

Lasers

lecture 3

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<http://www.optics.fuw.edu.pl/fizyka-laserow-2017-18>

laser amplifier

from lecture 2 we have

1. photon flux $F \equiv \frac{I}{\hbar\omega}$ $\left[\frac{1}{\text{cm}^2\text{s}} \right]$ evolves according to:

$$\frac{\partial}{\partial z} F = \sigma_{12}(\omega) \cdot \Delta N \cdot F$$

2. the populations inversion ΔN $\left[\frac{1}{\text{cm}^3} \right]$ evolution is governed by:

$$\frac{\partial}{\partial t} \Delta N = -\frac{\Delta N}{T_1} - 2\sigma_{12} \cdot \Delta b N \cdot F$$

note1: in more realistic models we will go beyond the two-level model and the second equation will be modified accordingly. In systems with a short lifetime of the lower level (the most common case) the factor 2 is missing.

note2: we switch from 0 to 12 when indexing the cross-section. From now on we will use σ_{12} to signify a typical energy level system with 1 and 2 being the lower and upper level of laser transition.

2 variables and 2 first order differential equations. The problem is that the equations are nonlinear – **there are no general analytic solutions.**

options:

- numerical integration
- approximate solutions

simple dynamics of the laser amplifier

$$\frac{\partial}{\partial z} F(t, z) = \sigma_{12} \Delta N(t, z) F(t, z)$$
$$\frac{\partial}{\partial t} \Delta N(t, z) = -\frac{\Delta N(t, z)}{T_1} - 2\sigma_{12} \Delta N(t, z) F(t, z)$$

Note that, for the amplifier to work, we need some initial population inversion ΔN_0 . This modifies the second equation which now reads (time and space dependence are dropped for clarity):

$$\frac{\partial}{\partial t} \Delta N = -\frac{\Delta N - \Delta N_0}{T_1} - 2\sigma_{12} \Delta N F$$

We formally transform the two equations as follows: From the first one we calculate $\Delta N = \frac{\partial F}{\partial z} / (\sigma_{01} F)$ and insert it into the second equation


$$\frac{\partial^2}{\partial t \partial z} \ln F + \sigma_{12} \frac{\partial}{\partial z} F + \frac{1}{T_1} \left(\frac{\partial}{\partial z} \ln F - \sigma_{12} \Delta N_0 \right) = 0$$

There are 2 characteristic time scales involved in this problem: (1) the population decay time T_1 and (2) light pulse duration (t_p). We will attempt to solve those equations in two limiting cases

„short” pulse laser amplifier

„short” pulses ($t_p \ll T_1$). They actually cannot be too short – we have previously neglected the transverse relaxation time so we need $t_p \gg T_2$

$$\frac{\partial^2}{\partial t \partial z} \ln F + 2\sigma_{12} \frac{\partial}{\partial z} F + \frac{1}{T_1} \left(\frac{\partial}{\partial z} \ln F - \sigma_{01} \Delta N_0 \right) = 0$$


this term is small;
we will drop it

solutions

$$F(z, t) = \frac{F(0, t)}{1 - e^{-S(t)}(1 - e^{-G(z)})}$$

$$\Delta N(z, t) = \frac{\Delta N(z, 0)e^{-G(z)}}{e^{S(t)} + e^{-G(z)} - 1}$$

$$S(t) = 2\sigma_{12} \int_{-\infty}^t F(0, t') dt'$$
$$G(z) = \sigma_{12} \int_0^z \Delta N(z', 0) dz'$$

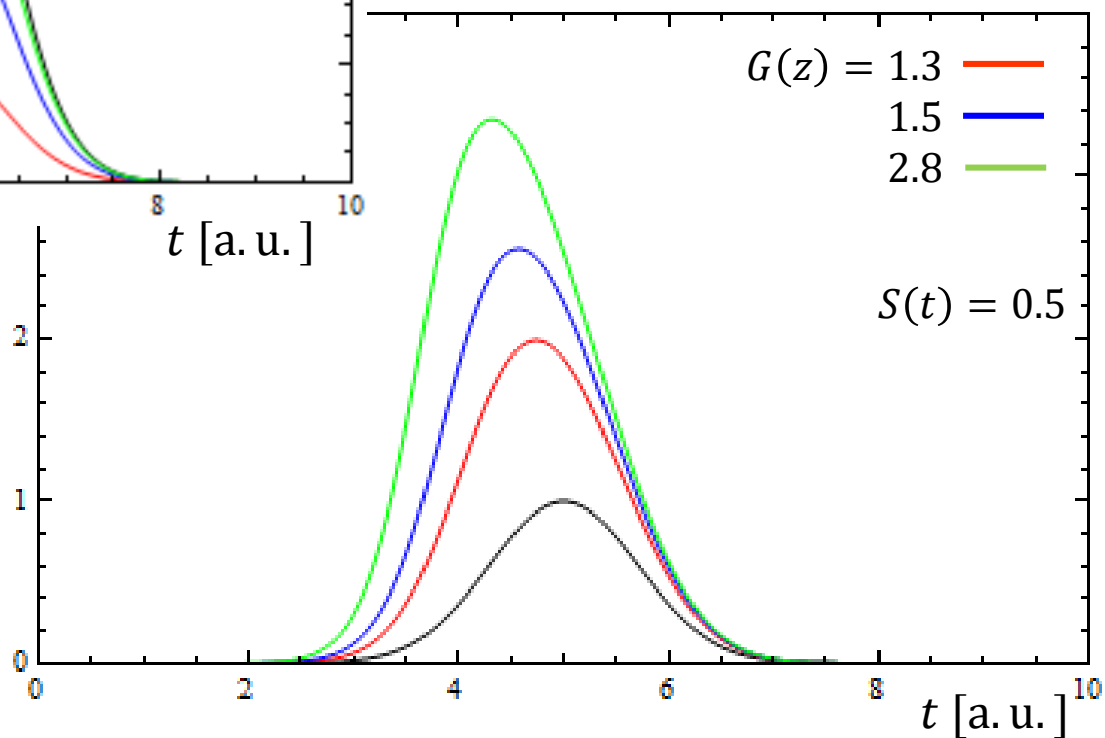
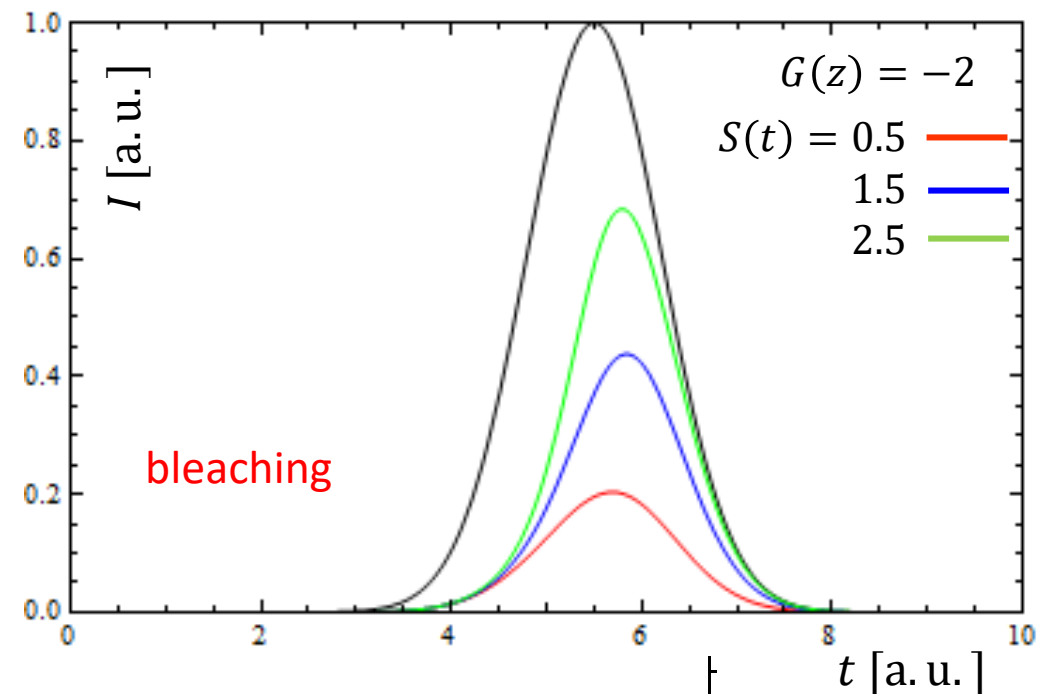
„short” pulse laser amplifier, 2

gain saturation leads to pulse shaping.

$$F(z, t) = \frac{F(0, t)}{1 - e^{-S(t)}(1 - e^{-G(z)})}$$

$$\Delta N(z, t) = \frac{\Delta N(z, 0)e^{-G(z)}}{e^{S(t)} + e^{-G(z)} - 1}$$

$$S(t) = 2\sigma_{12} \int_{-\infty}^t F(0, t') dt'$$
$$G(z) = \sigma_{12} \int_0^z \Delta N(z', 0) dz'$$



„long” amplified pulses ($\tau_p \gg T_1, T_2$).

„long” pulse laser amplifier

$$\underbrace{\frac{\partial^2}{\partial t \partial z} \ln F}_{\text{small - we drop it}} + \sigma_{12} \frac{\partial}{\partial z} F + \frac{1}{T_1} \left(\frac{\partial}{\partial z} \ln F - \sigma_{12} \Delta N_0 \right) = 0$$

small - we drop it

Note: no factor 2 in this eq. - explanation will be given later.

$$\frac{\partial}{\partial z} F = \frac{\gamma_0 F}{1 + F/F_s}$$

$$F_s = (\sigma_{12} T_1)^{-1}$$

the formal solution is

$$\ln \frac{F(z, t)}{F(0, t)} + \frac{F(z, t) - F(0, t)}{F_s} = \gamma_0 z$$

two limits:

$$I \ll I_s \Rightarrow F(z, t) = e^{\gamma_0 z} F(0, t) \quad \text{unsaturated laser amplifier}$$

$$I \gg I_s \Rightarrow F(z, t) = F(0, t) \quad \text{completely saturated laser amplifier}$$

$$\gamma_0 = \sigma_{12} \Delta N_0$$

- unsaturated gain coefficient

$$F_s \equiv \frac{1}{T_1 \sigma_{12}}$$

- saturating photon flux $\left[\frac{1}{\text{s} \cdot \text{cm}^2} \right]$

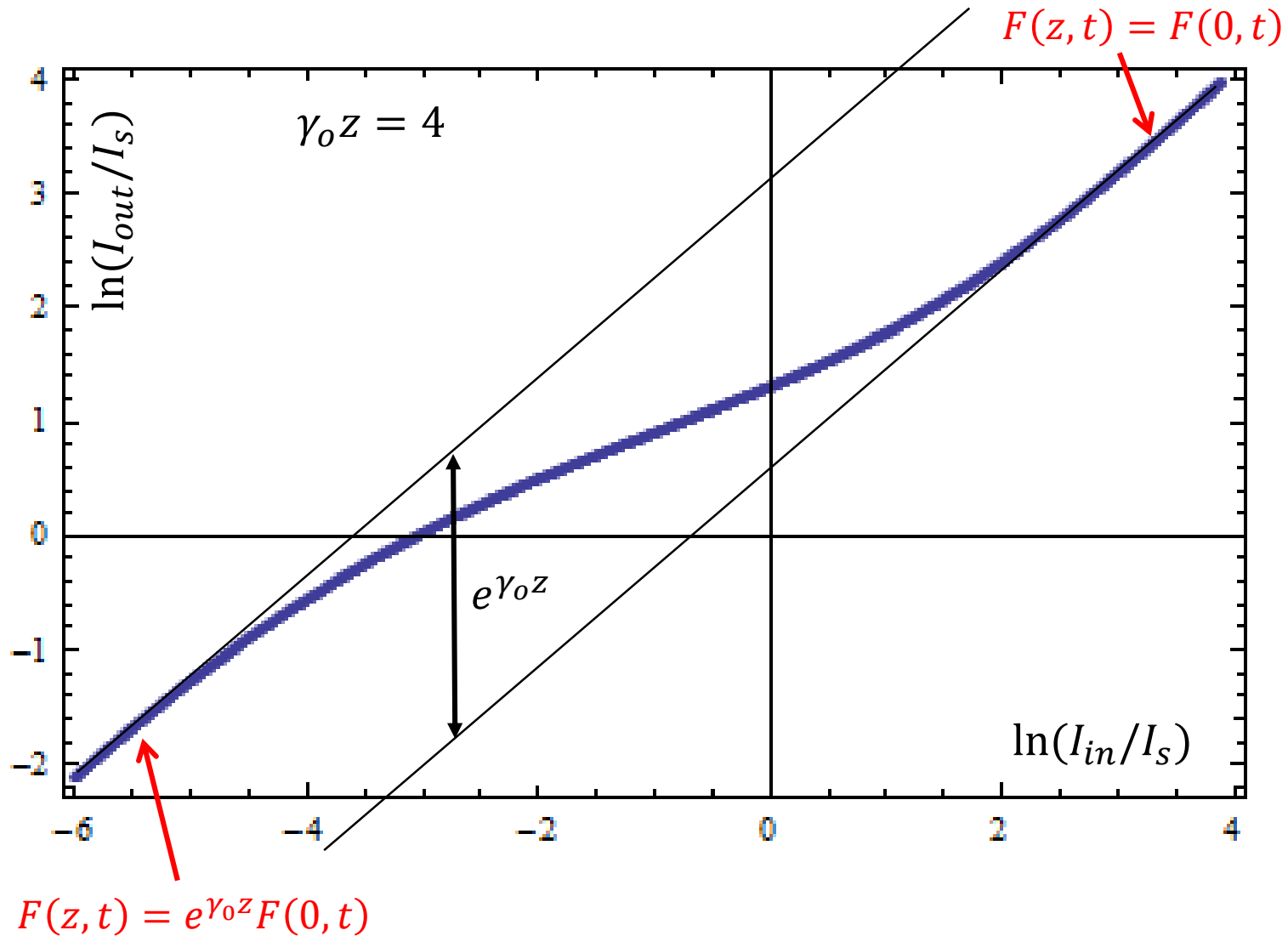
$$I_s = \hbar \omega F_s$$

- saturating intensity $\left[\frac{\text{W}}{\text{cm}^2} \right]$

„long” pulse laser amplifier, 2

$$\ln \frac{F(z, t)}{F(0, t)} + \frac{F(z, t) - F(0, t)}{F_s} = \gamma_0 z$$

$$F_s = (\sigma_{01} T_1)^{-1}$$



„intermediate” pulse laser amplifier

- „intermediate” pulses ($t_p \cong T_1$). The equation has to be integrated numerically in its full splendor

$$\frac{\partial^2}{\partial t \partial z} \ln F + 2\sigma_{01} \frac{\partial}{\partial z} F + \frac{1}{T_1} \left(\frac{\partial}{\partial z} \ln F - \sigma_{01} \Delta N_0 \right) = 0$$

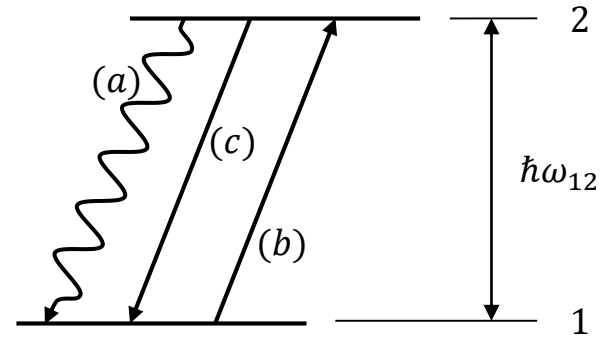
a simpler picture for light-matter interaction; Einstein coefficients

a two-level atom/ion. There are 3 radiative transitions:

- spontaneous emission
- absorption
- stimulated emission

populations: $N_1 + N_2 = N$

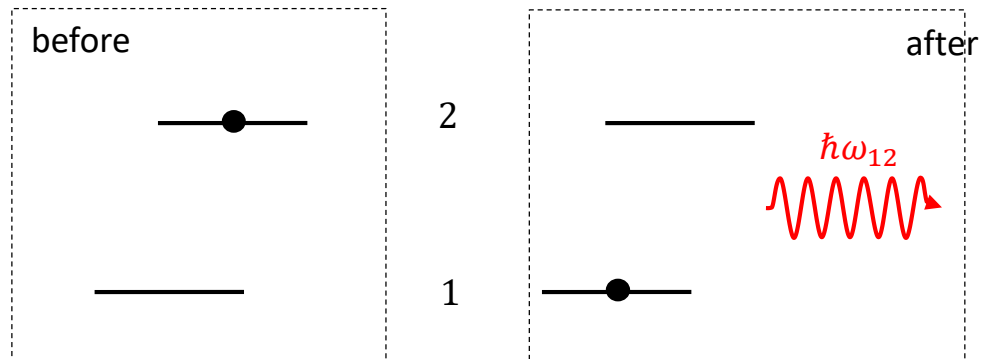
density of atoms/ions



Simple properties of the radiative transitions:

□ spontaneous emission

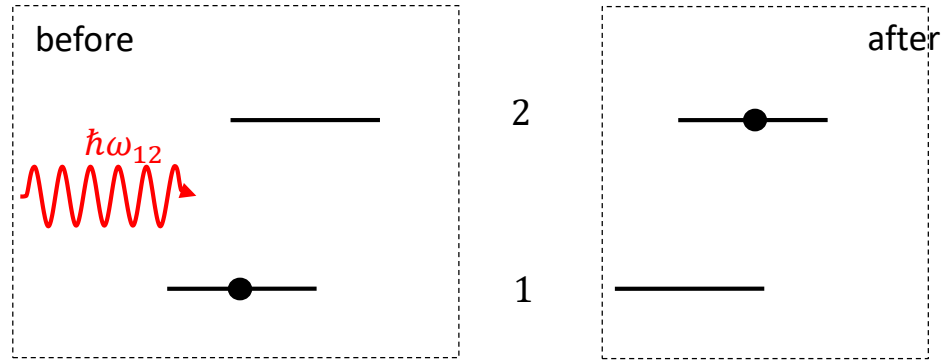
$$\frac{dN_2}{dt} = -A_{21}N_2, \quad A_{21} \text{ is a constant (coefficient)}$$



another picture

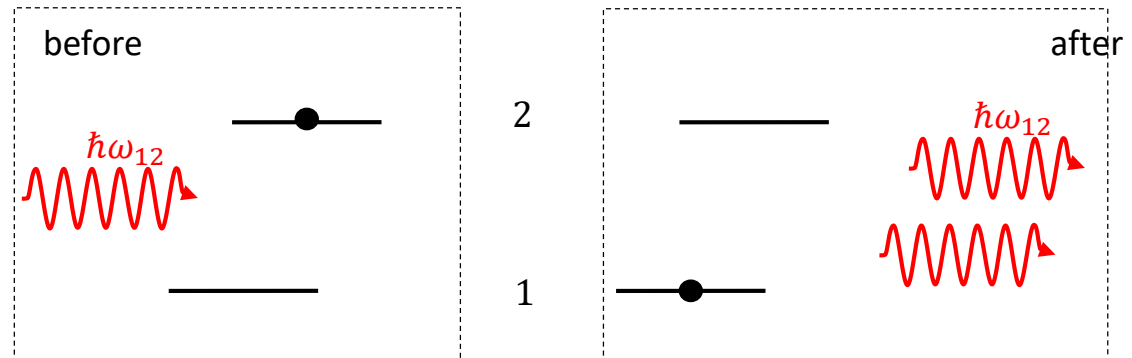
□ absorption

$$\frac{dN_2}{dt} = -\frac{dN_1}{dt} = B_{12} \rho(\omega_{12}) N_1 \quad B_{12} \text{ - coefficient, } \rho(\omega_{12}) \text{ power density of em field}$$



□ stimulated emission

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt} = B_{21} \rho(\omega_{12}) N_2$$

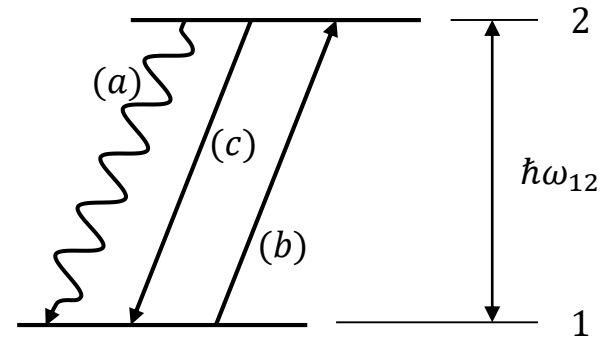


an alternative picture

relations between Einstein coefficients

$$B_{21} = B_{12}$$

$$A_{21} = \frac{\hbar\omega_{21}^3}{\pi^3 c^3} B_{21}$$



population evolution:

$$\frac{dN_2}{dt} = -A_{21}N_2 + B_{21}\rho(\omega_{21})(N_1 - N_2)$$

$$N_1 + N_2 = N$$

consequences:

- the same speed of stimulated transitions
- stimulated transitions dominate at low frequencies
- at high frequencies the spontaneous emission dominates

$\rho_{cr} \left[\frac{\text{J}}{\text{m}^3 \text{Hz}} \right]$ - critical spectral density and critical intensity of the em field:

$$A_{21} = B_{21}\rho_{cr}(\omega) = \frac{\hbar\omega_{21}^3}{\pi^3 c^3} B_{21} \Rightarrow \rho_{cr}(\omega) = \frac{\hbar\omega_{21}^3}{\pi^3 c^3}$$

$$I(\omega) = c \rho_{\omega}(\omega) d\omega \Rightarrow I_{cr}(\omega) = \frac{\hbar\omega_{21}^3}{\pi^3 c^2} d\omega$$

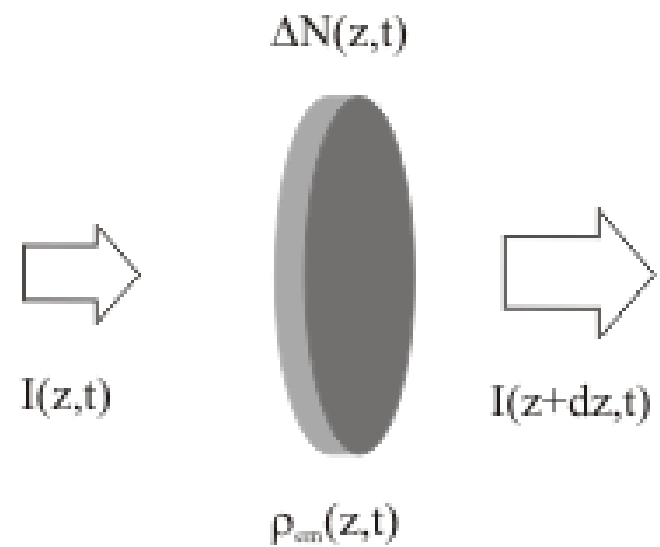
↑
in vacuum

energy transport equation

$$I(z, t) = \rho_{em}(z, t) d\omega \cdot v_g$$

\nearrow
 spectral density of
 em field $\left[\frac{\text{J}}{\text{m}^3 \text{Hz}} \right]$

\nearrow
 group velocity
 $v_g \equiv d\omega/dk$



em field propagates in the z direction. consider a slice with area S and thickness dz . em field energy change within the slice:

$$\underbrace{\frac{d\rho}{dt} d\omega S dz}_{\frac{dI}{dt} \frac{1}{v_g}} = [I(z, t) - I(z + dz, t)] S + \underbrace{\hbar\omega B_{21} \rho d\omega \Delta N(z, t) S dz}_{\sigma I \quad (\sigma = \hbar\omega B_{21}/c)} \quad / (S dz)$$

$$\frac{\partial I}{\partial z} + \frac{1}{v_g} \frac{\partial I}{\partial t} = \sigma \cdot \Delta N \cdot I$$

$$\frac{\partial F}{\partial z} + \frac{1}{v_g} \frac{\partial F}{\partial t} = \sigma \cdot \Delta N \cdot F \quad \text{- the same as in lecture 2}$$

populations

from Einstein's eqs.:

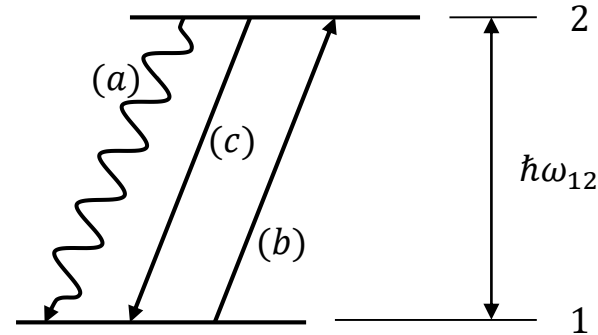
$$\frac{dN_2}{dt} = -A_{21}N_2 + B_{21}\rho(\omega_{21})(N_1 - N_2)$$

$$N_1 + N_2 = N$$

which gives

$$\Delta N = N_2 - N_1 = 2N_2 - N$$

$$\frac{d}{dt}\Delta N = 2 \frac{dN_2}{dt} = \underbrace{-A_{21}(\Delta N + N)}_{\text{spontaneous emission}} - \underbrace{2B_{21}\rho(\omega_{21})\Delta N}_{\sigma F}$$



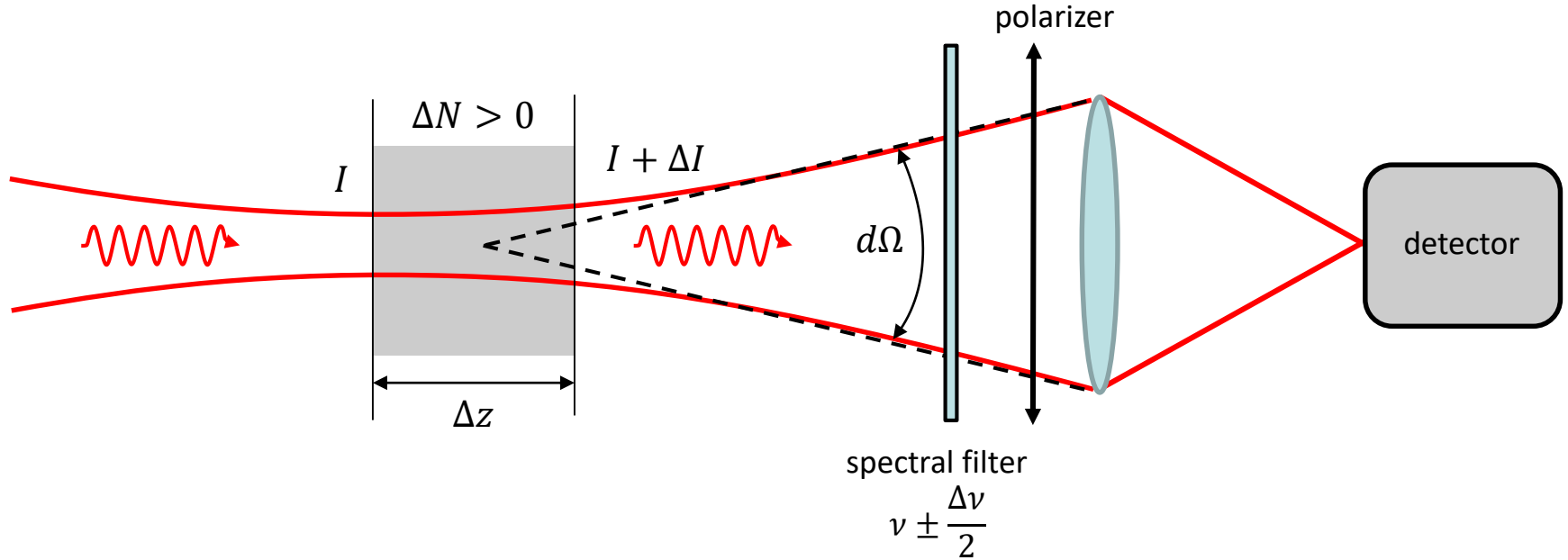
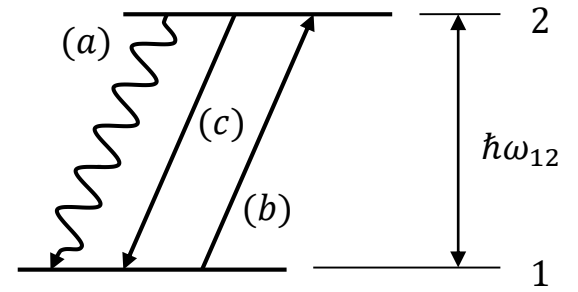
something is wrong; we know from lecture 2 that

$$\frac{d}{dt}\Delta N = -\frac{1}{T_1}(\Delta N - \Delta N_0) - 2\sigma\Delta NF$$

- the rate of stimulated emission is OK
- In lecture 2 we have ignored spontaneous emission!

spontaneous emission

Again, consider a thin slice of the amplifying medium. The thing we measure is intensity and the detector cannot distinguish between photons from stimulated and spontaneous emission.



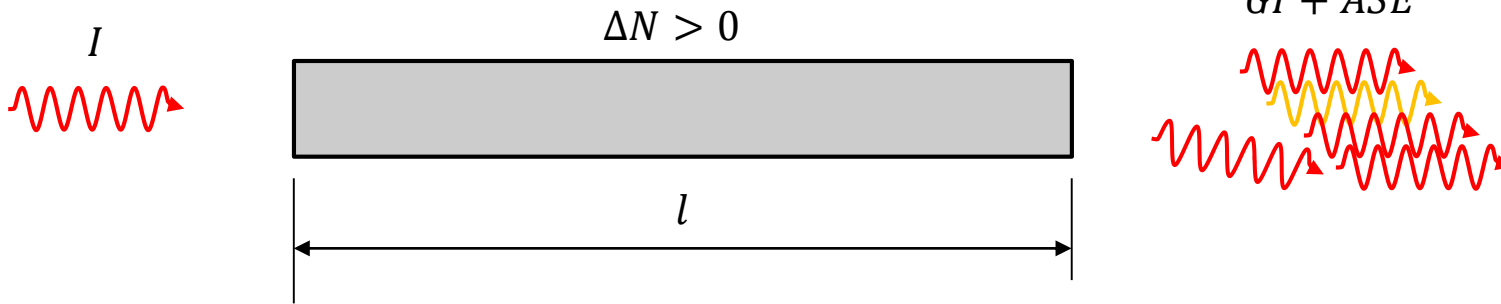
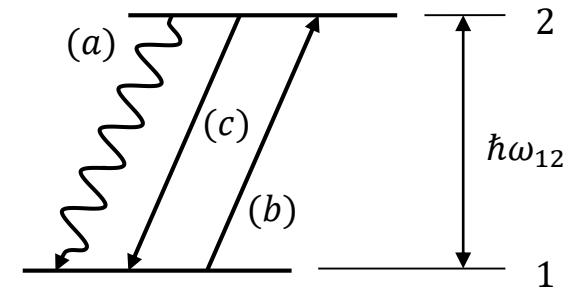
$$\Delta I(\nu) = \underbrace{\sigma(\nu)\Delta N I(\nu)}_{\text{coherent light - amplification}} + \underbrace{h\nu \times A_{21} \times g(\nu)\Delta\nu \times \frac{1}{2} \times \frac{d\Omega}{4\pi} \times N_2 dz}_{\text{incoherent light - spontaneous emission}}$$

energy of a photon transition rate spectral factor polarization solid angle number of atoms

in the laser amplifier:
spontaneous emission = noise

Amplified Spontaneous Emission (ASE)

when the medium is long and/or amplification coefficient large $\gamma_0 l \gg 1$ the spontaneous emission can be amplified to macroscopic intensities



No simple and convenient formulas for accounting for ASE. The spontaneous emission rate is given by $\frac{1}{2} h\nu$ for every spatio-temporal mode of the amplifier.

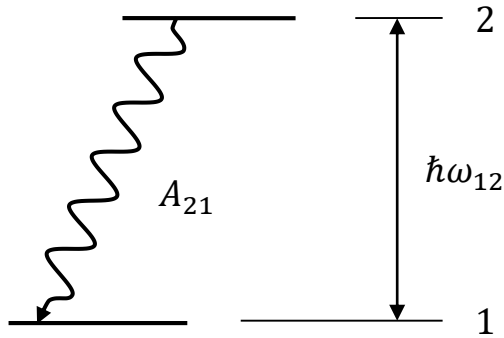
Consequences of ASE:

- ❑ Noise at the output of the amplifier
- ❑ In extreme cases ASE can saturate amplifier

Simple rule: to avoid problems with ASE **the input intensity has to be much larger than the spontaneous emission intensity**

homogenous line-broadening (we cannot address atoms by spectral methods)

- Natural broadening (always present). At least one of the two energy levels involved in light amplification corresponds to an excited state which has a finite life-time because atoms spontaneously drop to lower energy levels while emitting photons. In addition, in condensed phase, the life-time can be shortened by non-radiative transition which increases the total transition rate.

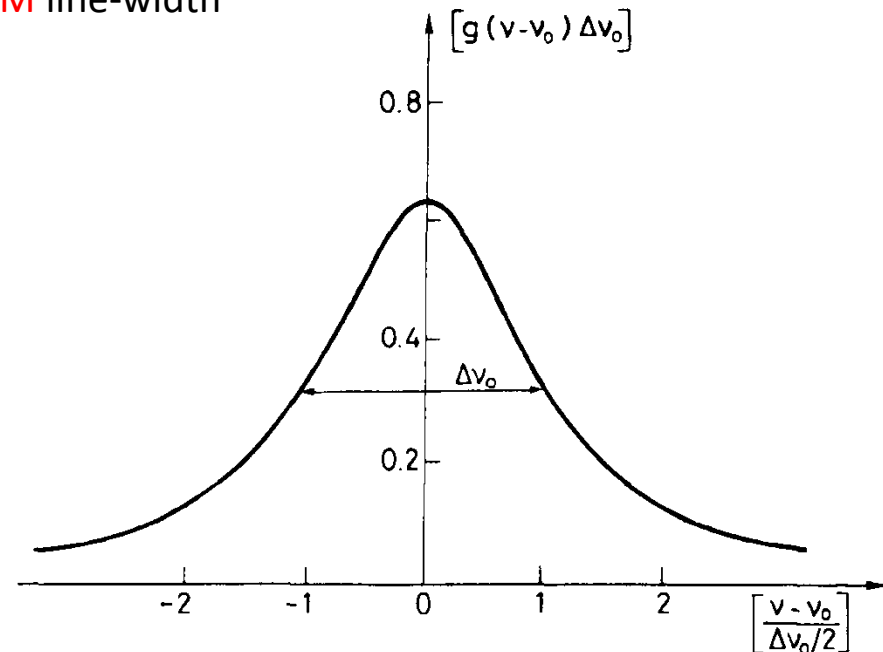


spontaneous emission leads to the Lorentzian line – shape

$$g(\nu) = \frac{\Delta\nu}{2\pi[(\nu-\nu_0)^2 + (\Delta\nu/2)^2]}, \quad g(\omega) = 2\pi g(\nu)$$

with **FWHM** line-width

$$\Delta\nu = \frac{A_{21}}{2\pi}$$



FWHM – Full Width at Half-Maximum

homogenous line-broadening, 2

□ Pressure broadening

probability density for atomic collisions in gas phase

$$p(\tau) = \frac{e^{-\tau/\tau_c}}{\tau_c}$$

$p(\tau)d\tau$ – the probability for that the atom to undergo a collision in the time interval $\tau, \tau + d\tau$

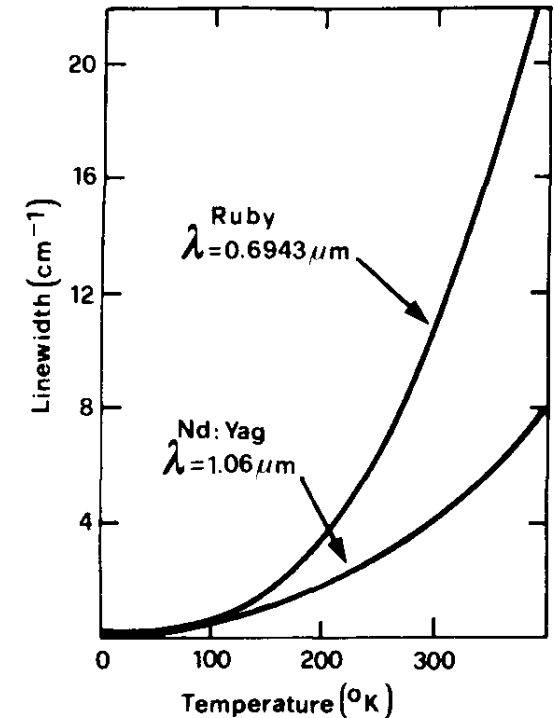
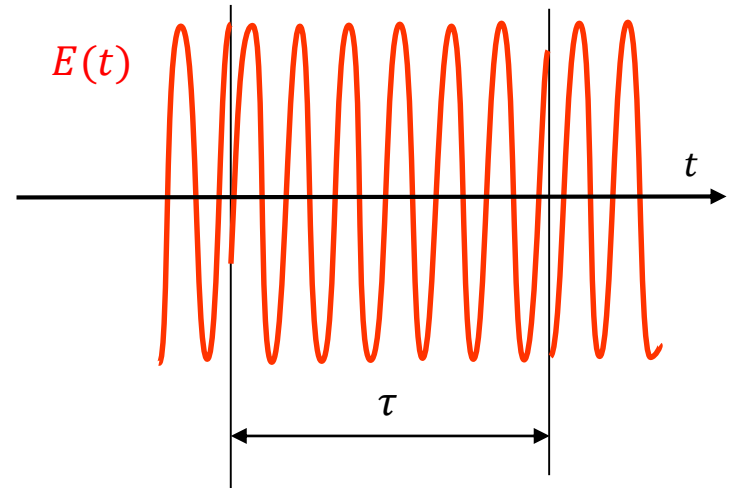
calculations.....

$$g(\nu) = \frac{\Delta\nu}{2\pi[(\nu-\nu_0)^2 + (\Delta\nu/2)^2]}$$

with $\Delta\nu = \pi/\tau_c$.

in glasses and crystals the interrupting events are phonons

for gases \cong MHz/mbar



inhomogenous line-broadening (we can address atoms by spectra methods)

□ Doppler broadening

Doppler shift; if the atoms moves slowly compared to the speed of light in vacuum ($v \ll c$) the largest shift comes from linear Doppler effect, which depends on the velocity component parallel to the direction of observation (we assume v_z):

$$v' = \left(1 + \frac{v_z}{c}\right)v \Rightarrow v_z = \frac{c}{v} (v' - v)$$

the convention is that $v_z > 0$ for atom moving towards the light source (absorption) or the observer (emission).

For gas at the temperature T the velocity distribution is given by Maxwell function

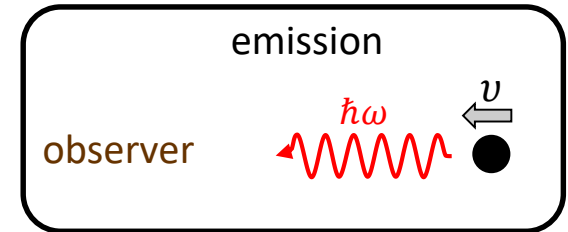
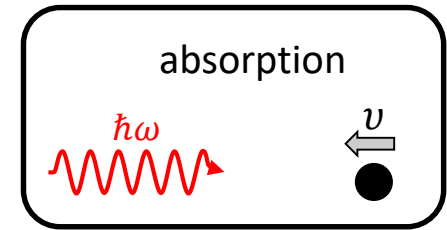
$$p(v) = \left(\frac{M}{2\pi kT}\right)^{1/2} \exp[-(Mv_z^2/T)]$$

Let's mark the resonant frequency in atom is by ν_0 and let's assume that homogenous line-broadening is small. Then the line-shape function is given by Gaussian function:

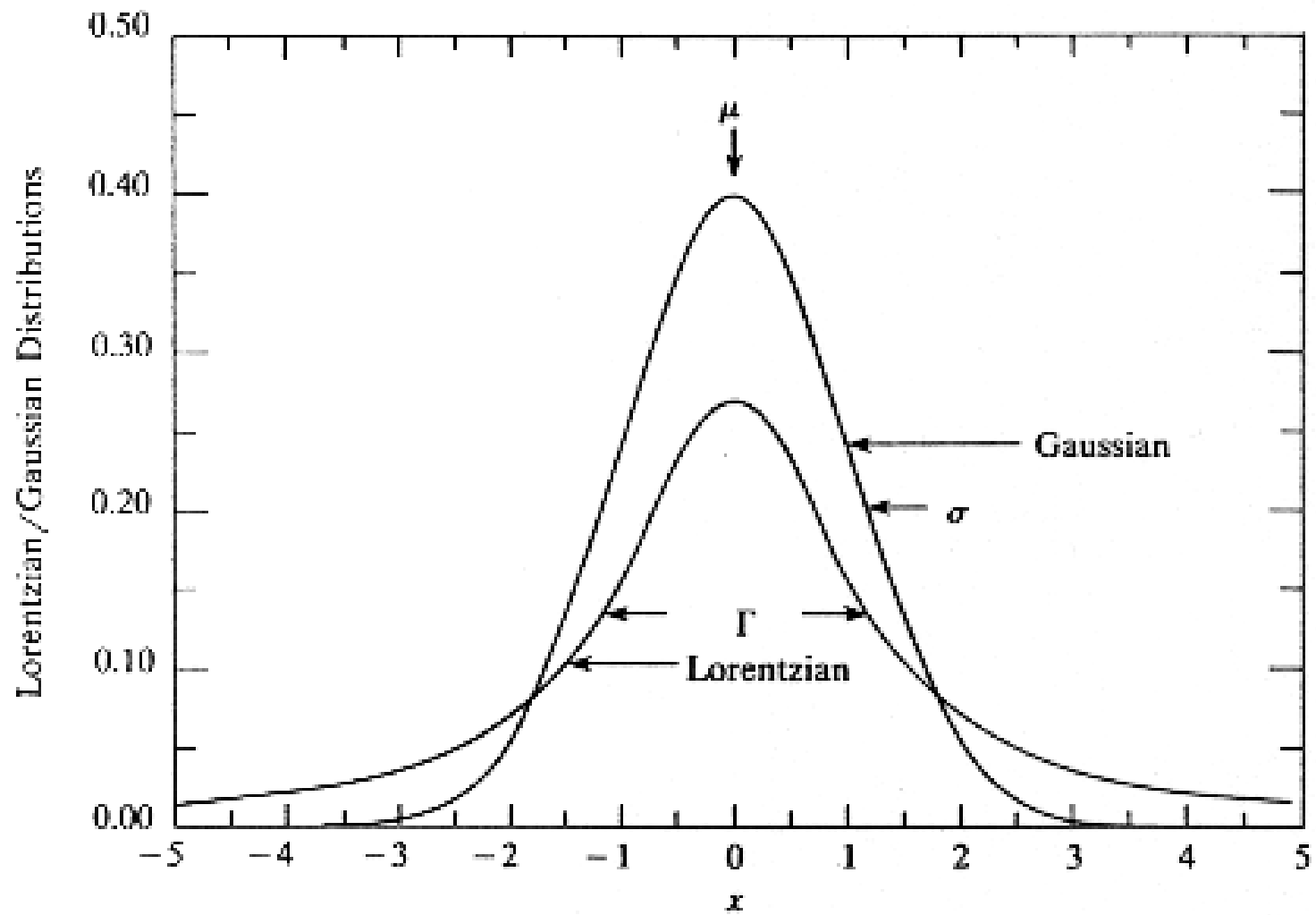
$$g(\nu) = \frac{1}{\nu_0} \left(\frac{M}{2\pi kT}\right)^{1/2} \exp\left[-\frac{Mc^2}{2kT} \frac{(\nu - \nu_0)^2}{\nu_0^2}\right]$$

with FWHM:

$$\Delta\nu = 2\nu_0 \left(\frac{2kT \ln 2}{Mc^2}\right)^{1/2}$$



Gauss vs Lorentz



mixed line-broadening

example: Doppler broadening + collisional broadening

Atoms with a given value of v_z are characterized by homogeneously broadened

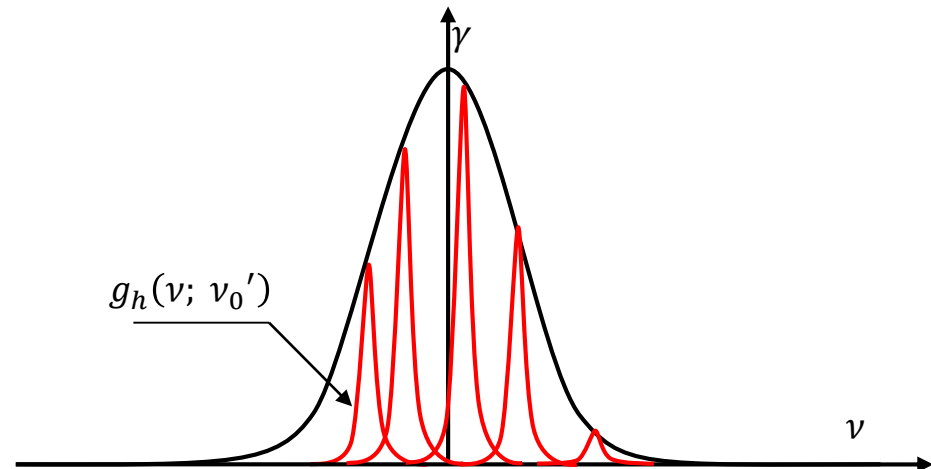
$$g_h(\nu') = \frac{\Delta\nu}{2\pi[(\nu' - \nu_0')^2 + (\Delta\nu/2)^2]}$$

Index h signifies homogenous (in this example collisional broadening), $\Delta\nu$ is the linewidth of homogenous broadening, and $\nu_0' = \left(1 + \frac{v_z}{c}\right)\nu_0$ the Doppler shifted resonance frequency .

The probability of atom having a given value of v_z is given by Maxwell's distribution , we integrate over the possible values of ν_0'

$$g_V(\nu) = \left(\frac{M}{2\pi kT}\right)^{1/2} \int_{-\infty}^{\infty} d\nu_0' \frac{\Delta\nu}{2\pi[(\nu - \nu_0')^2 + (\Delta\nu/2)^2]} \exp\left[-\frac{Mc^2}{2kT} \frac{(\nu - \nu_0')^2}{\nu_0^2}\right]$$

$g_V(\nu)$ is **Voigt's profile**.

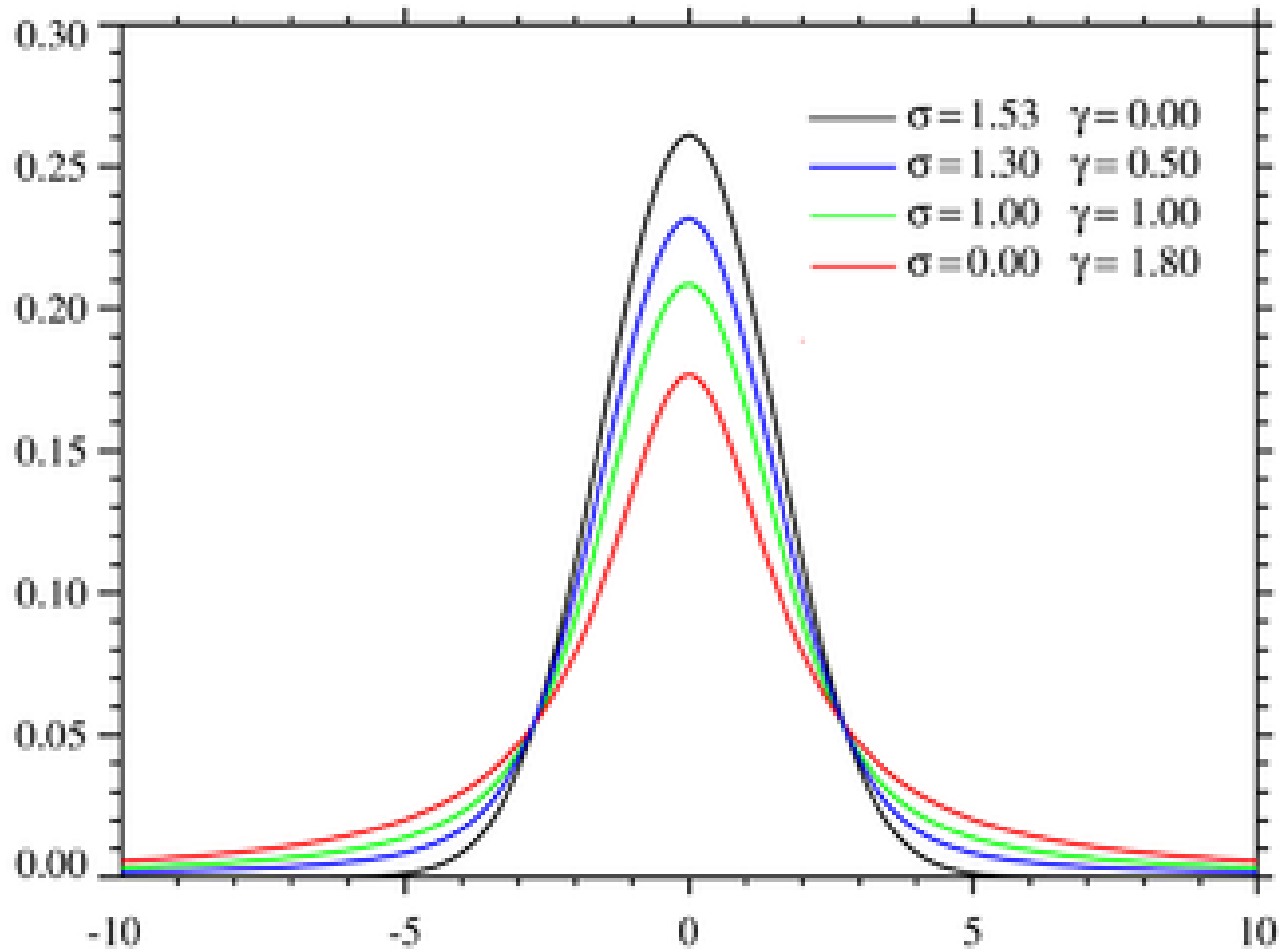


mixed line-broadening, 1

Voigt's profile is a convolution of Lorentz and Gauss functions

$$g_V(x) = \int_{-\infty}^{\infty} dx' G(x'; \sigma) L(x - x'; \gamma)$$

$$G(x; \sigma) \equiv \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sigma\sqrt{\pi}}, \quad L(x; \gamma) \equiv \frac{\gamma}{\pi(x^2 + \gamma^2)}$$



absorption coefficient α

$$\alpha \propto p \times 1/\Delta\nu$$

mixed line-broadening, 2

Doppler broadening: $\alpha \propto p$

pressure broadening: $\alpha \propto p \times 1/p = \text{const}$

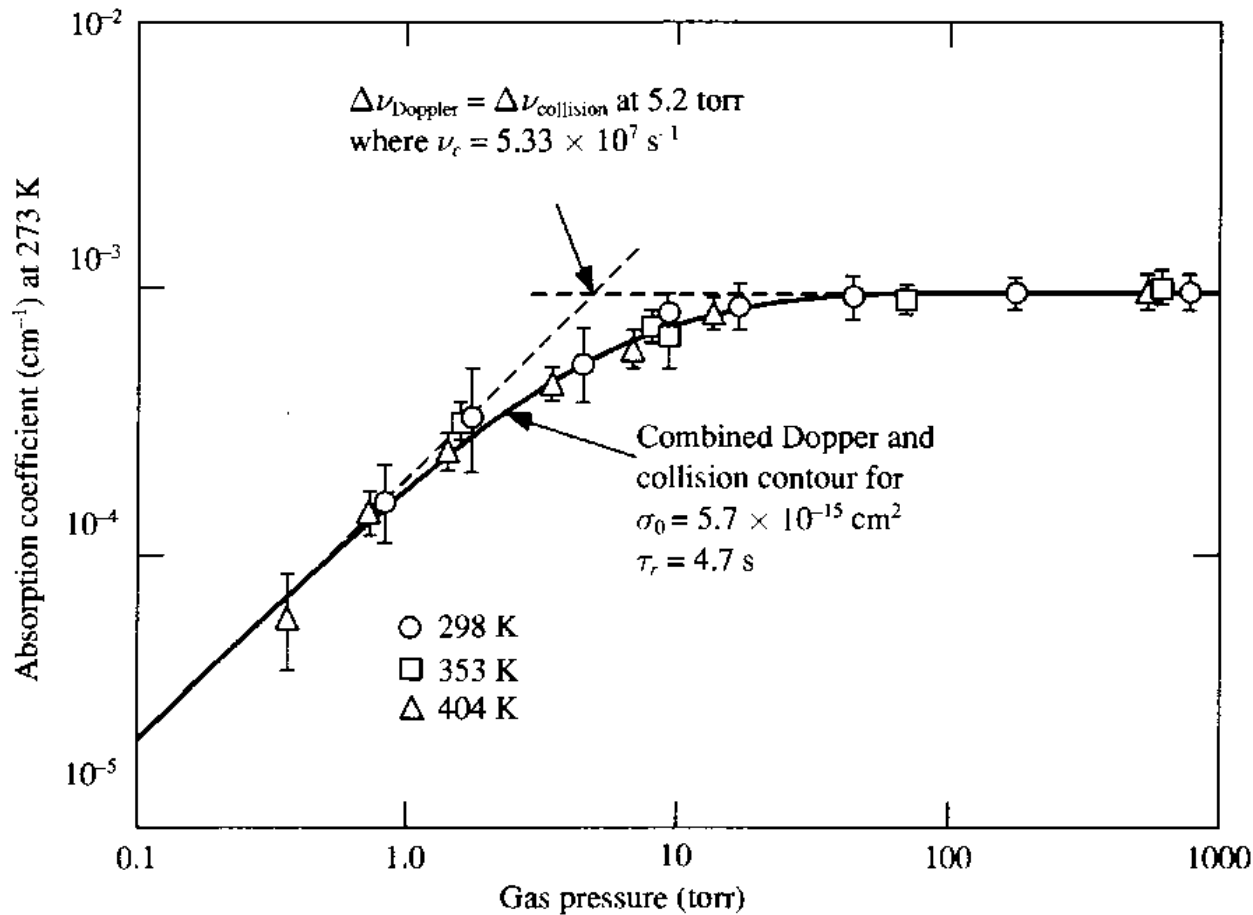


FIGURE 7.9. Absorption coefficient in CO_2 at $10.6 \mu\text{m}$ as a function of CO_2 pressure. (After E. T. Gerry and D. A. Leonard, *Appl. Phys. Lett.* 8, 227, 1966.)

"typical" linewidths

	effect	gas	liquid	condensed matter
homogenous	natural	0.001Hz-10MHz	n *	n
	atomic collision	5-10MHz/mbar	≈ 300 cm ⁻¹	----
	phonons	---	---	≈ 10 cm ⁻¹
inhomogeneous	Doppler	50MHz-1GHz	n	---
	Local fields	---	≈ 500 cm ⁻¹	1-500 cm ⁻¹

*n - negligible

cm⁻¹ units are often used in spectroscopy

$$\tilde{\nu} \equiv \frac{1}{\lambda[\text{cm}]}$$

$$\tilde{\nu} \equiv \frac{1}{\lambda} [\text{cm}^{-1}] = \frac{\nu}{c \left[\frac{\text{cm}}{\text{s}} \right]} = 10^{-2} \frac{\nu}{c}$$

numbers: $\lambda = 1 \mu\text{m} \Leftrightarrow 10\,000 \text{ cm}^{-1}$
 for $\lambda = 1 \mu\text{m}$: $1 \text{ cm}^{-1} = 30\text{GHz}$

gain saturation in media with different line broadening

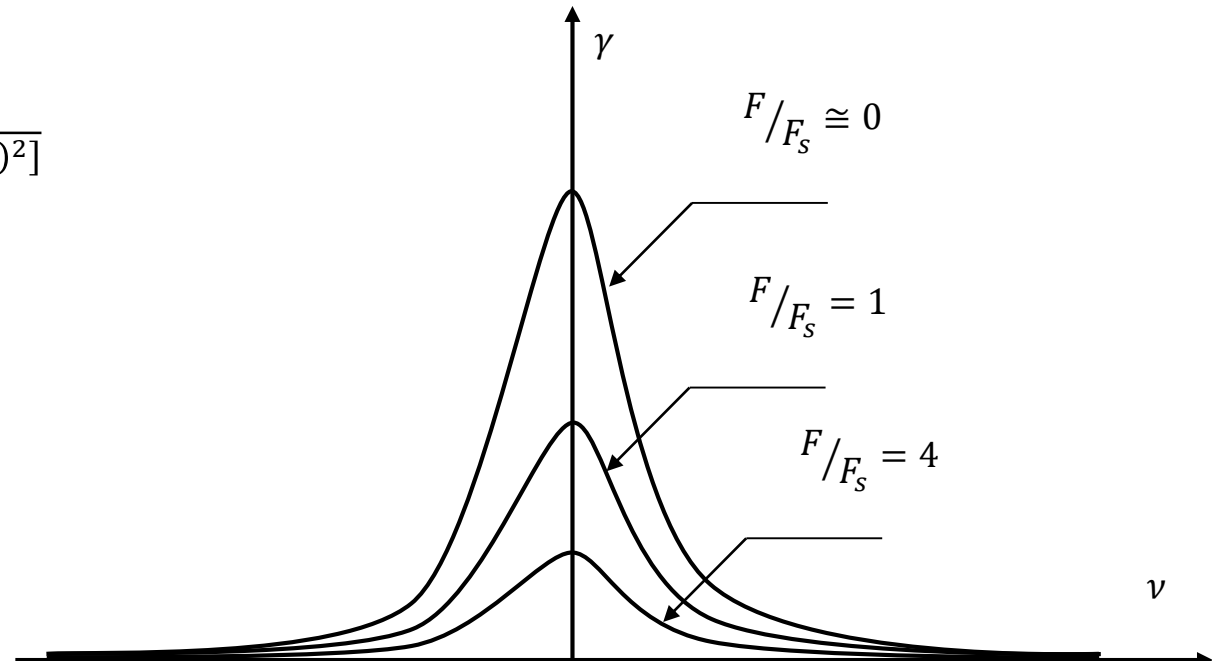
We will concentrate on the case $\tau_p \gg T_1$. Similar reasoning can be extended to other cases as well.

$$\gamma(F) = \frac{\gamma_0}{1 + F/F_s}$$

- Homogenous broadening dominates. As the population inversion decreases the gain drops for all frequencies because all atoms interact with the em wave in the same way. – Saturation requires higher intensities for frequencies far away from the resonance.

$$\gamma_0(\nu) \propto \frac{\Delta\nu}{2\pi[(\nu - \nu_0)^2 + (\Delta\nu/2)^2]}$$

$$\gamma(F, \nu) = \frac{\gamma_0(\nu)}{1 + F/F_s}$$



gain saturation in media with different line broadening, 2

- inhomogeneous broadening dominates. A monochromatic em wave of frequency ν interacts only with atoms that have their resonant frequencies close to ν (closer than homogenous linewidth). The saturation affects only this selected group of atoms – other groups „do not see” the em field.

$$\gamma_0(\nu) \propto \int_{-\infty}^{\infty} d\nu_0' \underbrace{\frac{\Delta\nu}{2\pi[(\nu - \nu_0')^2 + (\Delta\nu/2)^2]}}_{g_h(\nu, \nu_0')} g(\nu_0')$$

$g_j(\nu, \nu_0')$
homogeneously broaden line centered at ν_0' .

Saturation „burns a hole” in the gain profile. Its width corresponds to the homogenous linewidth. The depth of the hole scales with saturation (em field intensity).

