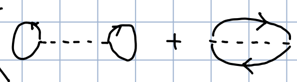
P.T. for the (thermal) Green's functions ctd.

$$G(\alpha_1\tau_1,\alpha_2\tau_2) = -\langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\rangle = -\frac{Z_0}{Z}\langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)e^{-S_{int}}\rangle_0$$

$$S_{int}[\psi^*,\psi] = \frac{1}{2}\int_0^\beta d\tau \sum_{\substack{\alpha_1\alpha_2\\ \alpha_1'\alpha_2'}} (\alpha_1\alpha_2|v|\alpha_1'\alpha_2')\psi^*_{\alpha_1}(\tau)\psi^*_{\alpha_2}(\tau)\psi_{\alpha_2'}(\tau)\psi_{\alpha_1'}(\tau)$$

To compute G to a given order in V expand both Z and $\langle\ldots\rangle_0$

Work it out at 1st order to begin with.

- Z - calculated before ()
- $-\langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)e^{-S_{int}}\rangle_0 \simeq G_0(\alpha_1\tau_1,\alpha_2\tau_2) + \langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)S_{int}\rangle_0 =$

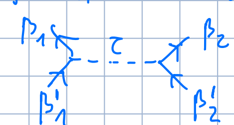
$$= G_0(\alpha_1\tau_1,\alpha_2\tau_2) + \frac{1}{2}\sum_{\substack{\beta_1\beta_2\\ \beta_1'\beta_2'}} (\beta_1\beta_2|v|\beta_1'\beta_2')\int_0^\beta d\tau \langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\psi^*_{\beta_1}(\tau)\psi^*_{\beta_2}(\tau)\psi_{\beta_2'}(\tau)\psi_{\beta_1'}(\tau)\rangle_0 = \circledast$$

Wick th.

$$\overbrace{\psi_\alpha(\tau)\psi^*_{\alpha'}}(\tau') = \langle \psi_\alpha(\tau)\psi^*_{\alpha'}(\tau')\rangle_0 = -G_0(\alpha\tau,\alpha'\tau')$$

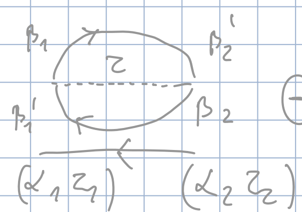
$$\langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\psi^*_{\beta_1}(\tau)\psi^*_{\beta_2}(\tau)\psi_{\beta_2'}(\tau)\psi_{\beta_1'}(\tau)\rangle_0 = \sum\left(\begin{array}{c}\text{COMPLETE}\\ \text{CONTRACTIONS}\end{array}\right)$$

four ψ's with indices from $(\beta_1\beta_2|v|\beta_1'\beta_2')$



→ CONTRACTIONS WHERE $\overbrace{\psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}}(\tau_2)$

$$\psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\,\psi^*_{\beta_1}(\tau)\psi^*_{\beta_2}(\tau)\psi_{\beta_2'}(\tau)\psi_{\beta_1'}(\tau)$$

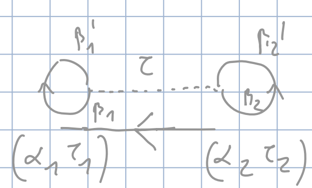


(-1)

SIGNS: $\overbrace{\psi\psi^*} \leftrightarrow -G_0$

$\overbrace{\psi\psi}{}^* \; \overbrace{\psi\psi^*} \leftrightarrow (-G_0)(-G_0)$

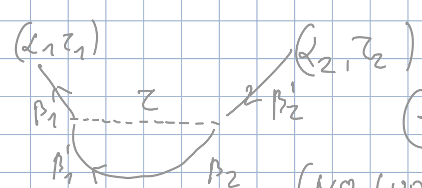
$$\psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\,\psi^*_{\beta_1}(\tau)\psi^*_{\beta_2}(\tau)\psi_{\beta_2'}(\tau)\psi_{\beta_1'}(\tau)$$



(-1)

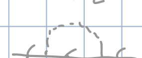
→ CONTRACTIONS WHERE $\overbrace{\psi_{\alpha_1}(\tau_1)\psi^*_{\beta_1}}(\tau)$

$$\psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\,\psi^*_{\beta_1}(\tau)\psi^*_{\beta_2}(\tau)\psi_{\beta_2'}(\tau)\psi_{\beta_1'}(\tau)$$

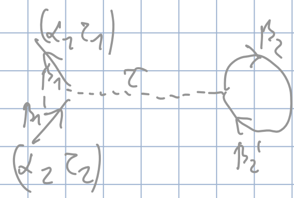


(-1)

(NO LOOPS FROM PROPAGATOR LINES)

(ANOTHER DRAWING: )

$$\psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\,\psi^*_{\beta_1}(\tau)\psi^*_{\beta_2}(\tau)\psi_{\beta_2'}(\tau)\psi_{\beta_1'}(\tau)$$



$(-1)\cdot\zeta$

$$= \zeta\, \psi_{\alpha_1}\psi^*_{\beta_1}\,\psi^*_{\alpha_2}\psi^*_{\beta_2}\psi_{\beta_2'}\psi_{\beta_1'} = \zeta\, \psi_{\alpha_1}\psi^*_{\beta_1}\,\psi_{\beta_1'}\psi^*_{\alpha_2}\,\psi_{\beta_2'}\psi^*_{\beta_2}$$

→ CONTRACTIONS WHERE $\psi_{\alpha_1}(\tau_1)\psi^*_{\beta_2}(\tau)$ GIVE THE SAME CONTRIBUTIONS AS THOSE WITH $\psi_{\alpha_1}(\tau_1)\psi^*_{\beta_1}(\tau)$

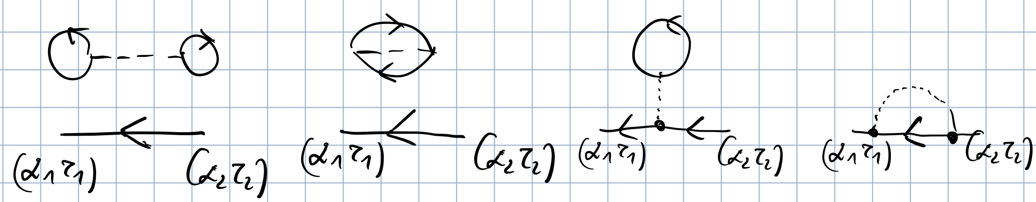
BECAUSE:

$$\langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\psi^*_{\beta_1}(\tau)\psi^*_{\beta_2}(\tau)\psi_{\beta_2'}(\tau)\psi_{\beta_1'}(\tau)\rangle_0 = \langle \psi_{\alpha_1}(\tau_1)\psi^*_{\alpha_2}(\tau_2)\psi^*_{\beta_2}(\tau)\psi^*_{\beta_1}(\tau)\psi_{\beta_1'}(\tau)\psi_{\beta_2'}(\tau)\rangle_0$$

AND THE INDICES β_i, β_i' ARE SUMMED OVER.

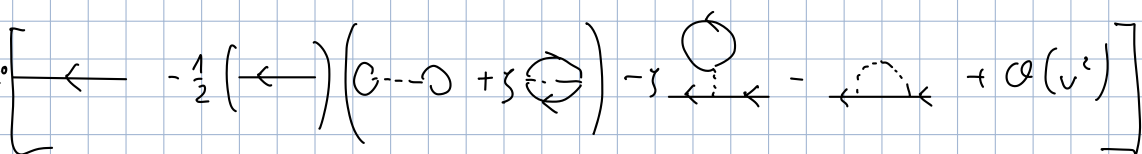
$$⊛ = G_0(\alpha_1\tau_1,\alpha_2\tau_2) - \sum_{\substack{\beta_1\beta_2\\ \beta_1'\beta_2'}} (\beta_1\beta_2|v|\beta_1'\beta_2')\int_0^\beta d\tau \Big\{ \tfrac{1}{2} G_0(\alpha_1\tau_1,\alpha_2\tau_2)\big[G_0(\beta_1'\tau,\beta_1\tau)G_0(\beta_2'\tau,\beta_2\tau) + \zeta G_0(\beta_1'\tau,\beta_2\tau)G_0(\beta_2'\tau,\beta_1\tau)\big] + G_0(\alpha_1\tau_1,\beta_1\tau)\big[G_0(\beta_2'\tau,\alpha_2\tau_2)G_0(\beta_1'\tau,\beta_2\tau) + \zeta G_0(\beta_1'\tau,\alpha_2\tau_2)G_0(\beta_2'\tau,\beta_2\tau)\big]\Big\}$$

$\Longrightarrow$



Only indices carried by the internal lines are summed over.

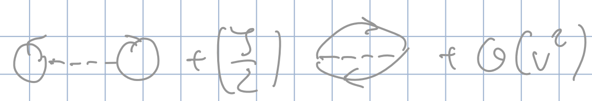
$G(\alpha_1\tau_1, \alpha_2\tau_2) = \frac{Z_0}{Z}$



Combine this with the expression for $\frac{Z}{Z_0}$ (at order $\mathcal{O}(v)$)

$\frac{Z}{Z_0} = 1 - \left(\frac{1}{2}\right)$

$\Longrightarrow \frac{Z_0}{Z} = 1 + \left(\frac{1}{2}\right)$



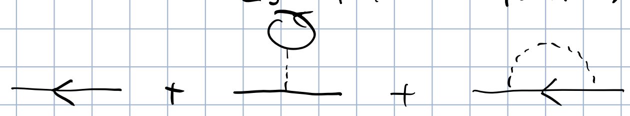
simply because $\frac{1}{1-x+\mathcal{O}(x^2)} = 1+x+\mathcal{O}(x^2)$

Disconnected contributions cancel.

③

$$\longrightarrow G(\alpha_1\tau_1,\alpha_2\tau_2) = G_0(\alpha_1\tau_1,\alpha_2\tau_2) - \sum_{\substack{\beta_1\beta_2\\ \beta_1'\beta_2'}} (\beta_1,\beta_2|v|\beta_1'\beta_2') \int_0^\beta d\tau\, G_0(\alpha_1\tau_1,\beta_1\tau)\times$$

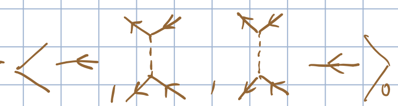
$$\times\Big[\zeta G_0(\beta_1'\tau,\alpha_2\tau_2)G_0(\beta_2'\tau,\beta_2\tau) + G_0(\beta_2'\tau,\alpha_2\tau_2)G_0(\beta_1'\tau,\beta_2\tau)\Big]$$

$\longrightarrow$ 

Only connected diagrams! (general result, valid at any order in v → proof by replica method)

Combinatorial factors → direct analysis of the number of contractions leading to a diagram.

e.g. 

Obtained from: $\frac{1}{2!\,2^2}$ 

- interchange of vertices → factor of 2
- connection of outgoing line to vertex → ×2 (o)
- connection of ingoing line to vertex → ×2.

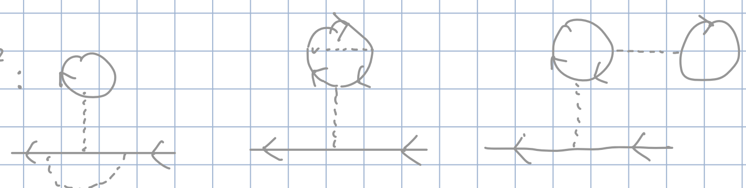
$\frac{1}{2!\,2^2}\cdot 2\cdot 2\cdot 2 = 1.$ (this turns out to be a general result proof → e.g. N-O.)

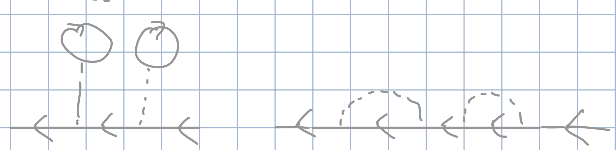
and for any $G^{(2n)}$

- Sign of a diagram ⇔ $(-1)^n \zeta^{n_c}$

Diagram rules — analogous to those for Z, but 2 external legs (1 ingoing and 1 outgoing).

At order v^2:





+ 5 more (IN TOTAL 10)

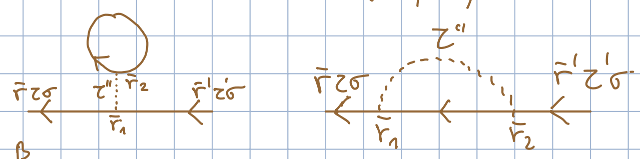
<u>Example</u>: ① 1-st order, position basis $(\bar r_1\sigma_1,\bar r_2\sigma_2|\hat V|\bar r_3\sigma_3,\bar r_4\sigma_4) = v(\bar r_1-\bar r_2)\,\delta(\bar r_1-\bar r_3)\,\delta(\bar r_2-\bar r_4)$ $\cdot\,\delta_{\sigma_1\sigma_3}\cdot\delta_{\sigma_2\sigma_4}$ (interaction diagonal)

$$S_0 = \int_0^\beta d\tau \int_0^\beta d\tau' \int d\bar r \int d\bar r' \sum_\sigma \psi_\sigma^*(\bar r,\tau)\left[-G_{0\sigma}^{-1}(\bar r\tau,\bar r'\tau')\right]\psi_\sigma(\bar r'\tau')$$ (quadratic, unperturbed S)

$$S_{int} = \frac{1}{2}\int_0^\beta d\tau \int d\bar r \int d\bar r' \sum_{\sigma\sigma'} v(\bar r-\bar r')\,\psi_\sigma^*(\bar r,\tau)\psi_{\sigma'}^*(\bar r',\tau)\psi_{\sigma'}(\bar r',\tau)\psi_\sigma(\bar r,\tau)$$

G_0 diagonal only in σ.

1st order correction to $G_{0\sigma}(\bar{r}\tau, \bar{r}'\tau')$:



$$-\varsigma\int_0^\beta d\tau''\int d\bar{r}_1\int d\bar{r}_2 \sum_{\sigma'} v(\bar{r}_1-\bar{r}_2)\, G_{0\sigma}(\bar{r}\tau,\bar{r}_1\tau'')\, G_{0\sigma}(\bar{r}_1\tau'',\bar{r}'\tau')\, G_{0\sigma'}(\bar{r}_2\tau'',\bar{r}_2\tau''^{+})$$

$$-\int_0^\beta d\tau''\int d\bar{r}_1\int d\bar{r}_2\; v(\bar{r}_1-\bar{r}_2)\, G_{0\sigma}(\bar{r}\tau,\bar{r}_1\tau'')\, G_{0\sigma}(\bar{r}_1\tau'',\bar{r}_2\tau''^{+})\, G_{0\sigma}(\bar{r}_2\tau'',\bar{r}'\tau')$$

(2) Momentum basis, assuming translational invariance

diagonal G, momentum/frequency conserved at each vertex.

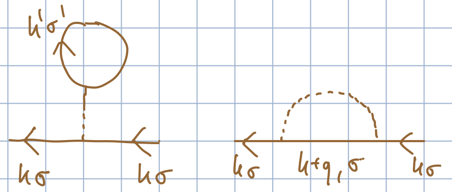
$$S_0 = \sum_{k\sigma} \psi_\sigma^*(k)\left[-G_{0\sigma}^{-1}(k)\right]\psi_\sigma(k) \qquad k:=(\vec{k},\omega)$$

$$S_{int} = \frac{1}{2\beta V}\sum_{\substack{\bar{k}\bar{k}'q\\ \sigma\sigma'}} v(\vec{q})\,\psi_\sigma^*(k+q)\,\psi_{\sigma'}^*(k'-q)\,\psi_{\sigma'}(k')\,\psi_\sigma(k)$$

1st order correction to $G_{0\sigma}(k)$

(frequency/momentum representation works completely like for Z)

↳ SEE PREVIOUS LECTURE



$\eta = 0^+$.

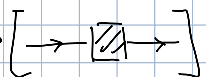
$$\frac{1}{\beta V}(-\varsigma)\sum_{k'\sigma'} v(0)\, G_{0\sigma}(k)\, G_{0\sigma}(k)\, G_{0\sigma'}(k')\, e^{i\omega_n'\eta}$$

$$+\frac{1}{\beta V}(-1)\sum_{q} v(\vec{q})\, G_{0\sigma}(k)\, G_{0\sigma}(k)\, G_{0\sigma}(k+q)\, e^{i\omega_{n+\nu}\eta}$$

- Self energy, Dyson's equation

Perturbation expansion for g $(\mathcal{G})$ → every (connected) diagram contains two external lines

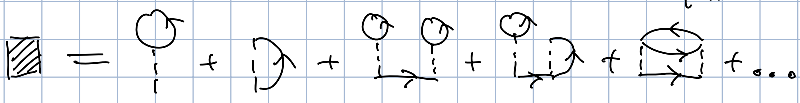
→ In momentum/frequency representation: (momentum/frequency conserved at each vertex)

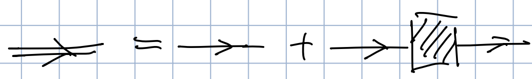
any diagram ⟶ 

→ the corresponding algebraic expression:

$$g_0(\vec{k}\sigma,\omega_n)\underbrace{A(\vec{k}\sigma,\omega_n)}_{\substack{\text{determined by the}\\ \text{structure of the diagram.}}}g_0(\vec{k}\sigma,\omega_n)$$

→ We may write: $g(\vec{k}\sigma,\omega_n) = g_0(\vec{k}\sigma,\omega_n) + g_0(\vec{k}\sigma,\omega_n)\underbrace{\tilde{\Sigma}(\vec{k}\sigma,\omega_n)}_{\text{from summing A over all possible connected diagrams}} g_0(\vec{k}\sigma,\omega_n)$

$\tilde{\Sigma}(\vec{k}\sigma,\omega_n) =$ 

$g(\vec{k}\sigma,\omega_n) =$ 

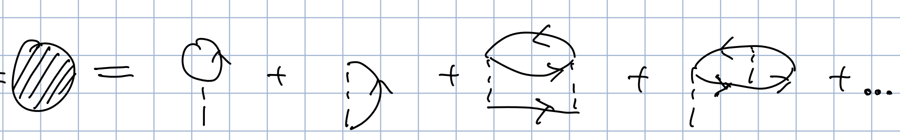
Two classes of diagrams in the expansion of $\tilde{\Sigma}$:

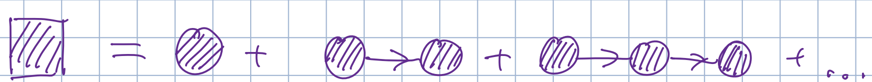
- reducible (can be separated into two pieces by cutting one propagator line)
- irreducible (cannot ...)

$\Sigma(\vec{k}\sigma,\omega_n) :=$ sum over all the contributions represented by irreducible diagrams

self energy ("proper", "irreducible").

$\tilde{\Sigma}(\vec{k}\sigma,\omega_n) = \Sigma(\vec{k}\sigma,\omega_n) + \Sigma(\vec{k}\sigma,\omega_n) g_0(\vec{k}\sigma,\omega_n)\Sigma(\vec{k}\sigma,\omega_n) + \Sigma(\vec{k}\sigma,\omega_n) g_0(\vec{k}\sigma,\omega_n)\Sigma(\vec{k}\sigma,\omega_n) g_0(\vec{k}\sigma,\omega_n)\Sigma(\vec{k}\sigma,\omega_n) + \dots$

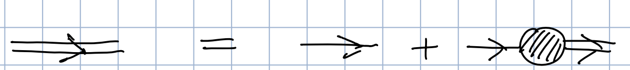
$\Sigma(\vec{k}\sigma,\omega) =$  (irreducible diagrams)



→ insert into $g(\vec{k}\sigma,\omega_n)$ $\qquad k = (\vec{k}\sigma\omega_n)$

$$g(\vec{k}\sigma,\omega_n) \overset{\text{notation}}{=} g(k) = g_0(k) + g_0(k)\Sigma(k)g_0(k) + g_0(k)\Sigma(k)g_0(k)\Sigma(k)g_0(k) + \dots$$

$$= g_0(k) + g_0(k)\Sigma(k)g(k)$$



$$\rightarrow g(k) = \frac{g_0(k)}{1 - g_0(k)\Sigma(k)} = \frac{1}{g_0^{-1}(k) - \Sigma(k)}$$

(Dyson's eq.) $\boxed{g^{-1}(k) = g_0^{-1}(k) - \Sigma(k)}$

$\Sigma(k)$ – represented by 1PI (irreducible) diagrams.

$$\rightarrow g(\vec{k}\sigma, \omega_n) = \frac{1}{i\omega_n - (\epsilon_{\vec{k}} - \mu) - \Sigma(\vec{k}\sigma, \omega_n)}$$