i\omega_n\eta}$
- For each independent momentum perform the integral $\int \frac{d^3k}{(2\pi)^d}$, sum over all indep. frequencies.
- Multiply by $\frac{1}{\beta^n} V^{n_c}$.

For each vertex include the corresponding matrix element ($\tilde{V}$)

Perturbation theory for the Green's function

$$\Big\{ Z = \int_{\psi(\beta)=\zeta\psi(0)} \mathcal{D}[\psi^*\psi]\, e^{-\int_0^\beta d\tau \left[\sum_\alpha \psi_\alpha^*(\tau)(\partial_\tau + \epsilon_\alpha - \mu)\psi_\alpha(\tau) + V(\psi_\alpha^*(\tau)\psi_\beta^*(\tau)\ldots\psi_\gamma(\tau)\psi_\delta(\tau))\right]}$$

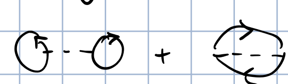
$$S_{int}[\psi^*,\psi] = \frac{1}{2}\int_0^\beta d\tau \sum_{\substack{\alpha_1\alpha_2\\ \alpha_1'\alpha_2'}} (\alpha_1\alpha_2|V|\alpha_1'\alpha_2')\,\psi_{\alpha_1}^*(\tau)\psi_{\alpha_2}^*(\tau)\psi_{\alpha_2'}(\tau)\psi_{\alpha_1'}(\tau)$$

$$G(\alpha_1\tau_1,\alpha_2\tau_2) = -\langle \psi_{\alpha_1}(\tau_1)\psi_{\alpha_2}^*(\tau_2)\rangle =$$

$$= -\frac{Z_0}{Z}\langle \psi_{\alpha_1}(\tau_1)\psi_{\alpha_2}^*(\tau_2)e^{-S_{int}}\rangle_0$$

To compute G to a given order in V expand both Z and $\langle\ldots\rangle_0$

Work it out at 1st order to begin with:

- Z - calculated before ()
- $-\langle \psi_{\alpha_1}(\tau_1)\psi_{\alpha_2}^*(\tau_2)e^{-S_{int}}\rangle_0 \simeq G_0(\alpha_1\tau_1,\alpha_2\tau_2) + \langle \psi_{\alpha_1}(\tau_1)\psi_{\alpha_2}^*(\tau_2)S_{int}\rangle_0 =$

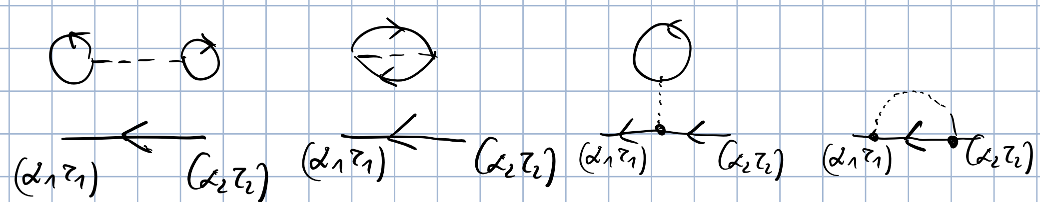
$$= G_0(\alpha_1\tau_1,\alpha_2\tau_2) + \frac{1}{2}\sum_{\substack{\beta_1\beta_2\\ \beta_1'\beta_2'}}(\beta_1\beta_2|V|\beta_1'\beta_2')\int_0^\beta d\tau \langle \psi_{\alpha_1}(\tau_1)\psi_{\alpha_2}^*(\tau_2)\psi_{\beta_1}^*(\tau)\psi_{\beta_2}^*(\tau)\psi_{\beta_2'}(\tau)\psi_{\beta_1'}(\tau)\rangle_0$$

$$= \Big\{\substack{\text{Wick}\\ \text{th}}\Big\} = G_0(\alpha_1\tau_1,\alpha_2\tau_2) - \sum_{\substack{\beta_1,\beta_2,\\ \beta_1',\beta_2'}}(\beta_1\beta_2|V|\beta_1'\beta_2')\int_0^\beta d\tau \Big\{\frac{1}{2}G_0(\alpha_1\tau_1,\alpha_2\tau_2)\big[G_0(\beta_1'\tau,\beta_1\tau)G_0(\beta_2'\tau,\beta_2\tau) +$$

$$+\zeta G_0(\beta_1'\tau,\beta_2\tau)G_0(\beta_2'\tau,\beta_1\tau)\big] +$$

$$+ G_0(\alpha_1\tau_1,\beta_1\tau)\big[G_0(\beta_2'\tau,\alpha_2\tau_2)G_0(\beta_1'\tau,\beta_2\tau) +$$

$$+\zeta G_0(\beta_1'\tau,\alpha_2\tau_2)G_0(\beta_2'\tau,\beta_2\tau)\big]\Big\}$$

$\Longrightarrow$ 

Only indices carried by the internal lines are summed over.