Classical field theory 2026 – homework

Problem 1

- From the definition of variation $\delta\phi$ of a field $\phi = \phi(x)$, show that: $\delta(\phi^2) = 2\phi\delta\phi,\ \delta(\sin\phi) = \cos(\phi)\delta\phi$.
- Show that variation $\delta\phi$ is a derivation (i.e. it is a linear operation and satisfies Leibniz rule), and it commutes with differentiation w.r.t. spacetime coordinates: $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$.

Problem 2

In four-dimensional Minkowski space with a diagonal metric of the form $(g_{\mu\nu})_{\mu,\nu=0,\dots,3} = \mathrm{diag}(1,-1,-1,-1)$, find components of the tensor $F_{\mu\nu}$, if the components of the (anti-symmetric) field strength tensor $F^{\mu\nu}$ take form $F^{i0} = E_i$ and $F^{ij} = -\epsilon_{ijk}B_k$, where $i,j,k = 1,2,3$ and E_i and B_j are components of electric and magnetic vector fields respectively, $\vec{E} = (E_1,E_2,E_3)$ and $\vec{B} = (B_1,B_2,B_3)$.

Problem 3

For appropriate values of indices $\lambda,\mu,\nu \in \{0,1,2,3\}$, Bianchi identity $\partial_{[\lambda}F_{\mu\nu]} = 0$ reproduces two of the Maxwell equations. Show that for all other values of indices there is no new information encoded in this identity.

Problem 4

Show that the general solution of Maxwell equations in $A_0(x) = 0$ gauge can be written as $A_\mu(x) = A^\perp_\mu(x) + B_\mu(\bar{x})$, where $B_0 = 0$ and $B_i(\bar{x}) = \partial_i\alpha(\bar{x})$.

Problem 5

Show that the equation $\partial_\mu F^{\mu\nu} = ej^\nu$ for the current four-vector $j^\nu = (\rho,\bar{j})$ reproduces Maxwell equations with sources.

Problem 6

Show that in the lagrangian for a massive vector field, $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu$, the parameter m has dimension of mass.

Problem 7

In a pseudo-Minkowski metric with signature q (i.e. with q minus signs on a diagonal of a metric tensor) prove the relation

$$\epsilon_{\alpha_1\dots\alpha_k\nu_1\dots\nu_l}\epsilon^{\alpha_1\dots\alpha_k\mu_1\dots\mu_l} = (-1)^q\, k!\, l!\, \delta^{[\mu_1}_{\nu_1}\cdots\delta^{\mu_l]}_{\nu_l}. \tag{1}$$

Problem 8

Show that for $A = ieA_\mu dx^\mu$ and $F = dA$, the condition $dF = 0$ is equivalent to Bianchi identity.