Conformal actions, Kummer tables and hypergeometric-type functions

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How the action of the symmetry group SO(6,C) and the choice of a nice set of parameters help one to understand and present logically a whole bunch of special functions.

History: early beginnings

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Wikipedia™ says

The term ,,hypergeometric series" was first used by **John Wallis** in his 1655 book *Arithmetica Infinitorum*. Hypergeometric series were studied by **Leonhard Euler**, but the first full systematic treatment was given by **Carl Friedrich Gauss (1813)**. Studies in the nineteenth century included those of **Ernst Kummer (1836)**, and the fundamental characterisation by **Bernhard Riemann** of the hypergeometric function by means of the differential equation it satisfies. Riemann showed that the second-order differential equation for ${}_2F_1(z)$, examined in the complex plane, could be characterised (on the Riemann sphere) by its three regular singularities.

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Beware

I will try to follow the historical path on which the subject was being discovered.

Euler Gamma function

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- Euler integral of the first kind

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$$\frac{1}{\Gamma(z)}=z\mathrm{e}^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\mathrm{e}^{-\frac{z}{n}} \tag{3}$$

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- $$\frac{1}{\Gamma(z)}=\frac{\sin \pi z}{\pi}\Gamma(1-z) \tag{4}$$

Hypergeometric series

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Hypergeometric series of type ${}_2F_1(a, b; c; z)$

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!} \tag{5} $$

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Pochhammer's symbol

$$ (a)_n=\prod_{k=0}^{n-1}(a+k)=a(a+1)\cdot\ldots\cdot(a+n-1) \tag{6} $$

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Examples

$$ (1)_n = n! \qquad (-k)_n = 0, \quad n \geq k \qquad (c)_n = \frac{\Gamma(c+n)}{\Gamma(c)} \tag{7} $$

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- Due to problems with the denominator one should introduce

$$ {}_2\mathbf{F}_1(a, b; c; z) = \frac{{}_2F_1(a, b; c; z)}{\Gamma(c)} = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{\Gamma(c+n) n!} \tag{6} $$

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- In fact, it is a subclass of the confluent function ${}_1F_1$
- It is classically known as the modified Bessel function,

$$ I_\alpha(w) = \left(\frac{w}{2}\right)^\alpha {}_0\mathbf{F}_1\left(\alpha+1; \frac{w^2}{4}\right) \tag{9} $$

Hypergeometric equation

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Hypergeometric operator $\mathcal{F}(a, b; c; z, \partial_z)$

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- Parameters a, b solve the index equation for $z=\infty$. The indices at 0 and 1 are respectively $1-c$ and $c-a-b$

A nightmarish zoo from Abramowitz & Stegun

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15.2.10

$$(c-a)F(a-1,b;c;z)+(2a-c-az+bz)F(a,b;c;z)+a(z-1)F(a+1,b;c;z)=0$$

15.2.11

$$(c-b)F(a,b-1;c;z)+(2b-c-bz+az)F(a,b;c;z)+b(z-1)F(a,b+1;c;z)=0$$

15.2.12

$$c(c-1)(z-1)F(a,b;c-1;z)+c[c-1-(2c-a-b-1)z]F(a,b;c;z)+(c-a)(c-b)zF(a,b;c+1;z)=0$$

15.2.13

$$[c-2a-(b-a)z]F(a,b;c;z)+a(1-z)F(a+1,b;c;z)-(c-a)F(a-1,b;c;z)=0$$

15.2.14

$$(b-a)F(a,b;c;z)+aF(a+1,b;c;z)-bF(a,b+1;c;z)=0$$

15.2.15

$$(c-a-b)F(a,b;c;z)+a(1-z)F(a+1,b;c;z)-(c-b)F(a,b-1;c;z)=0$$

15.2.16

$$[c-2b+(b-a)z]F(a,b;c;z)+b(1-z)F(a,b+1;c;z)-(c-b)F(a,b-1;c;z)=0$$

15.2.23

$$c[b-(c-a)z]F(a,b;c;z)-bc(1-z)F(a,b+1;c;z)+(c-a)(c-b)zF(a,b;c+1;z)=0$$

15.2.24

$$(c-b-1)F(a,b;c;z)+bF(a,b+1;c;z)-(c-1)F(a,b;c-1;z)=0$$

15.2.25

$$c(1-z)F(a,b;c;z)-cF(a,b-1;c;z)+(c-a)zF(a,b;c+1;z)=0$$

15.2.26

$$[b-1-(c-a-1)z]F(a,b;c;z)+(c-b)F(a,b-1;c;z)-(c-1)(1-z)F(a,b;c-1;z)=0$$

15.2.27

$$c[c-1-(2c-a-b-1)z]F(a,b;c;z)+(c-a)(c-b)zF(a,b;c+1;z)-c(c-1)(1-z)F(a,b;c-1;z)=0$$

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$$a = \frac{1}{2}(1 + \alpha + \beta + \mu) \quad b = \frac{1}{2}(1 + \alpha + \beta - \mu) \quad c = 1 + \alpha \tag{11}$$

Hypergeometric operator $\mathcal{F}_{\alpha,\beta,\mu}(z,\partial_z)$

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$$\mathcal{F}_{\alpha,\beta,\mu} = z(1-z)\,\partial_z{}^2 + ((1+\alpha)(1-z) - (1+\beta)z)\;\partial_z - \frac{(1+\alpha+\beta)^2 - \mu^2}{4} \tag{12}$$

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- We will see that the parameters really do have Lie-algebraic interpretation

$\mathcal{F}_{\alpha,\beta,\mu}(z, \partial_z)$ in balanced form

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Balanced form and Schrödinger form

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Balanced form and Schrödinger form

It is known that any ODE of our type can be shown in the form of a Schrödinger operator by simple substitutions. Similarily a so-called balanced form can always be obtained

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Canonical form of the hypergeometric case

$$\mathcal{F}_{\alpha,\beta,\mu}(z,\,\partial_z) =$$

$$= z^{-\alpha}(1-z)^{-\beta}\,\partial_z z^{\alpha+1}(1-z)^{\beta+1}\,\partial_z - \frac{(1+\alpha+\beta)^2-\mu^2}{4} \tag{13}$$

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Balanced form of the hypergeometric operator

$$z^{\frac{\alpha}{2}}(1-z)^{\frac{\beta}{2}}\mathcal{F}_{\alpha,\beta,\mu}(z,\,\partial_z)\,z^{-\frac{\alpha}{2}}(1-z)^{-\frac{\beta}{2}} = \\ = \partial_z z(1-z)\,\partial_z - \frac{\alpha^2}{4z} - \frac{\beta^2}{4(1-z)} - \frac{1-\mu^2}{4} \tag{13}$$

$\mathcal{F}_{\alpha,\beta,\mu}(z, \partial_z)$ in balanced form

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Notice!

The symmetries are becoming **even** more v-i-s-i-b-l-e.

A less nightmarish zoo from JD – Kummer table

$w=z:$			
		$\mathcal{F}_{\alpha,\beta,\mu}(z,\partial_z),$	
	$(-z)^{-\alpha}(z-1)^{-\beta}$	$\mathcal{F}_{-\alpha,-\beta,\mu}(z,\partial_z)$	$(-z)^{\alpha}(z-1)^{\beta}$
	$(z-1)^{-\beta}$	$\mathcal{F}_{\alpha,-\beta,-\mu}(z,\partial_z)$	$(z-1)^{\beta},$
	$(-z)^{-\alpha}$	$\mathcal{F}_{-\alpha,\beta,-\mu}(z,\partial_z)$	$(-z)^{\alpha};$
$w=1-z:$			
		$\mathcal{F}_{\beta,\alpha,\mu}(z,\partial_z),$	
	$(z-1)^{-\alpha}(-z)^{-\beta}$	$\mathcal{F}_{-\beta,-\alpha,\mu}(z,\partial_z)$	$(z-1)^{\alpha}(-z)^{\beta},$
	$(z-1)^{-\alpha}$	$\mathcal{F}_{\beta,-\alpha,-\mu}(z,\partial_z)$	$(z-1)^{\alpha},$
	$(-z)^{-\beta}$	$\mathcal{F}_{-\beta,\alpha,-\mu}(z,\partial_z)$	$(-z)^{\beta};$
$w=\frac{1}{z}:$			
	$(-z)^{\frac{1}{2}(\alpha+\beta+\mu+1)}$	$(-z)\mathcal{F}_{\mu,\beta,\alpha}(z,\partial_z)$	$(-z)^{\frac{1}{2}(-\alpha-\beta-\mu-1)},$
	$(-z)^{\frac{1}{2}(\alpha+\beta-\mu+1)}(z-1)^{-\beta}$	$(-z)\mathcal{F}_{-\mu,-\beta,\alpha}(z,\partial_z)$	$(-z)^{\frac{1}{2}(-\alpha-\beta+\mu-1)}(z-1)^{\beta},$
	$(-z)^{\frac{1}{2}(\alpha+\beta+\mu+1)}(z-1)^{-\beta}$	$(-z)\mathcal{F}_{\mu,-\beta,-\alpha}(z,\partial_z)$	$(-z)^{\frac{1}{2}(-\alpha-\beta-\mu-1)}(z-1)^{\beta},$
	$(-z)^{\frac{1}{2}(\alpha+\beta-\mu+1)}$	$(-z)\mathcal{F}_{-\mu,\beta,-\alpha}(z,\partial_z)$	$(-z)^{\frac{1}{2}(-\alpha-\beta+\mu-1)};$
$w=1-\frac{1}{z}:$			
	$(-z)^{\frac{1}{2}(\alpha+\beta+\mu+1)}$	$(-z)\mathcal{F}_{\mu,\alpha,\beta}(z,\partial_z)$	$(-z)^{\frac{1}{2}(-\alpha-\beta-\mu-1)},$
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$w=\frac{1}{1-z}:$			
	$(z-1)^{\frac{1}{2}(\alpha+\beta+\mu+1)}$	$(z-1)\mathcal{F}_{\beta,\mu,\alpha}(z,\partial_z)$	$(z-1)^{\frac{1}{2}(-\alpha-\beta-\mu-1)},$
	$(-z)^{-\beta}(z-1)^{\frac{1}{2}(\alpha+\beta-\mu+1)}$	$(z-1)\mathcal{F}_{-\beta,-\mu,\alpha}(z,\partial_z)$	$(-z)^{\beta}(z-1)^{\frac{1}{2}(-\alpha-\beta+\mu-1)},$
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$w=\frac{z}{z-1}:$			
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- $$h_1(z) = 1 - z \tag{14}$$
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These are enough to generate all possible permutations of the three singular points! There are six of those.

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$$\mathcal{F}_{\alpha,\beta,\mu}(z,\,\partial_z) = (1-z)^{-\beta}\mathcal{F}_{\alpha,-\beta,-\mu}(z,\,\partial_z)(1-z)^{\beta} \tag{17}$$

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- Three simple building blocks are enough for all of that!

A less nightmarish zoo from JD – Kummer table – again

$$
\begin{array}{llll}
w = z: & & & \\
& & \mathcal{F}_{\alpha,\beta,\mu}(z,\partial_z), & \\
& (-z)^{-\alpha}(z-1)^{-\beta} & \mathcal{F}_{-\alpha,-\beta,\mu}(z,\partial_z) & (-z)^{\alpha}(z-1)^{\beta} \\
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w = 1-z: & & & \\
& & \mathcal{F}_{\beta,\alpha,\mu}(z,\partial_z), & \\
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w = 1-\frac{1}{z}: & & & \\
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w = \frac{1}{1-z}: & & & \\
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Try to multiply

It is now clear that $2^3 \cdot 3! = 8 \cdot 2 \cdot 3 = 2 \cdot 24$ is the order of the Weyl group for $SO(6, \mathbb{C})$. We have briefly described the action of the discrete Weyl group on hypergeometric operators. Let us discover quickly how the whole group acts.

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It is not obvious that one can also ,,push forward" the Laplace operator. In fact it descends to the quotient space when one considers homogeneous functions of order $1-\frac{n}{2}$ on the quadric.

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- ...and three Cartan operators

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In this coordinates the bilinear form is $Q(\vec{z}) = r^2 + p^2$. Therefore the reduction to the quadric will be given by $p = \mathrm{i}r$.

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The four dimensional part with respect to coordinates w, u_1, u_2, u_3 gives the hypergeometric equation provided one makes a certain ansatz (roughly – the details of the reduction, even though most interesting, have been skipped for simplicity of this presentation)!

$$F(w,u_1,u_2,u_3) = u_1^{\alpha}\,u_2^{\beta}\,u_3^{\mu}\,F(w) \qquad (23)$$

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Further we will frequently use those operators as the hypergeometric functions are their eigenvectors.

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Note how obvious and ellegant the action of the Weyl group of $SO(6,\mathbb{C})$ looks using the above shown forms!

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The symmetries and relations for hypergeometric-type operators are visible here in the simplest way. Also this algebra can easily generate all possible formulae which arise through that construction! Please remember that the change of 1 into 2 involves the change of w into $1-w$ and that is all!

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- The solution was involving hypergeometric-type functions
- We looked at the geometric interpretation of the six exactly solvable Schrödinger potentials (by means of hypergeometric functions)
- That gave us interpretation, that Schrödinger operators, which are also special cases of the hypergeometric operator, are in fact Laplace-Beltrami operators on certain 3d surfaces embedded in 4d real subspaces of $\mathbb{C}^4$

A short outline of my PhD's work

- We have studied Gell-Mann – Low adiabatic limit approach to scattering, and calculated **rigorously** a time-ordered exponential for a **non-commutative** family of operators
- The solution was involving hypergeometric-type functions
- We looked at the geometric interpretation of the six exactly solvable Schrödinger potentials (by means of hypergeometric functions)
- That gave us interpretation, that Schrödinger operators, which are also special cases of the hypergeometric operator, are in fact Laplace-Beltrami operators on certain 3d surfaces embedded in 4d real subspaces of $\mathbb{C}^4$
- Jan introduced me to the global picture of the Lie group of symmetries for the hypergeometric equation which I liked very much and which finally helped me to understand the real beauty of the great plurality of possibilities.

A little about literature

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We do it the best way!

There is a great lot of literature on this subject. It would not be wise to present a long list here. For a condensed, full presentation, involving Lie groups of symmetries, I will show references to works of Jan Dereziński and recently also PM.

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- *M. Abramowitz, I. Stegun*, Handbook of Mathematical Fucntions, multiple editions, i.e. tenth printing, December 1972

Thank you for your attention!

It was a pleasure to give a talk to such an audience. ☺