Lasers

lecture 2

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electro-magnetic waves in vacuum (classical picture)

Maxwell's equations in vacuum → em wave equation in vacuum

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{E}(\boldsymbol{r},t) = 0$$

$\boldsymbol{E}(\boldsymbol{r},t)$ - electric field of em-wave

Linear polarization + plane wave propagating in the z direction:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E(z,t) = 0$$

c – speed od light in vacuum $\left(\sim 3 \cdot 10^8 \frac{\mathrm{m}}{\mathrm{s}}\right)$

Fourier transformation:

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right)\tilde{E}(z,\omega) = 0$$

$\tilde{E}(z,\omega)$ - spectral amplitude

solution:

$$\tilde{E}(\omega,z) = E_0 e^{ik(\omega)z}, \qquad k = \frac{\omega}{c}$$

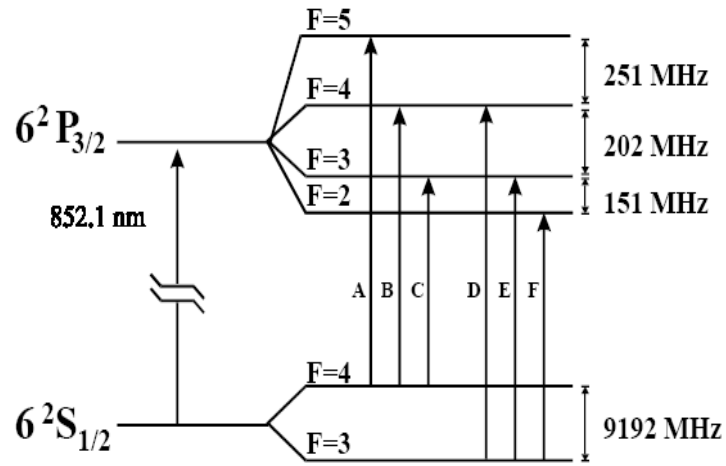
in time domain – monochromatic plane wave in vacuum

$$E(z,t) = E_0 e^{i(kz-\omega t)}, \qquad k(\omega) = \frac{\omega}{c}$$

SI units

^{133}Cs

since 1967:
1 second = the duration of 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.



Paris, Wendome Square

since 1983:
1 metr = the distance travelled by light in vacuum in 1/299792458 second.

We no longer try to measure the speed of light – it is fixed and, together with the second, used to define meter.

em waves in media, classical approach

Maxwell's equations + constitutive realtions ↔ em wave equation in medium

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{E}(\boldsymbol{r},t) = \mu_0 \frac{\partial^2}{\partial t^2}\boldsymbol{P}(\boldsymbol{r},t) \leftarrow$$

polarization of the medium induced by $\boldsymbol{E}$

Polarization is linear in E: $P(\omega) = \varepsilon_0 \chi(\omega) E$ + Fourier transformation

$$\left(\frac{\partial^2}{\partial z^2} + \varepsilon(\omega)\frac{\omega^2}{c^2}\right)\tilde{E}(z,\omega) = 0, \qquad \varepsilon(\omega) = 1 + \chi(\omega)$$

if $\chi(\omega)$ is real then $n = \sqrt{1+\chi},\ k = \frac{n\omega}{c}$

an example – monochromatic plane wave in the medium

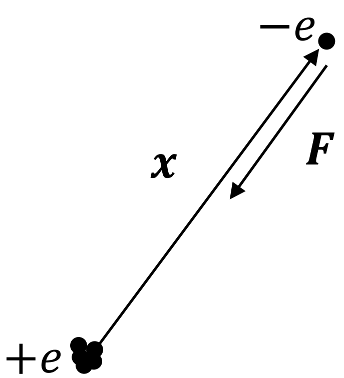
$$\tilde{E}(\omega,z) = E_0 e^{ik(\omega)z}, \qquad k = \frac{n(\omega)\omega}{c}, \qquad n(\omega) = \sqrt{1+\chi(\omega)}$$

and, in the time domain

$$E(z,t) = E_0 e^{i[k(\omega)z - \omega t]}, \qquad k(\omega) = \frac{n(\omega)\omega}{c}$$

phase velocity $\upsilon = c/n$

Example: Lorentz's model



assumptions:

atomic gas at low pressure

heavy nucleus at rest

light electron

interaction model: spring: $\boldsymbol{F} = -k\boldsymbol{x} = -m{\omega_0}^2\boldsymbol{x}$

atom is much smaller than λ: $\boldsymbol{F_E}(t) = -e\boldsymbol{E}(t)$

friction-type losses: $\boldsymbol{F}_T = -m\gamma\frac{dx}{dt}$

monochromatic wave: $\boldsymbol{E}(t) = \boldsymbol{E}_0 e^{i\omega t}$

$$m\frac{d^2\boldsymbol{x}}{dt^2} + m\gamma\frac{d\boldsymbol{x}}{dt} + m{\omega_2}^2\boldsymbol{x} = -e\boldsymbol{E}(t)$$

$$\boldsymbol{x} = \boldsymbol{x_0}e^{i\omega t}$$

$$\boldsymbol{x_0} = -\frac{e}{m}\frac{\boldsymbol{E}_0}{m{\omega_0}^2-\omega^2-i\gamma}$$

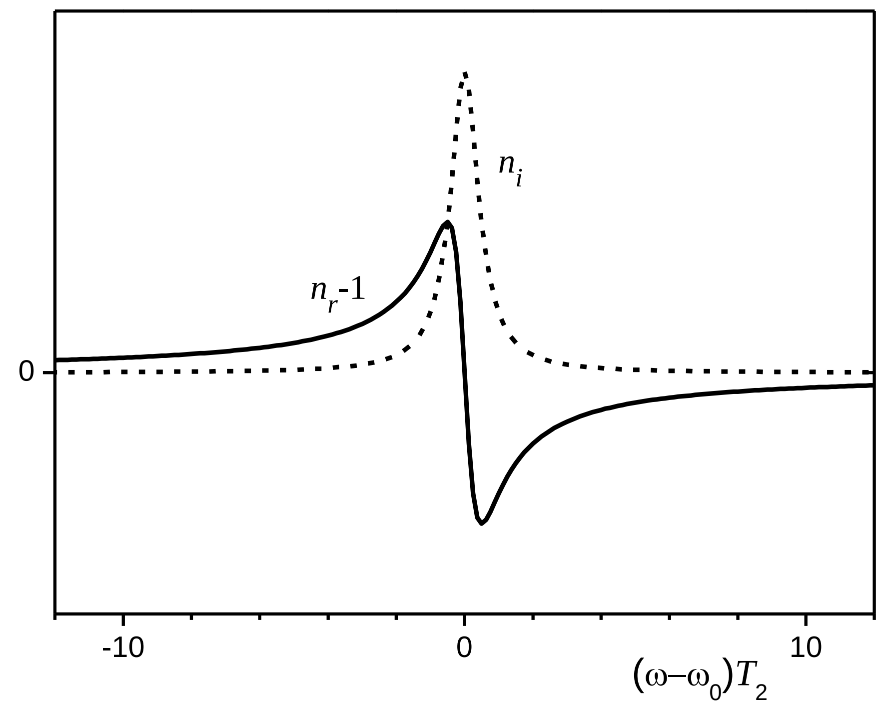
polarization of the medium:

$$\boxed{\boldsymbol{P} = \frac{Ne}{m}\frac{1}{m{\omega_0}^2 - \omega^2 - i\gamma}\boldsymbol{E}}$$

Lorentz's model – resonace

refractive index n is a complex number

$$n = n_r + in_i$$

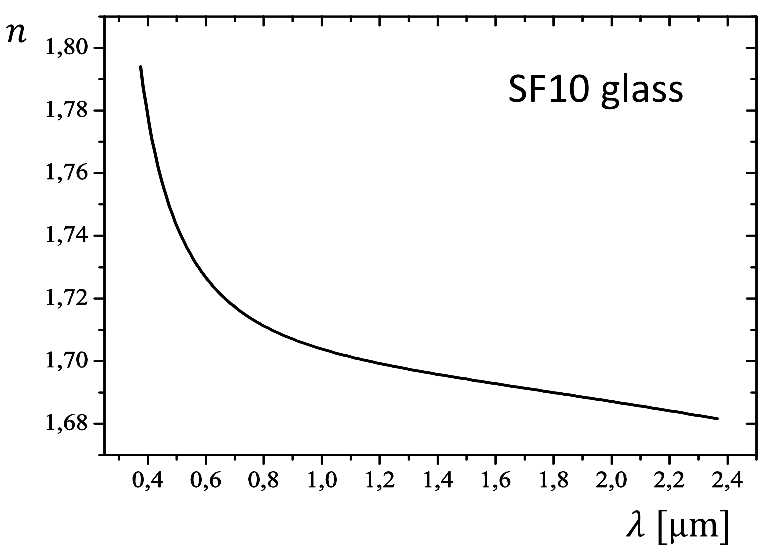


the amplitude of wave changes upon propagation

$$E(z,t) = E_0 e^{i(\omega t - kz)} = E_0 e^{-\frac{\alpha}{2}z} e^{i(\omega t - kz)},$$
$$I(z) = I_0 e^{-\alpha z}$$

$$\alpha = -2\frac{n_i \omega}{c}, \; k = \frac{n_r \omega}{c}$$

far from the resonance n is real



n_i - attenuation, amplification (?) of light
n_r - phase velocity different form c

assume scalar fields and slowly varying envelope:

$$E(\boldsymbol{r},t) = \mathcal{E}(\boldsymbol{r},t)e^{i(k_0 z-\omega_0 t)}, \quad P(\boldsymbol{r},t) = \mathcal{P}(\boldsymbol{r},t)e^{i(\omega_0 t-k_0 z)}, \quad k = \omega_0/c$$

some math

$$\frac{\partial^2 E}{\partial z^2} = \left(\frac{\partial^2 \mathcal{E}}{\partial z^2} - 2ik_0\frac{\partial \mathcal{E}}{\partial z} - k_0^2\mathcal{E}\right)e^{i(\omega_0 t-k_0 z)}$$

$$\frac{\partial^2 E}{\partial t^2} = \left(\frac{\partial^2 \mathcal{E}}{\partial t^2} + 2i\omega_0\frac{\partial \mathcal{E}}{\partial t} - \omega_0^2\mathcal{E}\right)e^{i(\omega_0 t-k_0 z)}$$

slowly varying envelope in the space domain:

$$\left|\frac{\partial^2 \mathcal{E}}{\partial z^2}\right| \ll 2k\left|\frac{\partial \mathcal{E}}{\partial z}\right|$$

leads to

$$\Delta = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z^2} = \Delta_t + \frac{\partial^2}{\partial z^2} \cong \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2ik_0\frac{\partial}{\partial z} - k_0^2\right)$$

and a stationary wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2ik_0\frac{\partial}{\partial z}\right)\mathcal{E} = 0$$ ← paraxial approximation, diffraction, Gaussian beams

slowly varying envelope in the time domain :

$\left|\frac{\partial^2 \mathcal{E}}{\partial t^2}\right|, 2\omega_0\left|\frac{\partial \mathcal{E}}{\partial t}\right| \ll {\omega_0}^2\mathcal{E}$ ← smooth temporal envelope, remember about Fourier limit ↔ $\Delta\omega \ll \omega_0$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2ik_0\frac{\partial}{\partial z} + 2i\omega_0\frac{\partial}{\partial t}\right)\mathcal{E}(\boldsymbol{r},t) = -\omega_0^2\mu_0\frac{\partial^2}{\partial t^2}\mathcal{P}(\boldsymbol{r},t)$$

Laser amplifiers using crystals and glasses.

The medium is a basically transparent medium (glassy, crystalline). It is called host and doped with some active ions. The dielectric susceptibility comes from both the host and the ions $\chi = \chi_h + \chi_j$.

The host is accounted for by its refractive index and, usually, a very small attenuation. The dielectric susceptibility of the host: $\chi_h = (n_r^2 - 1) - i\frac{c\alpha}{\omega}, \qquad \frac{c\alpha}{\omega} \ll n_r$.

The propagation equation now reads

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2ik_0\left(\frac{\partial}{\partial z} + \frac{\alpha}{2}\right) + 2in_r\omega_0\frac{\partial}{\partial t}\right]\mathcal{E}(\boldsymbol{r},t) = -\omega_0^2\mu_0\frac{\partial^2}{\partial t^2}\mathcal{P}_i(\boldsymbol{r},t)$$

where both the index of refraction and attenuation of the host medium are already included and polarization $\mathcal{P}_i(\boldsymbol{r},t)$ comes from the ions only.

for time being we will ignore diffraction and deal with a plane wave:

$$E(\boldsymbol{r},t) = \mathcal{E}(z,t)e^{i(k_0 z-\omega_0 t)}$$

and the propagation equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2ik_0\left(\frac{\partial}{\partial z} + \frac{\alpha}{2}\right) + 2in_r\omega_0\frac{\partial}{\partial t}\right]\mathcal{E}(\boldsymbol{r},t) = -\omega_0^2\mu_0\frac{\partial^2}{\partial t^2}\mathcal{P}_i(\boldsymbol{r},t)$$

now reads

$$2ik_0\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t} + \frac{\alpha}{2}\right)\mathcal{E}(z,t) = -\mu_0\omega_0^2\mathcal{P}_i(z,t)$$

$v = \frac{c}{n_r}$ - phase velocity in the host crystal

assume linear polarization

$$\mathcal{P}_i(\omega) = \varepsilon_0\chi_i\mathcal{E}$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t} + \frac{\alpha}{2}\right)\mathcal{E} = -i\frac{k_0\varepsilon_0}{2n_r}\chi_i\mathcal{E}$$

we take the field propagation equation

$$\left(\frac{\partial}{\partial z}+\frac{1}{v}\frac{\partial}{\partial t}+\frac{\alpha}{2}\right)\mathcal{E}=-i\frac{k_0\varepsilon_0}{2n_r}\chi_i\mathcal{E}$$

and do some mathematics

$$\left[\left(\frac{\partial}{\partial z}+\frac{1}{v}\frac{\partial}{\partial t}+\frac{\alpha}{2}\right)\mathcal{E}\right]\mathcal{E}^*=-i\frac{k_0\varepsilon_0}{2n_r}\chi_i\mathcal{E}\mathcal{E}^*$$

$$\frac{\partial\mathcal{E}}{\partial z}\mathcal{E}^*+\frac{1}{v}\frac{\partial\mathcal{E}}{\partial t}\mathcal{E}^*+\frac{\alpha}{2}\mathcal{E}\mathcal{E}^*=-i\frac{k_0\varepsilon_0}{2n_r}\chi_i\mathcal{E}\mathcal{E}^*$$

$$\mathrm{Re}\left[\frac{\partial\mathcal{E}}{\partial z}\mathcal{E}^*+\frac{1}{v}\frac{\partial\mathcal{E}}{\partial t}\mathcal{E}^*+\frac{\alpha}{2}\mathcal{E}\mathcal{E}^*\right]=-\frac{k_0\varepsilon_0}{2n_r}\mathcal{E}\mathcal{E}^*\mathrm{Im}[\chi_i]$$

$\frac{\partial(\mathcal{E}\mathcal{E}^*)}{\partial z}=\mathrm{Re}\left[\frac{\partial(\mathcal{E}\mathcal{E}^*)}{\partial z}\right]=\mathrm{Re}\left[\frac{\partial\mathcal{E}}{\partial z}\mathcal{E}^*+\frac{\partial\mathcal{E}^*}{\partial z}\mathcal{E}\right]=2\mathrm{Re}\left[\mathcal{E}^*\frac{\partial\mathcal{E}}{\partial z}\right]$ and thus $\mathrm{Re}\left[\mathcal{E}^*\frac{\partial\mathcal{E}}{\partial z}\right]=\frac{1}{2}\frac{\partial(I)}{\partial z}$

similarly: $\mathrm{Re}\left[\mathcal{E}\frac{\partial\mathcal{E}^*}{\partial t}\right]=\frac{1}{2}\frac{\partial(I)}{\partial t}$

and we end up with an equation describing the evolution of light intensity

$$\left(\frac{\partial}{\partial z}+\frac{1}{v}\frac{\partial}{\partial t}\right)I+\left(\alpha-\frac{k_0\varepsilon_0}{n_r}\mathrm{Im}[\chi_i]\right)I=0$$

propagation in a transparent medium

losses/gain

em waves in media, classical approach, cont.

in the stationary case ($\frac{\partial}{\partial t} = 0$)

$$\frac{\partial I}{\partial z} + \left(\alpha - \frac{k_0 \varepsilon_0}{n_r} \mathrm{Im}[\chi_j]\right) I = 0$$

we do not have to assume stationarity; instead we can change variables: $z' = z, t' = t - z/\upsilon$

$$\frac{\partial I}{\partial z'} = -\left(\alpha - \frac{k_0 \varepsilon_0}{n_r} \mathrm{Im}[\chi_j]\right) I$$

in the following we will drop apostrophes with z i t

To amplify light we need the coefficient on the right to be positive (negative absorption)

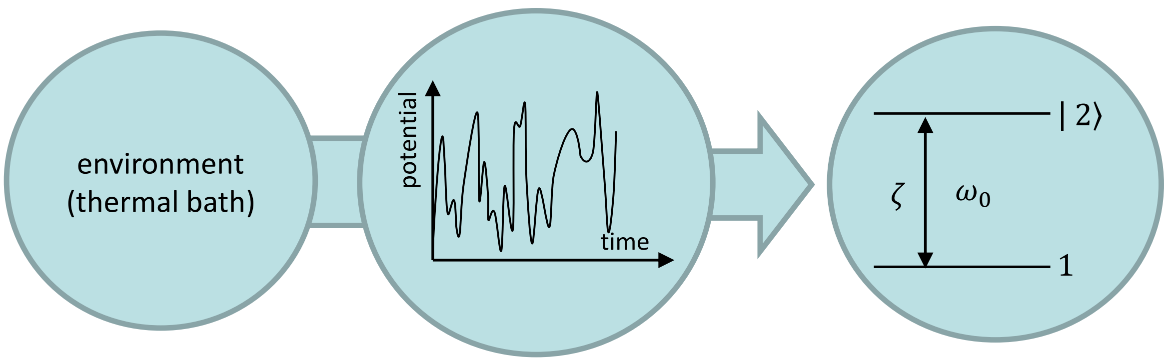
$$\alpha - \frac{k_0 \varepsilon_0}{n_r} \mathrm{Im}[\chi_j] < 0$$

we need to know $\mathrm{Im}[\chi_j]$ but ions are quantum objects

semi classical approach to light-matter interaction

assumptions:

- the gain medium consists of many identical atoms (ions) – atomic gas or ions in the host medium
- atoms/ions are treated quantum mechanically, em wave is described classically (thus semi classical approach)
- we cannot follow the dynamics of individual atoms, there are too many of them and they perturb each other and are perturbed by the environment as a whole
- mathematical tool: density matric: $\varrho_{ij} = |\, i\rangle\langle j|$

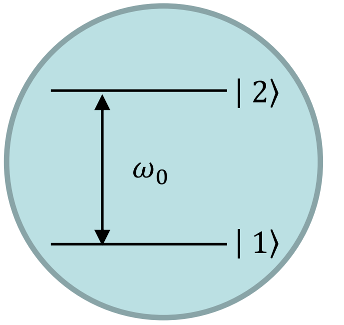


we start with a Hamiltonian $\hat{H}_a$ of an isolated two-level atom, for which the Heisenberg eq. reads:

$$\hat{H}_a \Psi(\boldsymbol{r}, t) = i\hbar \frac{d\Psi}{dt}$$

with stationary solutions: $\Psi_n(\boldsymbol{r}, t) = e^{-i\frac{E_n t}{\hbar}} \psi_n(\boldsymbol{r}) = e^{-i\omega_n t} \psi_n(\boldsymbol{r}), \quad \omega_n = {E_n}/{\hbar}$

and energy eigenvalues: $\hat{H}_a \psi_n(\boldsymbol{r}) = E_n \psi_n(\boldsymbol{r}), \quad \mathrm{n} = 1{,}2$



let's add an external time dependent perturbation $\hat{V}(t)$

$$\hat{H} = \hat{H}_a + \hat{V}(t)$$

and search for time-dependent solutions of the form

$$\Psi_n(\boldsymbol{r}, t) = C_1(t)\psi_1 e^{-i\omega_1 t} + C_2(t)\psi_2 e^{-i\omega_2 t}$$

plug this formulae into the Heiseberg eq. to get

$$C_1 V_{11} + C_2 V_{12} e^{-i\omega_0 t} = i\hbar \frac{dC_1}{dt}$$

$$C_2 V_{22} + C_1 V_{21} e^{i\omega_0 t} = i\hbar \frac{dC_2}{dt} \qquad \text{with} \quad \omega_0 = \omega_2 \text{-} \omega_1 \qquad \text{and} \quad V_{mn} = \int \psi_m^* \hat{V} \psi_n d\boldsymbol{r}$$

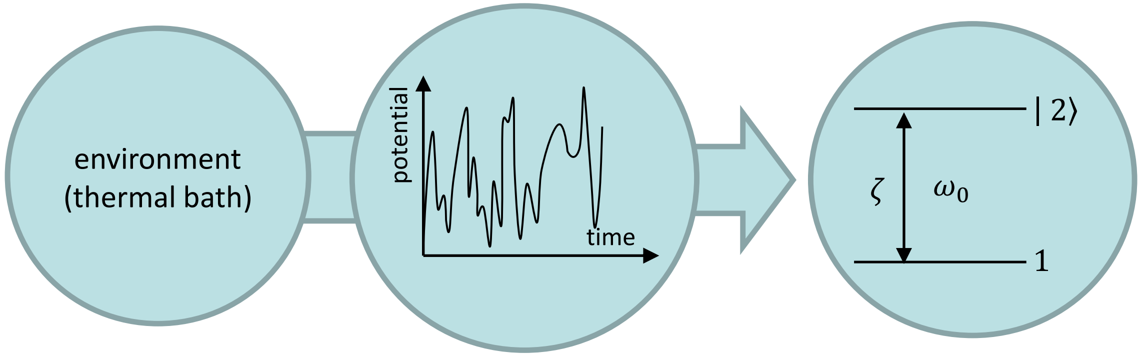
for em wave with the electric field $E(t) = \hat{\epsilon}\mathcal{E}(t) e^{-i\omega t}$ and dipole approx. we have

$$V_{nm}(t) = e r_{nm} \mathcal{E}(t) e^{i(\omega_n - \omega_m)t} = \mu_{nm} \mathcal{E}(t) e^{i\omega t} \quad \text{with} \quad r_{12} = \int \psi_1^* \hat{r} \cdot \hat{\epsilon}\, \psi_2 d\boldsymbol{r}$$

for $\mathcal{E}(t) = \mathrm{const} = \mathcal{E}_0$ we can define Rabi frequency $\Omega_0 = \frac{e\mathcal{E}_0 r_{12}}{\hbar}$ and $V_{12}(t) = \hbar\Omega_0 e^{i\omega t}$

Rabi oscillations, Bloch sphere, π-pulses, etc.

semi classical... cont.



the wave function description is great for a single atom although even then it does not allow for inclusion of spontaneous emission. For more realistic description we need a density matrix ρ. For our two-level atom in a state $|\psi\rangle = C_1|\psi_1\rangle + C_2|\psi_2\rangle$ the matrix is

$$\rho = \sum_{n,m=1,2} C_n C_m^* |n\rangle \langle m|$$

The density matrix evolves in time according to

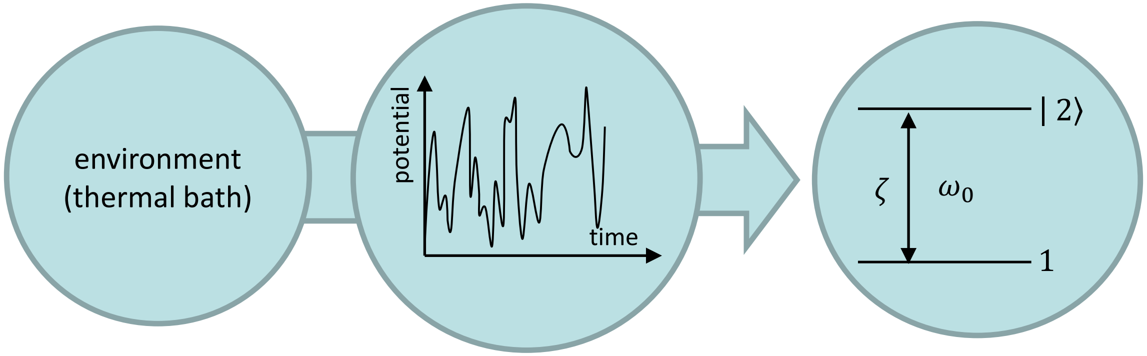
$$\frac{d\rho}{dt} = i\hbar\left[\hat{\rho}(t), \hat{H}\right]$$

which , for our Hamiltonian, gives

$$\frac{d\rho_{11}}{dt} = \frac{i}{\hbar}[\rho_{12}V_{21} - \rho_{21}V_{12}]$$
$$\frac{d\rho_{12}}{dt} = i\omega_0\rho_{12} + \frac{i}{\hbar}[V_{12}(\rho_{11} - \rho_{22})]$$
$$\frac{d\rho_{21}}{dt} = -i\omega_0\rho_{12} + \frac{i}{\hbar}[V_{12}(\rho_{22} - \rho_{11})]$$
$$\frac{d\rho_{22}}{dt} = \frac{i}{\hbar}[\rho_{21}V_{12} - \rho_{12}V_{21}]$$

- ϱ_{11} (ϱ_{22}) - probability of the atom occupying level 1(2)
- ϱ_{12} (ϱ_{21}) - coherence – the atom is in a state which is a coherent superposition of 1 and 2.

semi classical... cont.



Even a more realistic picture must include the environment which acts on the atom. Let's begin without the external perturbation (all V_{nm} are zero)

$$\frac{d}{dt}\varrho_{21} = -i\omega_0\varrho_{21} - \Gamma_2\varrho_{21}$$
$$\frac{d}{dt}\varrho_{22} = -\Gamma_1[\varrho_{22} - \varrho_{22}{}^e]$$
$$\varrho_{12} = \varrho_{21}{}^*$$
$$\varrho_{11} = 1 - \varrho_{22}$$

with $\varrho_{22}{}^e$ - probability of finding the system in level 1 in thermodynamics equilibrium (from now on we will assume that $\varrho_{22}{}^e \cong 0$)

$T_1 = 1/\Gamma_1$- longitudinal relaxation time affects populations
$T_2 = 1/\Gamma_2$- transverse relaxation time affects polarization

in the condensed mater: $T_2 \ll T_1$

semi classical... cont.

expectation value of an operator $\widehat{X}$: $X = Tr(\hat{\varrho}\hat{X})$

two quantities of interest are:

- polarization of the medium:

$$P = NTr(\hat{\varrho}\hat{p}) = N(\varrho_{12}\mu_{21} + \varrho_{21}\mu_{12}) = N(\varrho_{12}\mu_{21} + \rho_{12}^{*}\mu_{12}^{*})$$

with N – density of atoms/ions $\left[\frac{1}{\mathrm{cm}^3}\right]$ and $\mu_{12} = e\int \psi_1^{*}\hat{r}\cdot\hat{\epsilon}\,\psi_2 d\boldsymbol{r}$

- population of the level k: $N_k = NTr\left(\hat{\varrho}\widehat{N}_k\right) = N\varrho_{kk}, \quad k = 0,1$

let's see what happens if we start with some initial values of N_2 and P at $t = 0$: $N_2(0), P(0)$.

the equations from the previous transparency can be integrated to give:

$$P(t) \propto P(0)e^{-t/T_2}e^{-i\omega' t}$$

$$N_2(t) = N_2(0)e^{-t/T_1}$$

with ω' being the resonant frequency of a damped oscillator: $\omega' = \sqrt{\omega_0^2 + {T_2}^{-2}} \cong \omega_0.$

semi classical... cont.

now it is time to consider the full eqs. for the time evolution of the density matrix. We include relaxation and interaction terms:

$$\frac{d}{dt}\varrho_{12} = i\omega_0\varrho_{12} - \Gamma_2\varrho_{12} - i\frac{\mu_{12}}{\hbar}[\varrho_{11} - \varrho_{22}]\mathcal{E}(t)e^{-i\omega t}$$
$$\frac{d}{dt}\varrho_{11} = \Gamma_1\varrho_{22} + i\frac{\mu_{12}}{\hbar}[\varrho_{12} - \varrho_{21}]\mathcal{E}(t)e^{i\omega t}$$
$$\dots$$

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$$\frac{d\rho_{11}}{dt} = \frac{i}{\hbar}[\rho_{12}V_{21} - \rho_{21}V_{12}]$$
$$\frac{d\rho_{12}}{dt} = i\omega_0\rho_{12} + \frac{i}{\hbar}[V_{12}(\rho_{11} - \rho_{22})]$$
$$\frac{d\rho_{21}}{dt} = -i\omega_0\rho_{12} + \frac{i}{\hbar}[V_{12}(\rho_{22} - \rho_{11})]$$
$$\frac{d\rho_{22}}{dt} = \frac{i}{\hbar}[\rho_{21}V_{12} - \rho_{12}V_{21}]$$

apply rotating wave approximation

$\varrho_{12} = \zeta_{12}e^{i\omega t}$

and drop all the terms oscillating with 2ω:

$$\frac{d}{dt}\zeta_{12} = -i(\omega - \omega_0)\zeta_{12} - \Gamma_2\zeta_{12} - i\frac{\mu_{12}}{\hbar}[\zeta_{11} - \zeta_{22}]\mathcal{E}(\mathrm{t})$$
$$\frac{d}{dt}(\zeta_{22} - \zeta_{11}) = i\frac{2\mu_{12}}{\hbar}(\zeta_{21} - \zeta_{12}^*)\mathcal{E}(t) - \Gamma_1(\zeta_{22} - \zeta_{11})$$
$$\dots$$

those eqs can be integrated but we will look for specific solutions only

3 cases:

- stationary solutions - differential eqs. are turned into algebraic ones
- adiabatic solutions - we assume that the light pulse duration t_p is not long and not short $T_2 \leq t_p \ll T_1$, $\frac{d}{dt}\zeta_{12} \cong 0$
- full solutions - ????

semi classical… cont.

$$\frac{d}{dt}\zeta_{12} = -i(\omega - \omega_0)\zeta_{12} - \Gamma_2\zeta_{12} - i\frac{\mu_{12}}{\hbar}[\zeta_{22} - \zeta_{11}]\mathcal{E}(\mathrm{t})$$
$$\frac{d}{dt}(\zeta_{22} - \zeta_{11}) = i\frac{2\mu_{12}}{\hbar}(\zeta_{21} - \zeta_{12}^*)\mathcal{E}(t) - \Gamma_1(\zeta_{22} - \zeta_{11})$$

We follow the adiabatic approach:

$$\zeta_{12} = -i\frac{\mu\mathcal{E}(t)/\hbar}{\Gamma_2 + i(\omega - \omega_0)}[\zeta_{22} - \zeta_{11}]$$
$$\frac{d}{dt}(\zeta_{22} - \zeta_{11}) = 2\left(\frac{\mu}{\hbar}\right)^2\frac{\Gamma_2\mathcal{E}^2(t)}{\Gamma_2^2 + (\omega - \omega_0)^2}[\zeta_{22} - \zeta_{11}] - 2\Gamma_1\zeta_{22}$$

- atomic polarization follows the population difference $\zeta_{12} \propto [\zeta_{22} - \zeta_{11}]$. We ignore atomic coherences.
- a new quantity: population inversion $\Delta N \equiv N_2 - N_1 = N(\zeta_{22} - \zeta_{11})$
- $\zeta_{12} = -i\frac{\frac{\mu\mathcal{E}(t)}{\hbar}}{\Gamma_2 + i(\omega - \omega_0)}[\zeta_{22} - \zeta_{11}] = -\frac{\mu}{\hbar}\frac{\omega - \omega_0}{\Gamma_2^2 + (\omega - \omega_0)^2}[\zeta_{22} - \zeta_{11}] - i\frac{\mu}{\hbar}\frac{\Gamma_2}{\Gamma_2^2 + (\omega - \omega_0)^2}[\zeta_{22} - \zeta_{11}]$
- polarization of the medium

$$P = N\mathrm{Tr}(\hat{\varrho}\hat{p}) = N(\varrho_{12}\mu_{21} + \varrho_{21}\mu_{12}) = N\left(\zeta_{12}\mu_{21}e^{-i\omega t} + \zeta_{21}\mu_{12}e^{i\omega t}\right)$$

$$\mathrm{Im}\left[\chi_j\right] = \mathrm{Im}[\mu(\zeta_{22} - \zeta_{11})]N = \frac{\mu^2}{\epsilon_0\hbar}\frac{\Gamma_2}{\Gamma_2^2 + (\omega - \omega_0)^2}\Delta N$$

so we can now complete the transport equation

$$\frac{\partial I}{\partial z} + \left(\alpha - \frac{k_0}{n_r}\frac{\mu^2}{\hbar}\frac{\Gamma_2}{\Gamma_2^2 + (\omega - \omega_0)^2}\Delta N\right)I = 0$$

in the case of a transparent host ($\alpha = 0$)

$$\frac{\partial I}{\partial z} = \frac{k_0}{n_r}\frac{\mu^2}{\hbar}\frac{\Gamma_2}{\Gamma_2^2 + (\omega - \omega_0)^2}\Delta N \cdot I$$

semi classical… cont.

$$\boxed{\frac{\partial I}{\partial z} = \frac{k_0}{n_r}\frac{\mu^2}{\hbar}\frac{\Gamma_2}{\Gamma_2^2 + (\omega - \omega_0)^2}\Delta N \cdot I}$$

Let’s look closely at the coefficient in the last eq.

$$\frac{k_0}{n_r}\frac{\mu^2}{\hbar}\frac{\Gamma_2}{\Gamma_2^2 + (\omega - \omega_{10})^2}\Delta N = \frac{\pi k_0}{n_r \hbar} \cdot \mu^2 \cdot \frac{1}{\pi}\frac{\Gamma_2}{\Gamma_2^2 + (\omega - \omega_{10})^2} \cdot \Delta N$$

- The dipole transition moment $\mu = er_{12}$ describes interaction strength, for strong atomic lines this number is large; light is strongly absorber/amplified

- The interaction between light and atoms/ions is not restricted to a single frequency, the longitudinal damping manually put into density matrix evolution results in a finite linewidth. The normalized function

  $$g(\omega) = \frac{1}{\pi}\frac{\Gamma_2}{\Gamma_2^2 + (\omega - \omega_{10})^2}$$

  is called lineshape function. More on line broadening will follow.

- Whether resonant light is attenuated or amplified upon passage through a medium depends on population inversion ΔN. For positive ΔN we have amplification, for negative - attenuation

semi classical... cont.

Sometimes we will use photon flux instead of light intensity

photon flux $F \equiv \frac{I}{\hbar\omega} \qquad \left[\frac{1}{\mathrm{cm^2 s}}\right]$:

$$\frac{\partial F}{\partial z} = \frac{\pi k_0}{n_r}\frac{\mu^2}{\hbar}\cdot\frac{1}{\pi}\frac{\Gamma_2}{\Gamma_2^2 + (\omega - \omega_{10})^2}\cdot \Delta N \cdot F$$

$$\frac{\partial F}{\partial z} = \sigma_{12}(\omega)\cdot \Delta N \cdot F$$

with a new coefficient

$$\sigma_{12}(\omega) = \frac{\pi k_0}{n_r}\frac{\mu^2}{\hbar}\cdot\frac{1}{\pi}\frac{\Gamma_2}{\Gamma_2^2 + (\omega - \omega_{10})^2} = \sigma_{12} g(\omega) \qquad \sigma_{12} = [\mathrm{cm}^2]$$

frequency dependent cross-section for stimulated radiation transitions between atomic levels 1 and 2

Note 1: the line-shape function in the formulae above results form the finite value of the transversal relaxation time (T_2). In a general case, all line-broadening mechanism contribute to the shape and width of $\sigma_{21}(\omega)$.

Note 2: the definition of the cross-section includes line-shape function $g(\omega)$ which is normalized

$$\int_0^\infty g(\omega)\, d\omega = 1$$

the units of $\sigma_{12}(\omega)$ are $[\mathrm{cm}^2]$ and the units of σ_{12} are $\left[\mathrm{cm}^2/_{\mathrm{Hz}}\right]$

In most cases, we are interested in the light intensity and then the use of the equation

$$\frac{\partial F}{\partial z} = \sigma_{12}(\omega) \cdot \Delta N \cdot F$$

or its equivalent is well justified. Its advantage lies in simplicity. As we will see soon light intensity also drives population inversion.

To be more exact we might return to the field propagation (slide 10)

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t} + \frac{\alpha}{2}\right)\mathcal{E} = -i\frac{k_0\epsilon_0}{2n_r}\chi_j\mathcal{E}$$

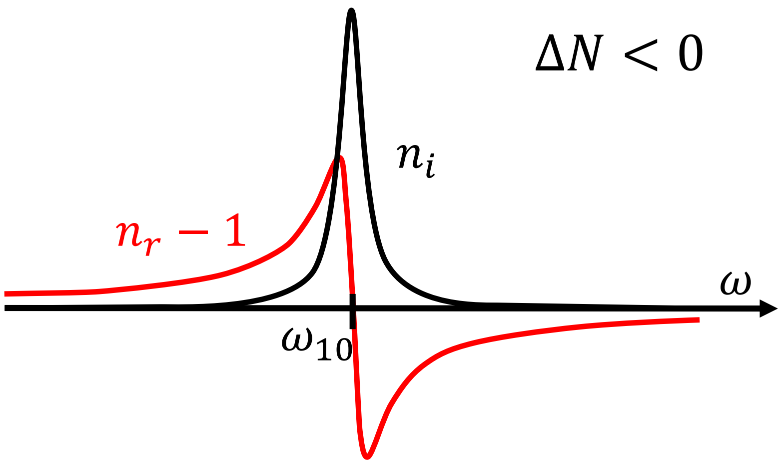
The term on the right side is a complex number and so the field envelope evolution is non-trivial: both amplitude and phase of the electric field are modified during propagation through a medium. The active ions contribute the total refractive index and this contribution is resonant. In most cases we do not care about it but there are instances when this effect matters. One example is „mode pulling" – shift of a laser mode frequency.

$$\chi_j = \mathrm{Re}[\chi_j] + \mathrm{i}\,\mathrm{Im}[\chi_j]$$

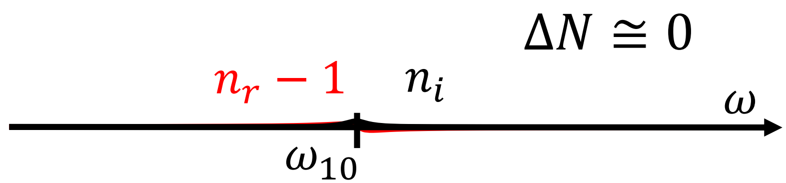
$$n_j = \sqrt{1+\chi_j} \cong 1 + \frac{\chi_j}{2} = 1 + n_r + \mathrm{i}\,n_i$$

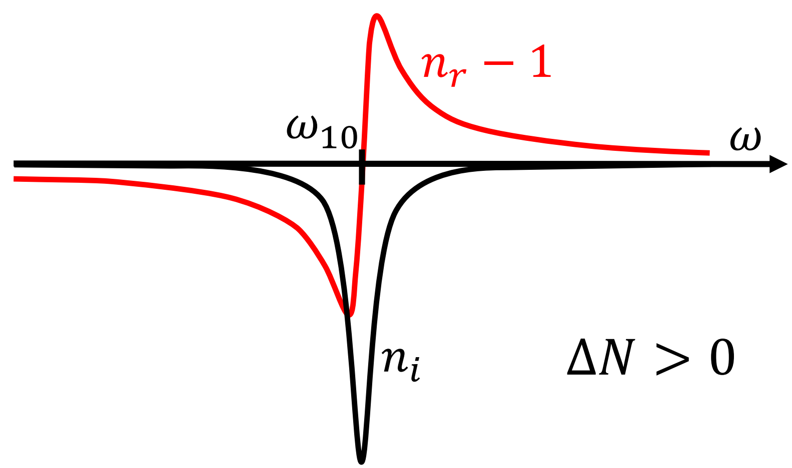
n_r refractive index

loss/gain



schematic drawings for
low pressure atomic gas
or low density ions





homogenous vs inhomogeneous line broadening

we deal with an assemble of atoms/ions

spectral lines are always broadened:

- All the atoms/ions have the same resonant frequency ω_0 - homogenous broadening
- $\mathrm{diffrenet\ sub-assemblies}$ of atoms/ions have different resonance frequencies – inhomogeneous broadening

In the last case the medium polarization is a sum of polarizations of the subassemblies

$$P(t)=\sum_s P'(t,\omega'_{s0})$$

We will discuss line broadening again

Sometimes we will be replacing light intensity with a photon flux

1. photon flux $F \equiv \frac{I}{\hbar\omega}$ $\left[\frac{1}{\mathrm{cm}^2\mathrm{s}}\right]$:

$$\frac{\partial}{\partial z}F = \sigma_{01}(\omega)\cdot\Delta N\cdot F$$

2. the populations evolution is governed by:

$$\frac{\partial}{\partial t}\Delta N = -\frac{\Delta N}{T_1} - 2\sigma_{12}\cdot\Delta N\cdot F$$

Note: in more realistic models we will go beyond the two-level model and this equation will be modified accordingly

2 variables and 2 differential equations. The problem is that the equations are nonlinear – there are no general analytic solutions.

options:

- numerical integration
- approximate solutions

simple dynamics for laser amplifier

$$\boxed{\begin{gathered}\frac{\partial}{\partial z}F(t) = \sigma_{12}\Delta N(t)F(t) \\ \frac{\partial}{\partial t}\Delta N(t) = -\frac{\Delta N}{T_1} - 2\sigma_{12}\Delta N(t)F(t)\end{gathered}}$$

Note: in more realistic models we will go beyond the two-level model and this equation will be modified accordingly

Note that for the amplifier to work we need some initial population inversion ΔN_0. This modifies the second equation which now reads:

$$\frac{\partial}{\partial t}\Delta N(t) = -\frac{\Delta N - \Delta N_0}{T_1} - 2\sigma_{12}\Delta N(t)F(t)$$

We formally transform the two equations as follows: From the first one we calculate $\Delta N = \frac{\partial F}{\partial z}/(\sigma_{01}F)$ and insert it into the second equation

$$\frac{\partial^2}{\partial t \partial z}\ln F + \sigma_{12}\frac{\partial}{\partial z}F + \frac{1}{T_1}\left(\frac{\partial}{\partial z}\ln F - \sigma_{12}\Delta N_0\right) = 0$$

There are 2 characteristic time scales involved in this problem: (1) the population decay time T_1 and (2) light pulse duration (t_p). Wee will attempt to solve those equations in two limiting cases

„short" pulse laser amplifier, 2

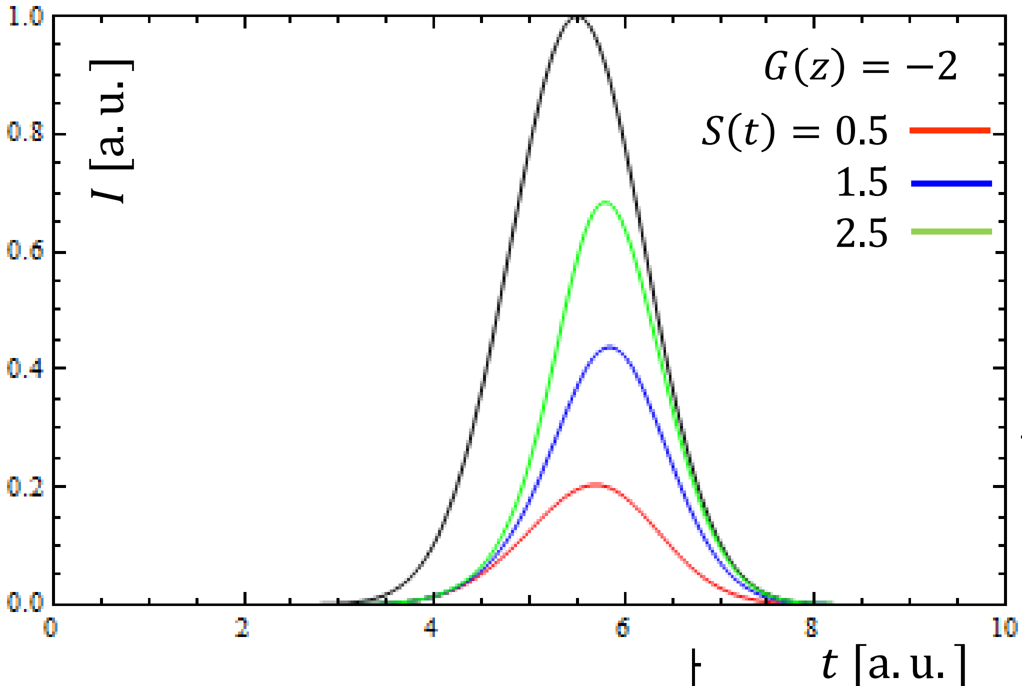
gain saturation leads to pulse shaping.

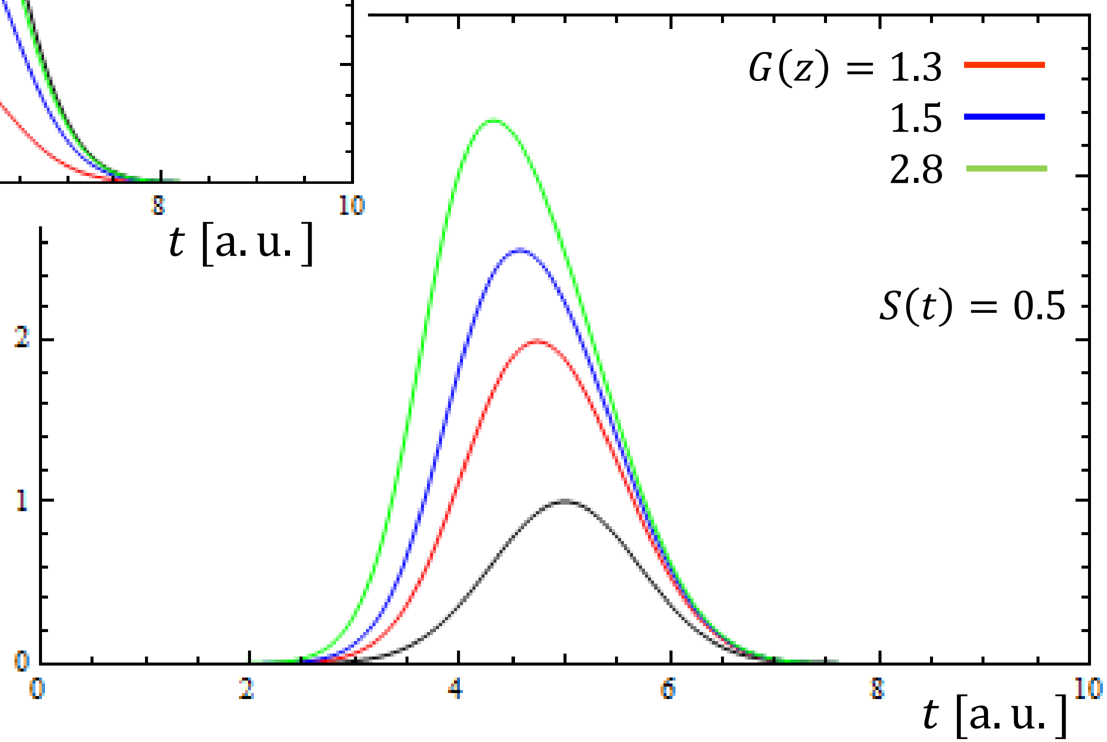
$$F(z,t) = \frac{F(0,t)}{1 - e^{-S(t)}\left(1 - e^{-G(z)}\right)}$$

$$\Delta N(z,t) = \frac{\Delta N(z,0)e^{-G(z)}}{e^{S(t)} + e^{-G(z)} - 1}$$

$$S(t) = 2\sigma_{12}\int_{-\infty}^{t} F(0,t')dt'$$

$$G(z) = \sigma_{12}\int_{0}^{z} \Delta N(z',0)dz'$$





„long” pulses ($\tau_p \gg T_1 , T_2$).

„long” pulse laser amplifier, 1

$$\underbrace{\frac{\partial^2}{\partial t \partial z} \ln F}_{\text{small –we drop it}} + 2\sigma_{12} \frac{\partial}{\partial z} F + \frac{1}{T_1}\left(\frac{\partial}{\partial z} \ln F - \sigma_{12} \Delta N_0\right) = 0$$

$$\frac{\partial}{\partial z} F = \frac{\gamma_0 F}{1 + F/F_s}$$

$$F_s = (2\sigma_{12} T_1)^{-1}$$

the solution is

$$\boxed{\ln \frac{F(z,t)}{F(0,t)} + \frac{F(z,t) - F(0,t)}{F_s} = \gamma_0 z}$$

$I \ll I_s \implies F(z,t) = e^{\gamma_0 z} F(0,t)$ unsaturated laser amplifier

$I \gg I_s \implies F(z,t) = F(0,t)$ completely saturated laser amplifier

$\gamma_0 = \sigma_{12} \Delta N_0$ - unsaturated gain coefficient

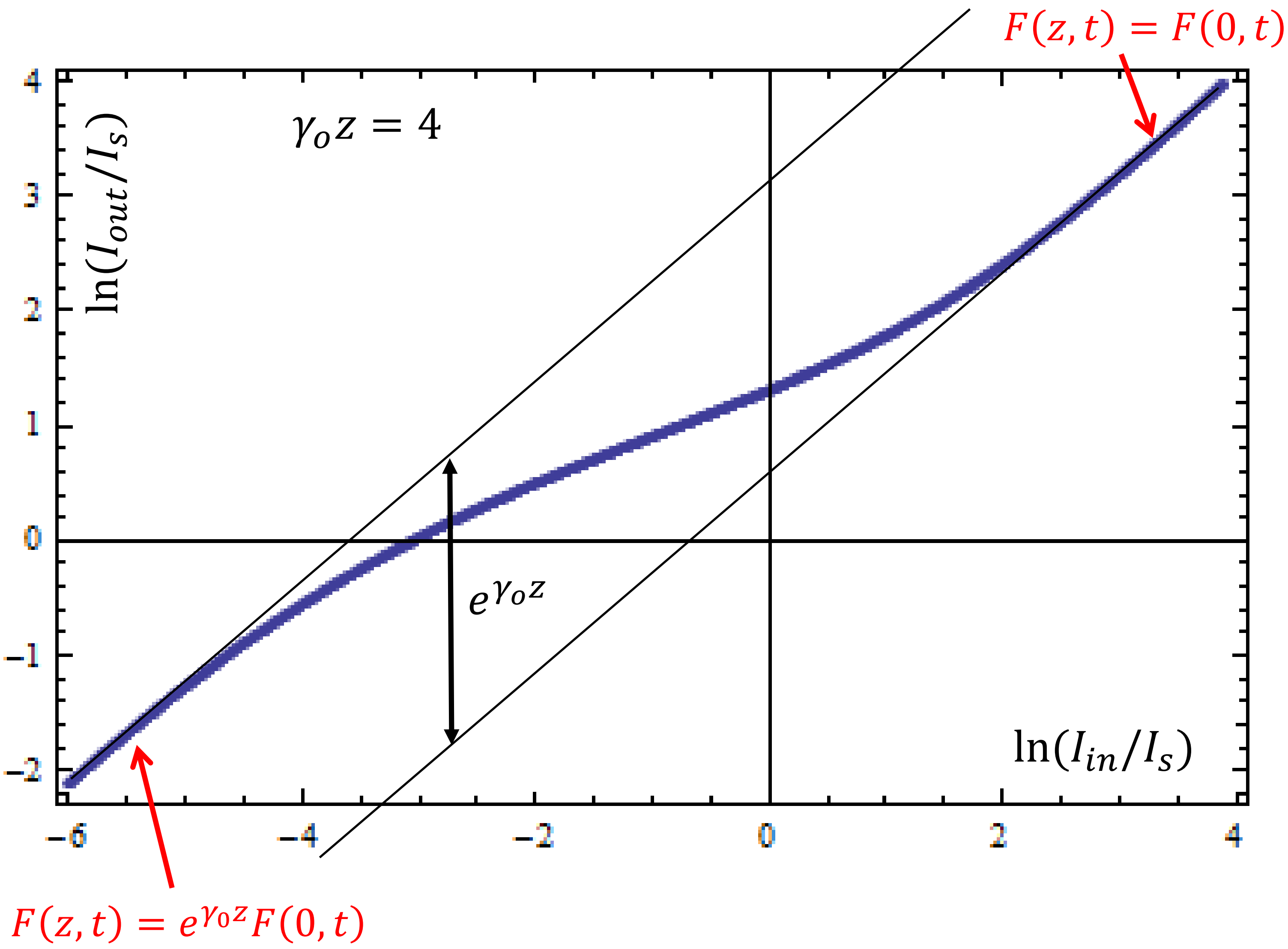
$F_s \equiv \frac{1}{T_1 \sigma_{12}}$ - saturating photon flux $\left[\frac{1}{\mathrm{s \cdot cm^2}}\right]$

$E_s = \hbar \omega F_s$ - saturating intensity $\left[\frac{\mathrm{W}}{\mathrm{cm^2}}\right]$

„long” pulse laser amplifier, 2

$$\ln\frac{F(z,t)}{F(0,t)} + \frac{F(z,t) - F(0,t)}{F_s} = \gamma_0 z$$

$$F_s = (2\sigma_{01}T_1)^{-1}$$



„intermediate” pulse laser amplifier

- „intermediate” pulses ($t_p \cong T_1$). The equation has to be integrated numerically

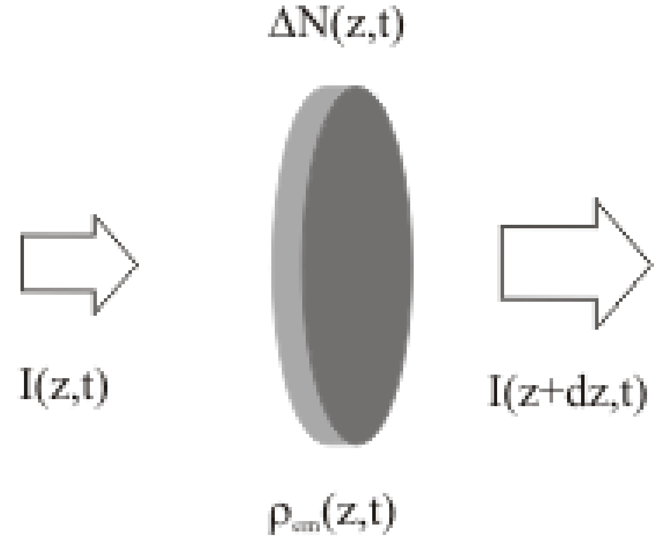
$$\frac{\partial^2}{\partial t \partial z} \ln F + 2\sigma_{01} \frac{\partial}{\partial z} F + \frac{1}{T_1} \left(\frac{\partial}{\partial z} \ln F - \sigma_{01} \Delta N_0 \right) = 0$$

równanie transportu energii

$$I(z,t) = \varrho_{em}(z,t) \cdot v_g$$

gęstość energii
fali em

prędkość grupowa
$v_g \equiv d\omega/dk$



Fala em rozchodzi się w kierunku z. Rozważamy plasterek o powierzchni przekroju A i grubości dz. Bilans energii fali wewnątrz plasterka:

$$\frac{d\varrho_{em}(z,t)}{dt} Adz = [I(z,t) - I(z+dz,t)]Adz + \varrho\Delta N(z,t) AdzI(z,t)$$

korzystamy z równania powyżej i przechodzimy do równania różniczkowego na natężenie światła

$$\frac{\partial I}{\partial z} + \frac{1}{v_g}\frac{\partial I}{\partial z} = \sigma \cdot \Delta N \cdot I$$