Analytical structure of the effective potential and quantum criticality for imbalanced Fermi mixtures

Piotr Zdybel and Paweł Jakubczyk

Institute of Theoretical Physics, Faculty of Physics, University of Warsaw

Pasteura 5, 02-093 Warsaw, Poland

Piotr.Zdybel@fuw.edu.pl, Pawel.Jakubczyk@fuw.edu.pl

1. Introduction

Understanding non-Fermi liquids is among the most interesting challenges to quantum many-body physics [1, 2]. Such systems may occur in a variety of physical contexts such as for example superfluidity in imbalanced Fermi gases. Non-Fermi liquid behaviour could appear in the proximity of a quantum critical point (QCP) [3, 4]. We study the possibility of generating a QCP for imbalanced Fermi gases [5–8] at the mean field level in two and three spatial dimensions.

2. Theoretical Model

We study the model defined by the hamiltonian [9]:

$$\hat{H} - \sum_{\sigma} \mu_\sigma \hat{n}_\sigma = \sum_{\vec{k},\sigma} \xi_{\vec{k},\sigma} c^\dagger_{\vec{k},\sigma} c_{\vec{k},\sigma} + \frac{g}{V} \sum_{\vec{k},\vec{k}',\vec{q}} c^\dagger_{\vec{k}+\vec{q}/2,\uparrow} c^\dagger_{-\vec{k}+\vec{q}/2,\downarrow} c_{\vec{k}'+\vec{q}/2,\downarrow} c_{-\vec{k}'+\vec{q}/2,\uparrow}$$

where $\xi_{\vec{k},\sigma} = \vec{k}^2/2m_\sigma - \mu_\sigma$, with $\sigma = \uparrow, \downarrow$, g is an attractive interspecies contact interaction constant and V is the volume of the system. In general, the masses m_σ and chemical potentials μ_σ can be different.

3. Grand Potential

The grand potential density at the mean-field level can be expressed as:

$$\omega = \frac{\Omega}{V} = \min_{\Delta} \left\{ -\frac{|\Delta|^2}{g} - \frac{1}{\beta} \int \frac{d^d\vec{k}}{(2\pi)^d} \sum_{i \in \{+,-\}} \ln\left(1 + e^{-\beta E_i}\right) \right\} = \min_{\Delta} \{\omega_L[\Delta]\},$$

where $\min_\Delta$ leads to the global minimum with respect to the order parameter Δ. We are denoting the Landau functional as $\omega_L[\Delta]$. Moreover, the elementary excitations energies are given by:

$$E_\pm = \frac{\xi_{\vec{k},\uparrow} - \xi_{\vec{k},\downarrow}}{2} \pm \sqrt{|\Delta|^2 + \left(\frac{\xi_{\vec{k},\uparrow} + \xi_{\vec{k},\downarrow}}{2}\right)^2}.$$

We use $\mu = \frac{\mu_\uparrow + \mu_\downarrow}{2}$ as the average chemical potential and $h = \frac{\mu_\uparrow - \mu_\downarrow}{2}$ as the average "Zeeman" field.

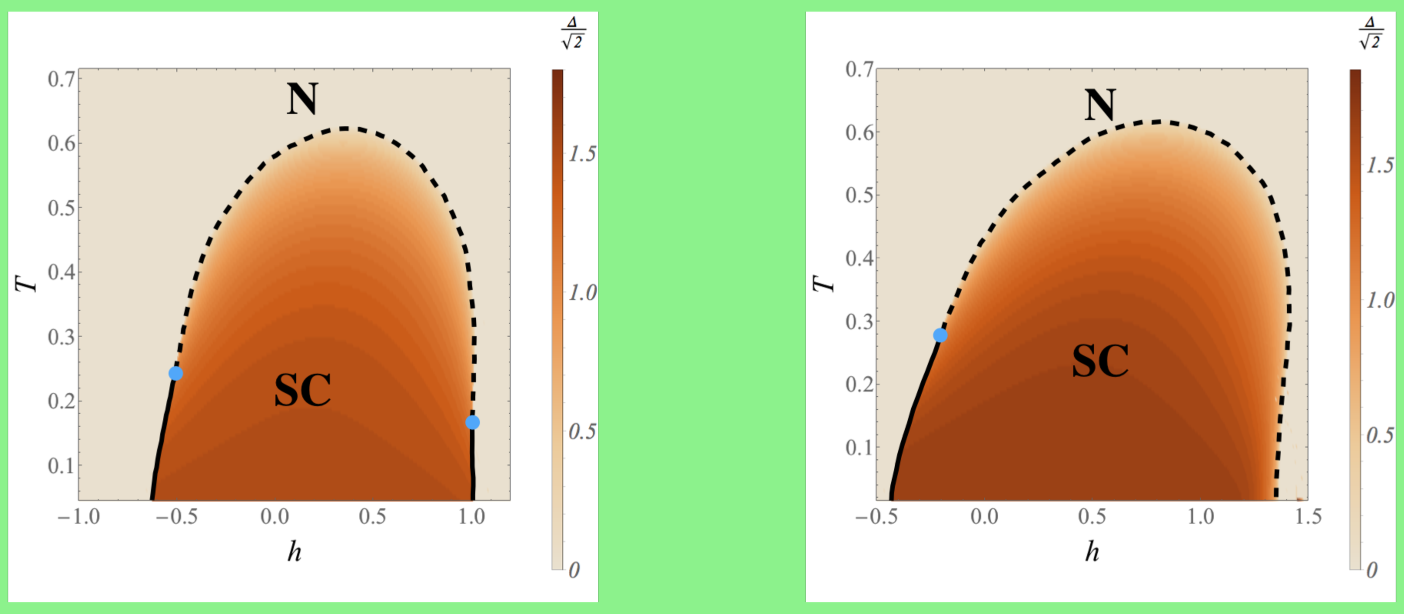

Figure 1: *(left) A typical mean-field phase diagram in d = 3. The low-temperature superfluid phase is separated from the normal phase with a first-order phase transition (bold line) at T sufficiently low, and with a second-order phase transition (dashed line) at T higher. The blue dots indicate the tricritical points.*
(right) A mean-field phase diagram in d = 3 displaying a quantum critical point. The colors refer to the value of the order parameter (Δ).

4. Landau Coefficients

We express Landau functional $\omega_L[\Delta]$ as a Taylor expansion in the order parameter Δ:

$$\omega_L[\Delta] = \omega_L(\Delta = 0) + a_2(T)|\Delta|^2 + a_4(T)|\Delta|^4 + O\left(|\Delta|^6\right),$$

where

$$a_2(T) = \left(\frac{\partial \omega_L}{\partial |\Delta|^2}\right)_{|\Delta|^2=0} \quad \text{and} \quad a_4(T) = \frac{1}{2}\left(\frac{\partial^2 \omega_L}{\partial \left(|\Delta|^2\right)^2}\right)_{|\Delta|^2=0}.$$

The QCP correspond to continuous phase transition at zero absolute temperature, so we compute Landau coefficients for $T \to 0$.

Since potential divergencies in a_4 come from the vicinity of $\xi_{\vec{k}} = 0$, we restrict the integration region in a_4 to a shell of width 2ϵ around $\xi_{\vec{k}} = 0$. Upon expanding the integrands, performing the integrations, and, at the end, considering $T \to 0^+$, we find that the limit is finite provided

$$\mu \neq h\frac{r+1}{r-1},$$

where we introduced $r = \frac{m_\downarrow}{m_\uparrow}$. The analysis can be extended to higher Landau coefficients. As a result we obtain a necessary and sufficient condition for the regularity of the Landau expansion in the limit $T \to 0^+$. The above result does not depend on the system dimensionality.

5. Coefficient a_2 for $T = 0$

Setting $\xi = \frac{\xi_{\vec{k},\uparrow} + \xi_{\vec{k},\downarrow}}{2}$ we have:

$$a_2(T = 0) = -\frac{1}{g} - \frac{1}{2} \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{\xi} \left[1 - \sum_{\sigma} \theta\left(-\xi_{\vec{k},\sigma}\right)\right],$$

where $\theta(\cdot)$ is the Heaviside step function. The first condition of the existence of the QCP requires the disappearance of the coefficient a_2 for that point of the phase diagram. The attractive interaction coupling $g < 0$ can always be tuned (both in $d = 2$ and $d = 3$) so that a_2 is zero (more about it in the work [10]). In an experimental situation this is achievable via Feshbach resonances.

6. Coefficient a_4 for $T = 0$

Existence of a QCP requires that $a_4 > 0$ in addition to $a_2 = 0$. As before we consider the zero-temperature case:

$$a_4(T = 0) = \frac{1}{8} \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{\xi^3} \left[1 - \sum_{\sigma} \left\{\theta\left(-\xi_{\vec{k},\sigma}\right) + \xi\delta\left(-\xi_{\vec{k},\sigma}\right)\right\}\right].$$

In two dimensions we get ($\mu > 0$)

$$a_4^{(2D)} = -\frac{m_r^3}{8\pi}\left[\frac{1}{\left(\Lambda^2 - \lambda_r^2\right)^2} + \frac{1}{\lambda_r^4} + \sum_{\sigma} \theta(\mu_\sigma) \frac{(m_\sigma/m_{\bar{\sigma}})}{\left(\lambda_\sigma^2 - \lambda_r^2\right)^2}\right].$$

where $\frac{2}{m_r} = \frac{1}{m_\uparrow} + \frac{1}{m_\downarrow}$ is the reduced mass, Λ is the UV cutoff, $\lambda_\sigma^2 = 2\mu_\sigma m_\sigma$ and $\lambda_r^2 = 2\mu m_r$. The coefficient $a_4^{(2D)}$ is always negative, so in two dimensions there is no QCP at the mean-field level.

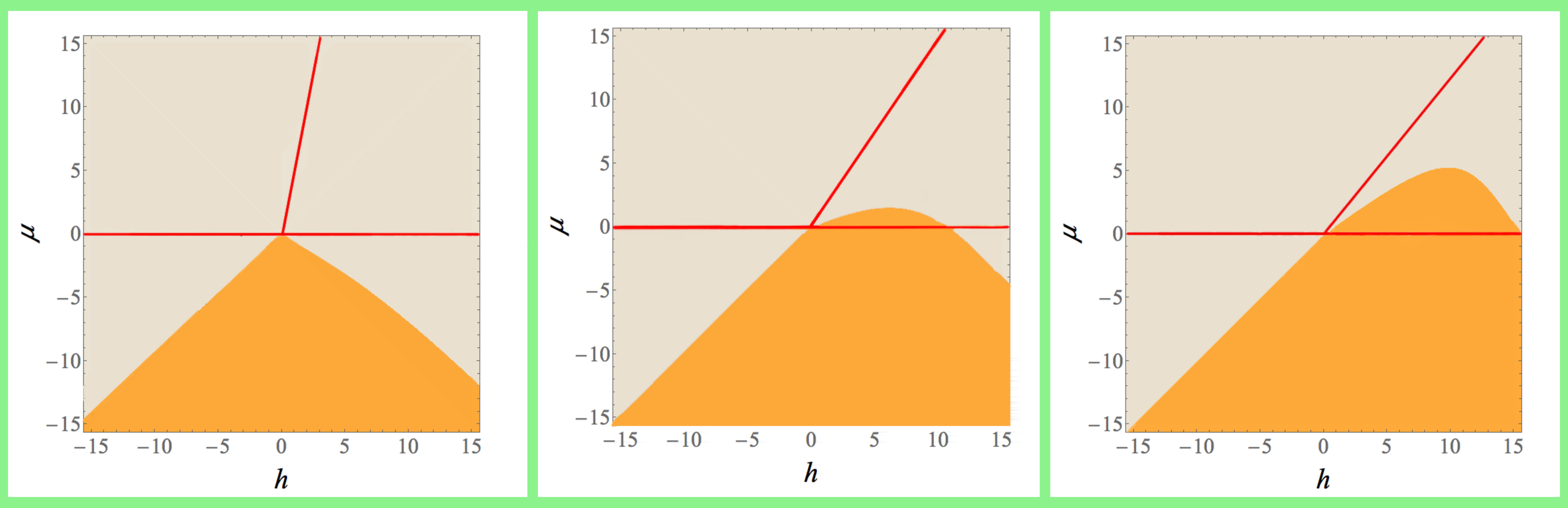

Figure 2: *Evolution of the subset of the $\mu - h$ plane characterized by $a_4^{(3D)} > 0$ upon varying r. The (light) beige area corresponds to negative $a_4^{(0)}$, while in the (darker) orange area $a_4^{(0)} > 0$. The coefficient $a_4^{(3D)}$ is singular along the red straight lines. The first diagram corresponds to $r = 1.5$, the second one to $r = 5$, the last one to $r = 10$. Upon increasing r towards $r \to \infty$ the orange region extends further to cover half of the $\mu - h$ plane located below the diagonal.*

Furthermore, the coefficient $a_4^{(3D)}$ in three dimensions is given by ($\mu > 0$)

$$a_4^{(3D)} = -\frac{m_r^2}{32\pi^2}\left[\frac{\Lambda\left(\Lambda^2 + \lambda_r^2\right)}{\mu\left(\Lambda^2 - \lambda_r^2\right)^2} + \frac{m_r}{\lambda_r^3}\ln\left(\frac{|\Lambda - \lambda_r|}{|\Lambda + \lambda_r|}\right) - \sum_{\sigma}\theta(\mu_\sigma)\left(\frac{\lambda_\sigma\left(\lambda_\sigma^2 + \lambda_r^2\right)}{\mu\left(\lambda_\sigma^2 - \lambda_r^2\right)^2} + \frac{m_r}{\lambda_r^3}\ln\left(\frac{|\lambda_\sigma - \lambda_r|}{|\lambda_\sigma + \lambda_r|}\right) - \frac{8m_\sigma\lambda_\sigma}{\left(\lambda_\sigma^2 - \lambda_r^2\right)^2}\right)\right].$$

For $d = 3$ we have that a second-order quantum phase transition is possible only between a fully-polarized gas and the superfluid phase (more details are included in the work [10]). Such a scenario is favorable at large mass imbalance ($r \gg 1$ or $r \ll 1$).

7. Finite temperature

We analyze the asymptotic shape of the T_c line in the vicinity of the QCP. The behavior observed in Fig. 1 (right) can be understood employing the Sommerfeld (low-temperature) expansion for the coefficient a_2. We obtain:

$$a_2(T, h) = a_2^{(0)}(h) - \alpha(h)T^2 + \ldots, \qquad \alpha(h) = \frac{m_r m_\uparrow^2(\lambda_\uparrow^2 + \lambda_r^2)}{12\lambda_\uparrow(\lambda_\uparrow^2 - \lambda_r^2)^2}.$$

The first term in the Sommerfeld expansion corresponds to the zero-temperature Landau coefficient and the second term is the low-temperature correction. We expand $a_2^{(0)}$ around the ($T = 0$) critical value h_c of the field h and find h_c from the condition $a_2^{(0)}(h_c) = 0$. This yields:

$$T_c(h_c + \delta h) \approx \sqrt{\frac{\partial_h a_2^{(0)}|_{h=h_c}\delta h}{\alpha(h_c)}} \propto \sqrt{h - h_c},$$

where δh is a small deviation from h_c. The MF T_c-line is described by a power law with the exponent $1/2$. Notably δh is positive, in agreement with the numerical results.

8. Summary

We have shown the Landau expansion to be well-defined at $T \to 0^+$ except for a subset of parameters described by condition in sec. 4. We have demonstrated that at mean-field level the occurrence of a QCP is generally excluded in $d = 2$. In $d = 3$ we have found and characterized a parameter regime admitting a QCP. This is restricted to situations where one of the chemical potentials is negative so that the quantum phase transition occurs between the superfluid phase and the fully polarized gas. The second-order transition turns out to be favorable at large mass imbalance r. We have performed a functional RG calculation showing stability of our conclusion with respect to fluctuation effects (details are included in the work [10]).

References

[1] A.J. Schofield, *Contemporary Physics* **40** (1999) 95

[2] C. M. Varma, et al., *Phys. Repts* **361** (2002) 267

[3] P. Coleman & A. J. Schofield, *Nature* **433** (2005) 226

[4] S. Sachdev & B. Kraimer, *Phys. Today* **64** (2011) 29

[5] L. Radzihovsky, *Pys. Rev. A* **84** (2011) 023611

[6] K. B. Gubbels & H. T. C. Stoof, *Phys. Rep.* **525** (2012) 255

[7] L. Radzihovsky & D. E. Sheehy, *Rep. Prog. Phys.* **73** (2010) 076501

[8] M. M. Parish et al., *Phys. Rev. Lett.* **98** (2007) 160402

[9] P. Jakubczyk & P. Strack, *Phys. Rev. X* **4**(2014) 021012

[10] P. Zdybel & P. Jakubczyk, **arXiv:1804.02937** (2018)