The origin of $\rho_0 = 1$ in the SM

1) Higgs doublet model

2) generalizations

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$$

$$D_\mu H = \left[\partial_\mu + i g W_\mu + i g' B_\mu \right] H$$

$$W_\mu = W_\mu^a T_L^a \quad , \quad B_\mu = B_\mu Y$$

$$W_\mu' = U_L \left[W_\mu + \partial_\mu \right] U_L^{-1}$$

$$U_L = e^{-i\theta^a(x) T_L^a}$$

Under global $SU_L(2)$:

$$W_\mu' = U_L W_\mu U_L^{-1}$$

$$U_L = e^{-i\theta^a T^a}$$

$$B_\mu' = e^{-i\theta Y} B_\mu e^{i\theta Y}$$; singlet under $SU_2(L)$

The Higgs Lagrangian

$$(D_\mu H)^\dagger (D^\mu H) - V(H^\dagger H)$$

$$V(H^\dagger H) = m^2 H^\dagger H + \frac{\lambda}{2}(H^\dagger H)^2$$

For $m^2 < 0$, the minimum of the potential is at

$$\langle H^\dagger H \rangle = -\frac{m^2}{\lambda} \equiv v^2$$

By $SU_L(2)$ rotation we can always redefine the vacuum so that only the vev of the lower component of the Higgs doublet is non-vanishing. The global $SU_L(2) \times U(1)$ symmetry is broken down to $U_{EM}(1)$, if $Q = T^3 + Y$

Indeed, for $\Omega = \begin{pmatrix} 0 \\ v \end{pmatrix}$,

$T^a \Omega \neq 0$ for $a = 1, 2, 3$

$(T^3 + Y)\Omega = 0$, $Y = \frac{1}{2}$ for H

so the vacuum is invariant under $U_{EM}(1)$ rotations

By analogy with the already discussed $U(1)$ case, we can use the "angular" variables for the Higgs field

$$H(x) = \frac{1}{\sqrt{2}} \exp\left[i \frac{\eta^a(x)}{v} T^a \right] \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

It is clear that the fields η^a are Goldstone bosons because they drop out from the Higgs potential.

Choosing the gauge rotations *)

$\theta_L^a(x) = -\eta^a(x)/v$ we get

rid of the three Goldstone bosons,

and the three degrees of freedom

are exchanged for the longitudinal

components of the gauge bosons

*) unitary gauge

Origin of W, Z masses: vacuum

$$\Omega = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

The condition that the photon remains massless is equivalent to the vacuum being electrically neutral:

$$Q\,|\Omega\rangle = (T_L^3 + Y)\,|\Omega\rangle = 0$$

satisfied, if the hypercharge $H,\ Y = \frac{1}{2}$

We rewrite the Higgs kinetic terms $(D_\mu H)^\dagger (D_\mu H)$ in terms of the defined previously W^+, W^-, Z, A fields

(unitary gauge)

Interaction with $v\Omega$ gives **vector boson masses**

$$M_W^2 W^+_\mu W^-_\mu = g^2 \, |T_L^+ v\Omega|^2 \, W^+_\mu W^{-\mu}$$

$$\Rightarrow \quad M_W^2 = \frac{1}{2} g^2 v^2$$

$$\frac{1}{2} M_Z^2 Z_\mu Z^\mu = \left| \left[g \cos\theta_W T^3_L - g' \sin\theta_W Y_H \right] \Omega \right|^2 Z_\mu Z^\mu$$

because it's neutral

$Q|\Omega\rangle = 0$

$$= (g\cos\theta_W + g'\sin\theta_W)^2 \left| T^3_L \Omega \right|^2$$

$$\Rightarrow \quad M_Z^2 = \frac{1}{2} g^2 \frac{v^2}{\cos^2\theta_W}$$

$$\rho_0 = 1 \; !$$

Why $\rho_0 = 1$ in this case?

1) $Q|\Omega\rangle = 0$

2) $|T_L^1|\Omega\rangle|^2 = |T_L^2|\Omega\rangle|^2 = |T_L^3|\Omega\rangle|^2 = \frac{1}{4}v^2$

$$T_L^{\pm} = T_L^1 \pm T_L^2$$

Symmetry reason for 2 : imagine $SU_V(2)$ symmetry with T^a, such that

$[T_V^a, T_L^k] = i\epsilon_{akl} T_L^l$ $\quad (W_\mu \to U_V W_\mu U_V^{-1})$

$$[T_V^a, T_L^k]\,|\Omega\rangle = i\epsilon_{akl}\, T_L^l\, |\Omega\rangle$$

$$T_V^a\left(T_L^k|\Omega\rangle\right) - \underbrace{T_L^k\left(T_V^a|\Omega\rangle\right)}_{\text{if } T_V^a|\Omega\rangle = 0} = i\epsilon_{akl}\, T_L^l\, |\Omega\rangle$$

then $\left|T_L^k|\Omega\rangle\right|^2 = \left|T_L^l|\Omega\rangle\right|^2$

We need a vacuum that is invariant under some SU(2) symmetry; then $\rho_0 = 1$.

Before showing that the vacuum of the Higgs doublet is invariant under some $SU_V(2)$ (called custodial symmetry of the Higgs potential) let's see what happens for other Higgs representations, with Ω_i vacua

We assume, again,

$$Q|\Omega_i\rangle = \left(T^3 + Y\right)|\Omega_i\rangle = 0$$

Thus, the result for M_Z^2 is similar

$$\frac{1}{2} M_Z^2 = \frac{g^2}{\cos^2\theta_W} \sum_i |t_i^3 v_i|^2$$

where $\Omega_i = \begin{pmatrix} 0 \\ \vdots \\ v_i \end{pmatrix}$ is neutral $(Q = 0)$

But

$$M_W^2 W_\mu^+ W_\mu^- = \sum_i \frac{g^2}{2} \left(T_i^+ T_i^- + T_i^- T_i^+\right) \Omega_i^i$$

$$T^\pm T^\mp = \vec{T}^2 - T_3^2 \mp T^3$$

$$M_W^2 = g^2 \sum_i \left[T_i(T_i+1) - T_{3i}^2\right] v_i^2$$

$$\rho_0 = \frac{\sum_i \left[t_i(t_i+1) - (t_i^3)^2 \right] v_i^2}{2 \sum_i (t_i^3)^2 v_i^2}$$

Origin of $SU_V(2)$ in the Higgs doublet model:

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \qquad H^c = i\sigma_2 H^* = \begin{pmatrix} H_0^* \\ -H^- \end{pmatrix}$$

Define $$\Phi = \begin{pmatrix} H_0^* & H^+ \\ -H^- & H^0 \end{pmatrix}$$

The Higgs Lagrangian

$$\partial_\mu \phi\, \partial^\mu \phi^\dagger + \frac{1}{2}\lambda\left[\left(\frac{1}{2}\mathrm{Tr}\,\phi^\dagger \phi\right) - v^2\right]^2$$

has $SU_L(2) \times SU_R(2)$ global symmetry, defined by transforming

$$\phi' = U_L \phi U_R^\dagger \,, \quad U_{L,R} = e^{-i\theta^a_{L,R} T^a_{L,R}}$$

We want to show that the vacuum $\langle\Omega\rangle = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$ breaks it to $SU_V(2)$

Of course, for a scalar field T_L and T_R are Pauli matrices.

If chiral fermions are included then

$$T_L^a = T^a \times P_L \qquad P_L = \frac{1}{2}(1-\gamma_5)$$

$$T_R^a = T^a \times P_R \qquad P_R = \frac{1}{2}(1+\gamma_5)$$

and we can define vector and axial vector generators

$$T_V^a = \frac{1}{2}(T_L^a + T_R^a), \quad T_A^a = \frac{1}{2}(T_R^a - T_L^a)$$

Transformation properties

$$[T_V^a, T_V^b] = [T_A^a, T_A^b] = i\epsilon^{abc} T_V^c$$

$$[T_V^a, T_A^b] = i\epsilon^{abc} T_A^c$$

$$[T_{L,R}^a, T_{L,R}^b] = i\epsilon^{abc} T_{L,R}^c$$

If only scalars are present and no γ_5 involved, all of them are Pauli matrices

For infinitesimal transformations

$$\delta\phi = -i\,\theta_L^a T_L^a \phi + i\,\phi\,\theta_R^a T_a^R$$

We can also write ϕ as

$$\phi = \sigma + i\,T^a \tilde{\pi}^a$$

We can introduce

$$\theta^a = (\theta_R^a + \theta_L^a)/2$$

$$\theta_5^a = (\theta_R^a - \theta_L^a)/2$$

One can check the following transformations

$$\delta\sigma = \frac{1}{2}\theta_5^a \pi^a$$

$$\delta\pi^a = -\epsilon_{abc}\theta^b \pi^c - \theta_5^a \sigma$$

$$\left(H^0 = \sigma + i\pi^3 \right)$$

$$(H^0)' = H^0 - \frac{i}{2}\left[\theta_5^3 H^0 + \theta_R^+ H^- + \theta_L^- H^+\right]$$

$$(H^+)' = H^+ - \frac{i}{2}\left[\theta^3 H^+ - \theta_R^+ H^0 + \theta_L^+ H^{0*}\right]$$

We defined $\Theta^{\pm}_{L,R} = \Theta^{1}_{L,R} \pm i\Theta^{2}_{L,R}$

We see that for $\Theta^{a}_{L} = \Theta^{a}_{R} = \frac{1}{2}\Theta^{a}$

the vacuum $\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$ $\left(H_0 = v \right)$

remains invariant $\Rightarrow SU_L(2) \times SU_R(2) \to SU_V(2)$

We also see that the T^3 generator of $SU_V(2)$ should be interpreted as the generator of $U(1)_{EM}$ because only $H^{\pm}$ transform under such rotations (we put $\Theta^{+}_{R} = \Theta^{-}_{L} = 0$)

Indeed, for $\theta^a_L = \theta^a_R = \frac{1}{2}\theta^a$

$$\Omega' = U \, \Omega \, U^{-1} = \Omega \, , \quad U = e^{-i\theta^a T^a}$$

By the way, since $Q = T^3_L + Y$ and now $Q = T^L_3 + T^R_3$, we see that we have an identification

$$Y = T^3_R$$

Identification of Goldstone bosons

Again

$$\phi = \sigma + i T^a \pi^a$$

$$\delta \sigma = \frac{1}{2} \theta_5^a \pi^a$$

$$\delta \pi_a = - \epsilon_{abc} \theta^b \pi^c - \theta_5^a \sigma$$

(σ, π^a) transform into themselves, of course

Potencjał wybraliśmy tak, żeby symetria $SU_L(2) \times SU_R(2)$ była spontanicznie naruszona, gdyż minimum potencjału jest dla

$$\sigma^2 + \pi_a^2 = F_\pi^2 \equiv v^2$$

Zawsze możemy tak obrócić pola, żeby

$$\langle \sigma \rangle = F_\pi \quad , \quad \langle \pi_a \rangle = 0$$

Definiujemy pola fizyczne

$$\sigma' = \sigma - F_\pi \quad , \quad \langle \sigma' \rangle = 0$$

we can calculate to get

$$\mathcal{L} \sim \frac{1}{2}\partial_\mu \sigma' \partial^\mu \sigma' + \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a$$

$$- \frac{\lambda}{4}\left(\sigma'^2 + \pi_a^2 + 2v\sigma'\right)^2$$

π^a 's are massless (Goldstone bosons)

since $\phi = \sigma + i\pi^a T^a$, it is clear

that π^a transform as a triplet of

the unbroken $SU_V(2)$

We can calculate the current

$$j^{\mu}_{\phi} = \frac{1}{2}\left(\phi^{*}\partial_{\mu}\phi - \phi\,\partial_{\mu}\phi^{*}\right)$$

in terms of (σ, π^{a})

It contains the piece

$$j^{\mu}_{5a} \equiv (\partial^{\mu}\pi_{a})\sigma - (\partial^{\mu}\sigma)\pi^{a}$$

and for $\sigma = v + \sigma'$ we get

$$j^{\mu}_{5a} = v\,\partial^{\mu}\pi^{a}$$

We can do it in an even simpler way by using the non-linear parametrization for the field Φ:

$$\Phi = e^{i\pi_a T^a} \begin{pmatrix} \sigma + v & 0 \\ 0 & \sigma + v \end{pmatrix}$$

It is clear that the π^a's are 3 Goldstone bosons because they don't enter into the potential

$$V = m^2 \Phi^\dagger \Phi + \frac{\lambda}{2} (\Phi^\dagger \Phi)^2, \quad v = -\frac{m^2}{\lambda}$$

We calculate the current

$$\frac{1}{2}\left(\phi\,\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right) = i\,\partial_\mu\pi^a T^a \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$$

$$= i v\,\partial_\mu \pi^a T^a$$

Hence $j^a_\mu = i v\,\partial_\mu \pi^a$

SU(2) invariant

From now on, we can exactly follow the discussion as for the U(1) case