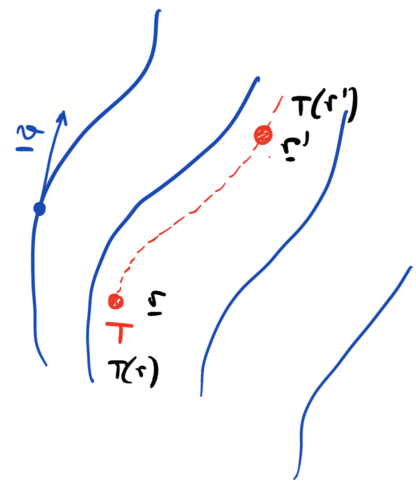
Equations of motion for a fluid

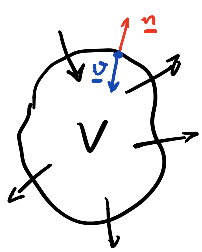
Material derivative



$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \underline{v}\cdot\underline{\nabla} T$$

Mass conservation



$$\frac{d}{dt} M_V = -\oint_{\partial V} \rho\, \underline{v}\cdot\underline{n}\, dS$$

divergence theorem

$$\frac{d}{dt}\int_V \rho\, dV = -\int_V \underline{\nabla}\cdot(\rho\, \underline{v})\, dV$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\, \underline{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\underline{\nabla}\cdot\underline{v}) + \underline{v}\cdot\nabla\rho = 0$$

$$\frac{D\rho}{Dt} = -\rho\,(\nabla\cdot\underline{v})$$

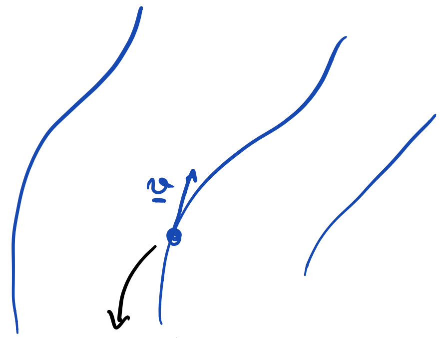
incompressible fluid

$$\frac{D\rho}{Dt} = 0 \Rightarrow (\nabla\cdot\underline{v}) = 0$$

Momentum conservation

- momentum advection

momentum gets carried (along streamlines)

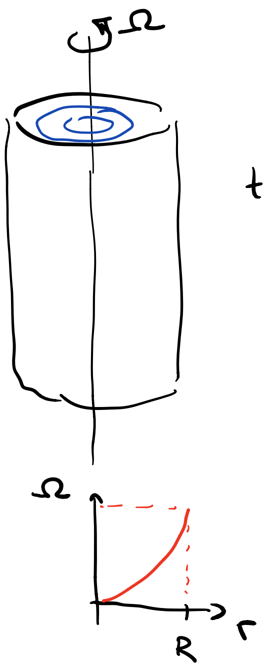


flux of momentum $\longleftarrow$ momentum

$(\rho \underline{v}) \cdot \underline{v}$ $\qquad$ $\rho \underline{v}$

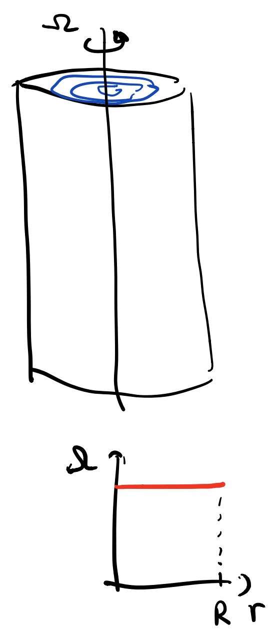
$\sim \dfrac{\rho v^2}{2}$

- Momentum diffusion

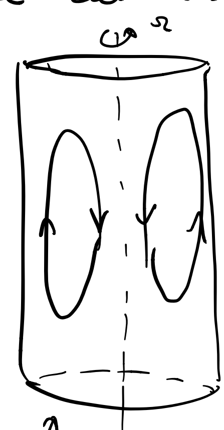


$t \to \infty$

$\Rightarrow$



In the real world:



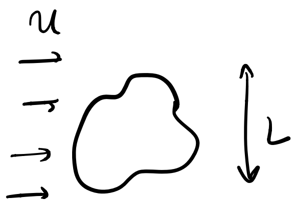
bottom

(secondary flow)

Navier - Stokes equation

$\underline{u}, p, \mu, \rho$

$$\rho \frac{D\underline{u}}{Dt} = \underbrace{-\nabla p + \mu \nabla^2 \underline{u} \quad = \rho \left[\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} \right]}$$



Introduce dimensionless variables:

$$x^* = \frac{x}{L} \qquad \underline{u}^* = \frac{\underline{u}}{U}, \quad t^* = \frac{t}{T}, \quad p^* = \frac{p}{\frac{\mu U}{L}}$$

$$\rho\left[\frac{U}{T}\frac{\partial \underline{u}^*}{\partial t^*} + \frac{U^2}{L}(\underline{u}^*\cdot\nabla^*\underline{u}^*)\right] = -\frac{\mu U}{L^2}\nabla^* p^* + \frac{\mu U}{L^2}\nabla^{*2}\underline{u}^*$$

$$\underbrace{\frac{\rho L^2}{\mu T}}_{Re\cdot St}\frac{\partial \underline{u}^*}{\partial t^*} + \underbrace{\rho\frac{UL}{\mu}}_{Re}(u^*\cdot\nabla u^*) = -\nabla^* p^* + \nabla^{2*}\underline{u}^*$$

$Re = \frac{\rho U L}{\mu}$ Reynolds number

$St = \frac{L}{UT}$ Strouhal number

$$Re\left[St\frac{\partial u}{\partial t} + (u\cdot\nabla\underline{u})\right] = -\nabla p + \nabla^2 u$$

(i) $St \ll 1$ quasi-steady limit → ok to ignore $\frac{\partial u}{\partial t}$

(ii) $St \gg 1$ rapidly changing flow → ok to ignore $\underline{u}\cdot\nabla\underline{u}$

(iii) $Re \ll 1$ viscosity dominated → ok to ignore LHS

(iv) $Re \gg 1$ inertially dominated → ok to ignore $\nabla^2 u$ far from boundaries