$$\int_0^{2\pi} \frac{dx}{5-3\cos x} = \frac{\pi}{2}$$

Ogólnie: $I = \int_0^{2\pi} Q(\cos(x), \sin(x))\,dx$ gdzie Q wymierna dwóch zmiennych i ciągła na $C(0,1) = \{(x,y) : x^2+y^2 = 1\}$.

$z = e^{ix}$, $dz = iz\,dx$, $x \in [0, 2\pi]$, $\cos x = \frac{z + \frac{1}{z}}{2}$, $\sin x = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$$I = \oint_{|z|=1} Q\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{dz}{iz} = 2\pi i \sum_{|z|<1} \mathrm{Res}_z\, Q\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{1}{iz}.$$

Lemat $f : \mathcal{O} \to \mathbb{C}$ hol., $a \in \mathcal{O}$ jest biegunem f rzędu 1, $\alpha < \beta < 2\pi$.

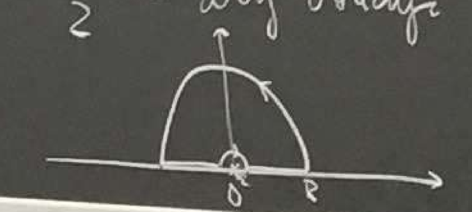
Wówczas $\lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon,\alpha,\beta}} f(z)\,dz = i(\beta - \alpha)\cdot \mathrm{Res}_a f.$

$\Gamma_{\varepsilon,\alpha,\beta} = \{a + \varepsilon e^{i\varphi} : \varphi \in [\alpha, \beta]\}$

$$\int_{\Gamma_{\varepsilon,\alpha,\beta}} f(z)\,dz = \int_\alpha^\beta f(a+\varepsilon e^{i\varphi})\,\varepsilon e^{i\varphi}\, i\,d\varphi = \left\{ \mathrm{Res}_a f = \lim_{z\to a} f(z)(z-a) \overset{\forall\varphi}{=} \lim_{\varepsilon\to 0} f(a+\varepsilon e^{i\varphi})\,\varepsilon e^{i\varphi} \right\} \xrightarrow[\varepsilon\to 0]{} i(\beta-\alpha)\cdot\mathrm{Res}_a(f).$$

$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ – aby obliczyć tą całkę rozważmy całkę $\int_{\Gamma_{R,\varepsilon}} \frac{e^{iz}}{z}\,dz$

gdzie



• $\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_\varepsilon^R \frac{e^{ix}}{x}\,dx = 2i\int_\varepsilon^R \frac{\sin x}{x}\,dx.$

•• $\int \frac{e^{iz}}{z}\,dz \xrightarrow[\varepsilon\to 0]{} -i\pi\, \mathrm{Res}_0 \frac{e^{iz}}{z} = -i\pi.$

••• Jeśli $\int \frac{e^{iz}}{z}\,dz \xrightarrow[R\to\infty]{} 0$ to $2i\int_0^\infty \frac{\sin x}{x}\,dx - i\pi = 0.$

Lemat Jordana. Niech $f : \mathbb{R}_{>0} \to \mathbb{C}$ będzie funkcją ciągłą t. że $\lim_{|z|\to\infty} f(z) = 0$. Jeśli $a > 0$ to $\int_{\Gamma_R} f(z) e^{iaz}\,dz \xrightarrow[R\to\infty]{} 0$.

$\Gamma_R = \{Re^{i\varphi} : \varphi \in]0, \pi[$

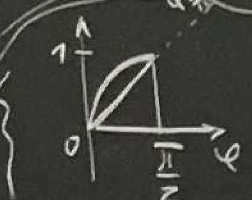
Dowód:

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \le \int_0^\pi \left|f(Re^{i\varphi})\, e^{iaR(\cos\varphi + i\sin\varphi)}\, Re^{i\varphi} i\right| d\varphi$$

$$\le \sup_{\varphi\in[0,\pi]} |f(Re^{i\varphi})| \int_0^\pi e^{-aR\sin\varphi} R\,d\varphi \xrightarrow[R\to\infty]{} 0.$$

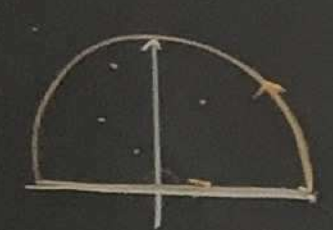
$0 \xleftarrow[R\to\infty]{}$ – ograniczone ze względu na R

$$\int_0^\pi \ldots d\varphi = 2R\int_0^{\frac{\pi}{2}} e^{-aR\sin\varphi}\,d\varphi =$$



$$\le 2R\int_0^{\frac{\pi}{2}} e^{-aR\varphi\cdot\frac{2}{\pi}}\,d\varphi = \frac{2R\cdot\pi}{-2aR} e^{-aR\varphi\frac{2}{\pi}}\Big|_0^{\frac{\pi}{2}} = -\frac{\pi}{a}\left(e^{-aR} - 1\right) \xrightarrow[R\to\infty]{} \frac{\pi}{a}$$

- $\int_{-\infty}^{+\infty} Q(x) e^{iax}$ gdzie Q - wymierna, $a > 0$.
  bieguny $Q \in \mathbb{C} \setminus \mathbb{R}$.
  $\lim_{z \to \infty} Q(z) = 0$



$f(z) = Q(z)$

$$\int_{\Gamma_R} f(z) e^{iaz} dz + \int_{-R}^{R} f(x) e^{iax} dx \xrightarrow[R \to \infty]{\text{Lemat Jordana}} \int_{-\infty}^{+\infty} Q(x) e^{iax} dx$$

$\Gamma_R = \{ Re^{i\varphi} : \varphi \in [0, \pi] \}$

$$2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res}_z \left(f(z) e^{iaz} \right)$$

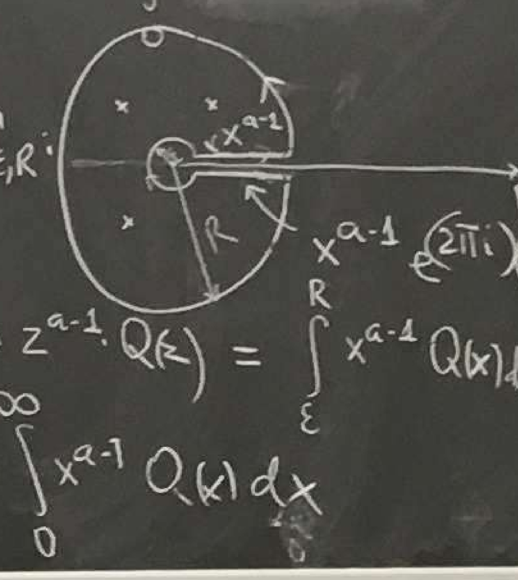
dla dost. dużych R

Przykład: $\int_0^\infty \frac{\sqrt[6]{x}\, dx}{(x+1)(x+2)} = ?$

Ogólnie: $\int_0^\infty x^{a-1} Q(x) dx$ gdzie $a \in \mathbb{R} \setminus \mathbb{Z}$, Q - wymierna, bieguny $Q \subset \mathbb{C} \setminus \mathbb{R}_+$.

$\lim_{z \to 0} z^a Q(z) \overset{*}{=} \lim_{z \to \infty} z^a Q(z) \overset{*}{=} 0$

Kontur $\Gamma_{\varepsilon, R}$:



$f(z) = z^{a-1} \cdot Q(z)$

$z^{a-1} = e^{(a-1) \log z}$ $\quad \log z = \log|z| + i \arg z$, $\arg z \in]0, 2\pi[$

$x^{a-1} e^{(2\pi i)(a-1)}$

$x = x e^{2\pi i} \Rightarrow x^{a-1} \rightsquigarrow x^{a-1} e^{2\pi i (a-1)}$

$$2\pi i \sum \operatorname{Res} z^{a-1} Q(z) = \int_\varepsilon^R x^{a-1} Q(x) dx + \underbrace{\int_0^{2\pi} (Re^{i\varphi})^{a-1} Q(Re^{i\varphi}) Re^{i\varphi} i\, d\varphi}_{* \Rightarrow \to 0} - e^{2\pi i (a-1)} \int_\varepsilon^R x^{a-1} Q(x) dx - \underbrace{\int_0^{2\pi} (\varepsilon e^{i\varphi})^{a-1} Q(\varepsilon e^{i\varphi}) \varepsilon e^{i\varphi} i\, d\varphi}_{(*) \Rightarrow \to 0}$$

$$\left(1 - e^{2\pi i (a-1)}\right) \int_0^\infty x^{a-1} Q(x) dx$$

W szczególności: $a - 1 = \frac{1}{6}$ $\quad Q = \frac{1}{(x+1)(x+2)}$

$$\left(1 - e^{2\pi i \cdot \frac{1}{6}}\right) \cdot \int_0^\infty \frac{x^{\frac{1}{6}}\, dx}{(x+1)(x+2)} = \left\{ \begin{array}{l} \text{bieguny} \\ x = -1 = e^{i\pi} \\ x = -2 = 2 e^{i\pi} \end{array} \right\} = 2\pi i \left(\frac{e^{\frac{i\pi}{6}}}{1} + \frac{2^{\frac{1}{6}} e^{\frac{i\pi}{6}}}{-1} \right)$$

..... i t. d.

$$\int_0^\infty \frac{x^{\frac{1}{6}}\, dx}{(x+1)(2+x)} = \frac{e^{\frac{i\pi}{6}} 2\pi i}{-1 + e^{\frac{2\pi i}{6}}} \left(2^{\frac{1}{6}} - 1\right) = \frac{2\pi i \left(2^{\frac{1}{6}} - 1\right)}{-e^{-\frac{i\pi}{6}} + e^{\frac{i\pi}{6}}} = \frac{\pi \cdot \left(2^{\frac{1}{6}} - 1\right)}{\sin \frac{\pi}{6}} = 2\pi \cdot \left(2^{\frac{1}{6}} - 1\right).$$

Sfera Riemanna:

$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Mówimy, że $z_n \to \infty$ jeśli $\forall M > 0\ \exists N : n > N : |z_n| > M$.

Otoczenia $\infty \in \overline{\mathbb{C}}$: zbiory postaci $\{ z : |z| > M \} \cup \{\infty\}$.

Rzut stereograficzny: np.