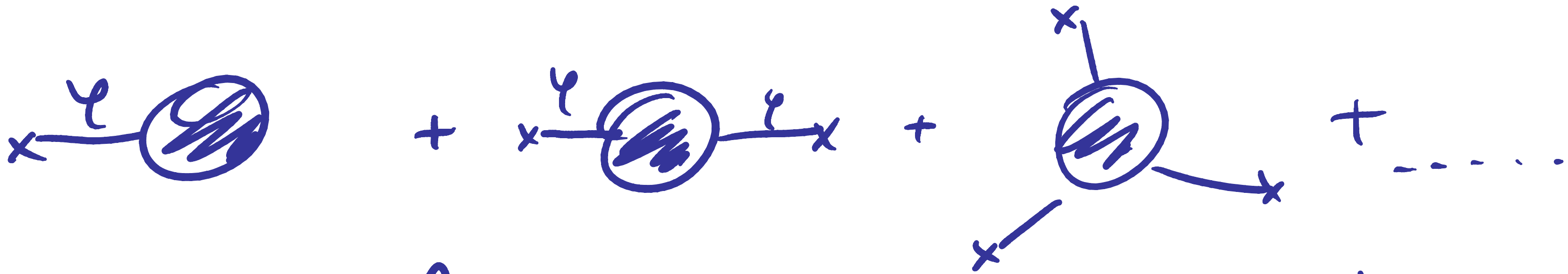
LHCphysics7

Effective potential

Define $V(\varphi)$ as the expectation value of the vacuum energy density when the expectation value of the field Φ is φ. Clearly

$$\frac{dV}{d\varphi} = 0$$ gives us the "minimal" vev of Φ: $\varphi = v$, i.e. the ground state

From its definition, $V(\varphi)$ is given by the sum of diagrams



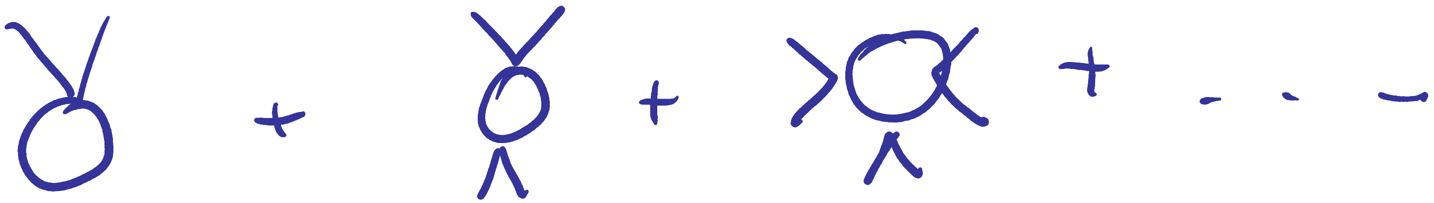
for vanishing external momenta and the external legs denoting the field value φ.

For a scalar theory

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 + \left(\frac{\lambda}{4!}\right) \varphi^4$$

at the tree level

At one loop level, one gets the sum of diagrams:



So,
$$V_1(\varphi) = i \sum_{n=1}^{\infty} \frac{1}{2n} \int \frac{d^4k}{(2\pi)^4} \left[\lambda \frac{1}{k^2 - m^2 + i\varepsilon} \frac{\varphi^2}{2} \right]^n$$

$$= -\frac{1}{2} i \int \frac{d^4k}{(2\pi)^4} \ln\left(1 - \frac{\frac{1}{2}\lambda\varphi^2}{k^2 - m^2 + i\varepsilon}\right) =$$

$dk_0 = i\, dk_4$ and Wick rotation

$$= \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_4}{2\pi} \ln\left(1 + \frac{\frac{1}{2}\lambda\varphi^2}{k_4^2 + \vec{k}^2 + m^2}\right)$$

$$= \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d_4 k}{2\pi} \ln\left(\frac{k_4^2 + \vec{k}^2 + V_0''(\varphi)}{k_4^2 + \vec{k}^2 + m^2}\right)$$

$$= \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left\{ k_4 \ln(\ldots)\Big|_{-\infty}^{+\infty} - 2 \int_{-\infty}^{\infty} \frac{dk_4}{2\pi} \, * \right.$$

$$\left. * \left(\frac{k_4^2}{k_4^2 + k^2 + V_0''(\varphi)} - \frac{k_4^2}{k_4^2 + \vec{k}^2} \right) \right\} =$$

$$= \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_4}{2\pi} \left(\frac{\vec{k}^2 + V''}{k_4^2 + \vec{k}^2 + V''} - \frac{\vec{k}^2}{k_4^2 + \vec{k}^2} \right)$$

$$= \frac{1}{2} \underbrace{\int \frac{d^3k}{(2\pi)^3} \sqrt{\vec{k}^2 + \underbrace{V''(\varphi)}_{m^2 + \frac{1}{2}\lambda\varphi^2}}}_{\text{energy fluctuations around the classical field}}$$

Another form

$$V_1(\varphi) = \frac{1}{2}\int \frac{d^4 k_E}{(2\pi)^4} \ln\left(\frac{k_E^2 + V_0''(\varphi)}{k_E^2 + m^2}\right)$$

$$= \frac{1}{2}\,\frac{1}{(4\pi)^2}\int_0^{\Lambda^2} dk^2\, k^2 \ln\left(\frac{k^2 + V_0''(\varphi)}{k^2 + m^2}\right)$$

we have regularised the integral by a cut-off Λ

$$V_1(\varphi) = \frac{1}{2(4\pi)^2}\left\{\Lambda^2 V_0''(\varphi) + \frac{1}{2}\left[V_0''(\varphi)\right]^2\left(\ln\frac{V_0''(\varphi)}{\Lambda^2} - \frac{1}{2}\right)\right\}$$

$+$

$$+\underbrace{\frac{1}{2}\Lambda^4 \ln \Lambda^2 - \frac{1}{4}\Lambda^4}_{\text{"cosmological constant"}}$$

often used notation

$$V_0''(\varphi) \equiv M^2(\varphi)$$

masses, including the contribution from the vevs

$$V_{eff} = V_0(\varphi) + \Lambda^2 M^2(\varphi) + \frac{1}{2} M^4(\varphi) \, * $$

$$* \left(\ln \frac{M^2(\varphi)}{\Lambda^2} - \frac{1}{2} \right)$$

for $V_0 = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4$

we get

$$V_1(\varphi) \cong \Lambda^2 \left(m^2 + \frac{1}{2}\lambda \varphi^2\right) - \frac{1}{2}\left(m^2 + \frac{1}{2}\lambda\varphi^2\right)^2 \ln \Lambda^2$$

$$+ \text{ finite terms}$$

$$\delta m^2 = \frac{d^2 V_1}{d\varphi^2} = \underbrace{\Lambda^2 \lambda}_{\text{quadratic divergence}} + \ldots$$

One can, of course, also use dimensional regularisation and $\overline{MS}$ to calculate V_{eff}. One gets then

$$V_{eff} = V_0(\varphi) + \frac{1}{4(4\pi)^2} M^4(\varphi)\left[\ln\frac{M^2(\varphi)}{4\pi\mu^2} - \frac{3}{2}\right]$$

μ^2 - renormalisation scale

Suppose, there are several fields that couple to Φ (as in SM: W, Z, fermions).

Then the result is

$$V_{eff}(\varphi) = V(\varphi) + \frac{1}{64\pi^2}\sum_J (-1)^{2J}(2J+1)\mathcal{M}_J^4(\varphi)\left[\ln\frac{\mathcal{M}_J^2(\varphi)}{4\pi\mu^2} - \frac{3}{2}\right]$$

Our old example with two scalar fields: φ, Φ, $\langle\Phi\rangle = 0$, $\langle\varphi\rangle \neq 0$

$m_\varphi \ll M_\Phi$

One has

$$(m^2+\lambda_1\varphi^2)^4 \ln\frac{m^2+\lambda_1\varphi^2}{\mu^2} +$$

$$+(M^2+\lambda_2\varphi^2)^4 \ln\frac{M^2+\lambda_2\varphi^2}{\mu^2} =$$

$$=\left(2m^2\lambda_1\varphi^2+m^4+\lambda_1^2\varphi^4\right)\left(\ln\frac{m^2}{\mu^2}+\ln\left(1+\frac{\lambda_1\varphi^2}{m^2}\right)\right)$$

$$+\left(2M^2\lambda_2\varphi^2+M^4+\lambda_2^2\varphi^4\right)\left(\ln\frac{M^2}{\mu^2}+\ln\left(1+\frac{\lambda_2\varphi^2}{M^2}\right)\right)$$

We can discuss the hierarchy problem in the presence of two or three scales (as in the previous lecture).

We can discuss the "little hierarchy" problem in supersymmetric models.

Effective potential - a compact and convenient tool.

Vacuum stability in the SM :

$$V_0 = m^2 (H^+ H) + \frac{\lambda}{2} (H^+ H)^2$$

$$\langle H^+ H \rangle_{min} = -\frac{m^2}{\lambda} \equiv \frac{v^2}{2}$$

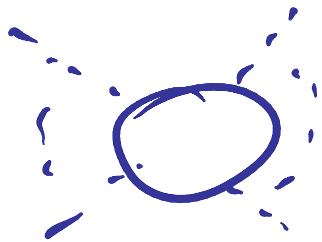
What happens after inclusion of 1-loop quantum corrections ? Calculate $V_{eff}(H)$ remembering that Yukawa couplings $Y_i \bar{Q}_i u_i H$ give quark masses $m_q = m_q(v) = Y_q v$

and similarly $M_W^2 = \frac{1}{2} g^2 v^2$

$$M_Z^2 = \frac{1}{2}(g^2 + g'^2)\, v^2$$

$v \sim \langle H \rangle$ (our φ from earlier pages)

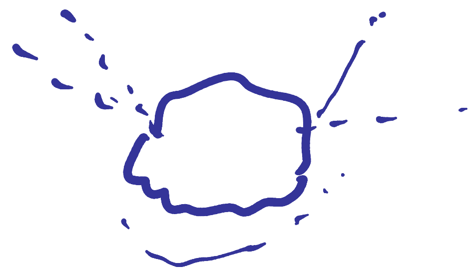
Fermion loops



give

$$V_1^{\text{fermion}} = -\frac{3\, y^4}{64\pi^2}\,(H^\dagger H)^2 \ln \frac{H^\dagger H}{\mu^2}$$

Vector boson loops



$$V_1^{vector} = \frac{3}{16}\,\frac{\left[2g^4 + (g^2+g_1^2)^2\right]}{64\pi^2}\,(H^\dagger H)^2 \ln\frac{H^\dagger H}{\mu^2}$$

and

$$V_1^{scalar} = \frac{1}{64\pi^2}\left(m^2 + \lambda H^\dagger H\right)^2 \ln\frac{H^\dagger H}{\mu^2}$$

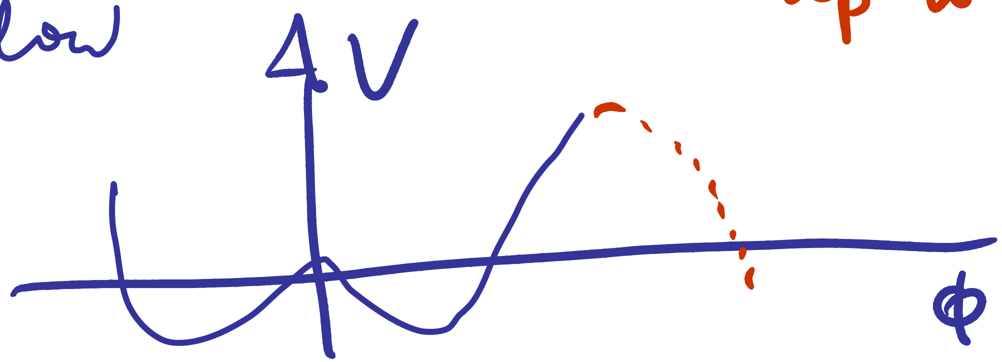
(Goldstone bosons do not contribute)

Together,

$$V_{eff} = V_0 + V_{scal} + \frac{1}{64\pi^2} B \varphi^4 \ln \frac{\varphi^2}{\mu^2}$$

$$B = \frac{1}{\varphi^4}\left[6 M_W^4 + 3 M_Z^4 - 12 m_t^4 \right]$$

The potential must be bounded from below up to some required values of Φ (the maximal scale of validity of the theory)



For $\mu^2 = \phi^2$ (our choice) we get $M_h^2 \gtrsim 12\, m_t^4 - 6\, M_W^4 - 3\, M_Z^4$

We get a lower bound on the Higgs boson mass.

For a precise calculation, we need a renormalisation group calculation (recent paper by Rychkov et al)

$M_h \gtrsim 110$ GeV