SELECTED TOPICS LECTURE 2

{ Waves and capillary phenomena

IDEAL FLUID

$\eta = 0$

Feynman → "dry water"

1° continuum mech. : no viscosity → energy & momentum conserved

2° ideal fluid flow - persistence of irrotational flow

irrotational flow $\underline{\omega} = \nabla \times \underline{v} = 0$ ← vorticity

3° N-S reduce to Euler eq.

$$\rho \frac{D\underline{v}}{Dt} = -\nabla p + \rho \underline{f}$$

↑ body force density

4° $\frac{D\underline{\omega}}{Dt} = 0$

5° Irrotational flow $\underline{\omega} = 0$ is automatically satisfied if

$$\underline{v} = \nabla \phi$$ ← velocity potential

Such flows are called <u>potential flows</u>

6° Incompressible fluid $\underline{\nabla} \cdot \underline{v} = 0$

$$\nabla \cdot \nabla \phi = \underbrace{0 = \nabla^2 \phi}$$ Laplace's eq.

<u>Bernoulli's theorem</u>

- Ideal fluid, potential flow $\rightarrow \phi(\underline{r})$
- conservative force with potential ψ

$$\rho \frac{D}{Dt} \nabla \phi = -\nabla p + \rho \nabla \psi$$

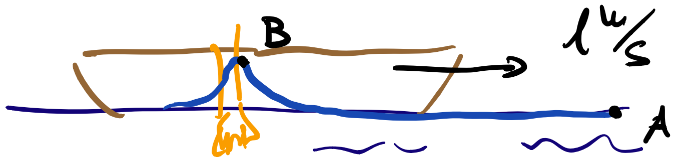
$\rho = \text{const.}$

$$\rho \left(\frac{\partial}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \right) \nabla \phi = -\nabla p + \rho \nabla \psi$$

$$\nabla \left(\rho \frac{\partial \phi}{\partial t} \right) + \rho \underbrace{\underline{v} \cdot \nabla \underline{v}}_{\nabla \left(\frac{1}{2} \rho v^2 \right)} = -\nabla p + \rho \nabla \psi$$

$$\nabla \left(\rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho v^2 + p + \rho \psi \right) = 0$$

$$\underbrace{\rho \frac{\partial \phi}{\partial t} + \frac{\rho v^2}{2} + p + \rho \psi = 0}_{\text{Bernoulli's theorem}}$$ throughout the fluid volume



stationary $\frac{\partial \phi}{\partial t} = 0$

A) $z=0$, $p=p_a$, $v = 1\frac{m}{s}$

B) $z=h$, $p=p_a$, $v=0$

$$\frac{\rho v^2}{2} + p + \rho g z = \text{const.}$$

$$\rightarrow \quad h = \frac{1}{2g} \sim 5\,\text{cm}$$

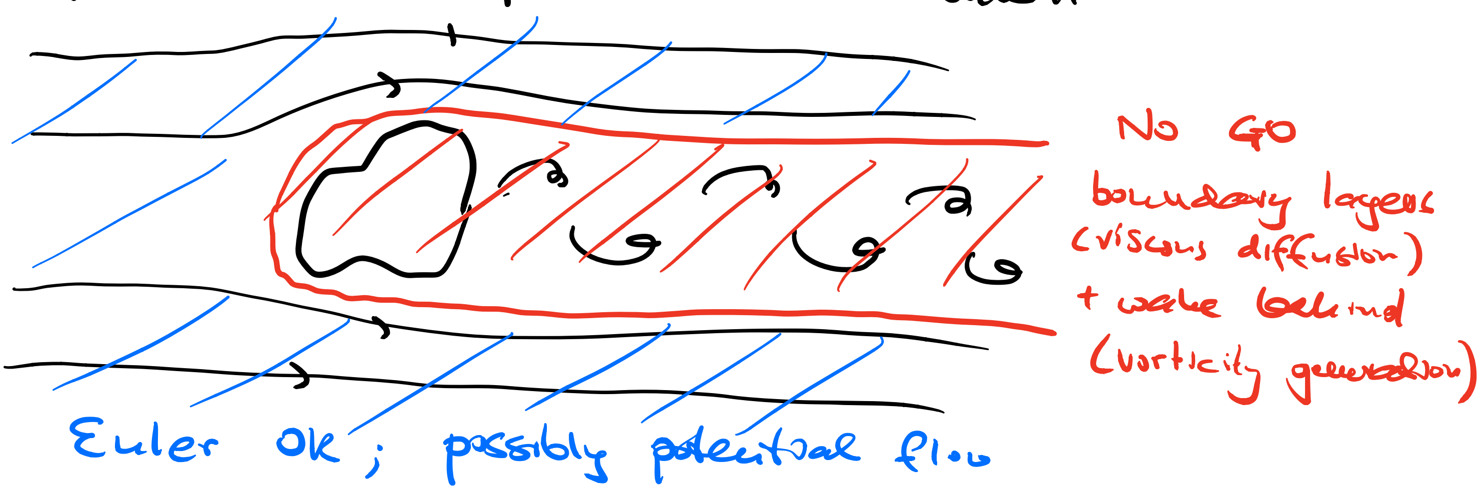
When is the ideal fluid approximation valid?

a) a fluid with $\eta = 0$

liquid ^{4}He below 2.712 K is a mixture of two fluids, one of which has zero viscosity.

b) $$Re = \frac{\rho u L}{\eta} = \frac{\text{'inertial'}}{\text{'viscous'}}$$

large Re flows far away from walls, if the bulk flow is not turbulent

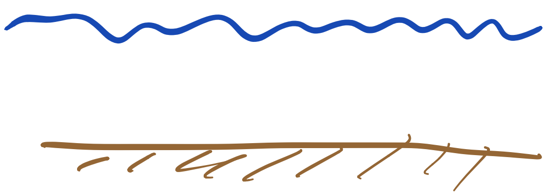


Potential flows → Acheson

SURFACE WAVES

- can be treated in the ideal fluid approximation

→ neglect flow attenuation effects from viscosity



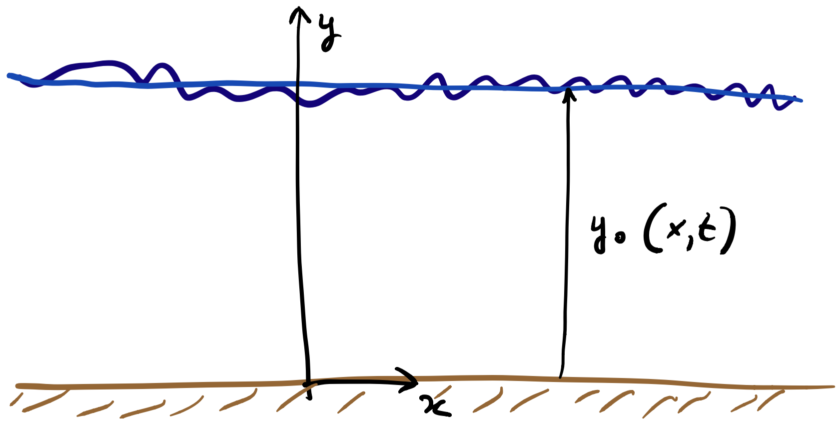
surface deformation + two restoring mechanisms:

- gravity counteracts deviations from the horizontal
- capillary forces (surface tension) oppose any curvature of the surface

Dispersion - dependence of the wave speed on the wave vector (direction & frequency)

Waves are generally dispersive and non-linear (speed depends on the amplitude). For small amplitudes we linearise the equations.

Dispersion relation for surface waves



- avg. depth h
- 2D
- seek solutions for waves

- assume potential flow $\phi(x,y,t)$
- assume small amplitude, so terms

$$|(\underline{v}\cdot\underline{\nabla})\underline{v}| \text{ are small} \quad \ll \frac{\partial \underline{v}}{\partial t} \quad \ll \nabla p$$

- small curvature of the interface

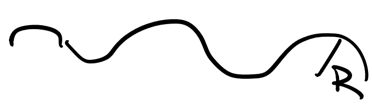
y_0 - instantaneous position of the surface

$$\frac{\partial y_0}{\partial x} \ll 1$$

We will use

$$\underbrace{\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gy = \text{const.}}_{} \qquad (**)$$

The pressure on the surface



$$p = p_0 - \frac{\gamma}{R} \qquad R = 1\Big/\left(\frac{\partial^2 y}{\partial x^2}\right)$$

γ - surface tension coefficient

R - local radius of curvature of the interface

Boundary conditions:

(a) bottom wall – no normal velocity $\underline{v} = \nabla\phi$

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=0} = 0$$

(b) top surface – vertical component at interf. equal to interface velocity

$$v(y=y_0) = \left.\frac{\partial\phi}{\partial y}\right|_{y=y_0} = \frac{\partial y_0}{\partial t}$$ (approximately, within linear theory)

We seek a solution to incompressible potential flow

$$\left.\begin{aligned} \underline{\nabla}\cdot\underline{v} &= 0 \\ \underline{v} &= \underline{\nabla}\phi \end{aligned}\right\} \quad \nabla^2\phi = 0 \qquad (*)$$

+ b.c. as above

Separation of variables

- need a solution in the form of a travelling wave

$$f(u) = f(x-ct)$$

(with speed c in the x-direction)

amplitude $q(y)$

$$\Rightarrow \quad \phi(x,y,t) = f(u)\,q(y) = f(x-ct)\,q(y)$$

From (*)

$$\frac{\partial^2 f}{\partial^2 u}\,q + f\,\frac{\partial^2 q}{\partial y^2} = 0 \Rightarrow \frac{1}{f}\frac{\partial^2 f}{\partial u^2} = -\frac{1}{q}\frac{\partial^2 q}{\partial y^2}$$

So we must have

$$\frac{1}{f}\frac{\partial^2 f}{\partial u^2} = -\frac{1}{q}\frac{\partial^2 q}{\partial y^2} = -k^2 = \text{const.}$$

B.C. (a) $q'(0) = 0$

(b) $q'(y_0) = y_0'(t)$

$q'' = k^2 q$

$q = Ae^{ky} + Be^{-ky}$

$q' = kAe^{ky} - kBe^{-ky}$

$q'(0) = 0 \Rightarrow A = B$

$q = A\cosh(ky)$

We must have

$$\phi = f(x-ct)\,q(y) =$$
$$= A e^{i(kx-\omega t)}\cosh(ky)$$

$\frac{\partial}{\partial t}$ (Bernoulli eq) gives

$$\frac{\partial^2 \phi}{\partial t} + g\frac{\partial y_0}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial t} = 0$$

now use $p = p_0 - \frac{\gamma}{R} = p_0 - \gamma\left(\frac{\partial^2 y}{\partial x^2}\right)$

and (b) $\left.\frac{\partial \phi}{\partial y}\right|_{y_0} = \frac{\partial y_0}{\partial t}$

We finally get:

$$\left[\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} - \frac{\gamma}{\rho}\frac{\partial^3 \phi}{\partial x^2 \partial y}\right]_{y=y_0} = 0$$

using the solution for ϕ, we get: $\phi = Ae^{i(kx-\omega t)}\cosh(ky)$

$$-A\omega^2 \cosh(ky_0) + Agk\sinh(ky_0) + \frac{\gamma}{\rho}\sinh(ky_0)k^3A = 0$$

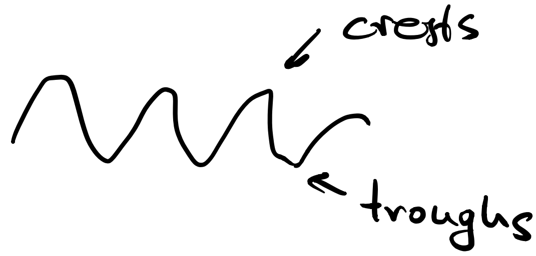
We get the dispersion relation

avg. value of y_0

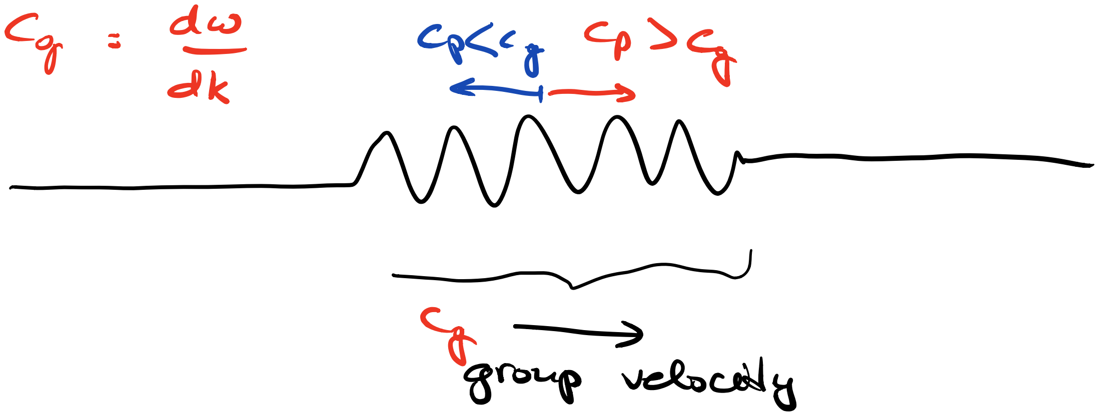
$$\boxed{\omega^2 = \left(gk + \frac{\gamma k^3}{\rho}\right)\tanh(kh)}$$

Dispersion relation for water waves

Phase velocity: $c_p = \dfrac{\omega}{k}$



Group velocity: $c_g = \dfrac{d\omega}{dk}$



Phase velocity of water waves

$$c = \frac{\omega}{k} = \frac{g}{k}\tanh(kh)\left(1 + \frac{\gamma k^2}{\rho g}\right)$$

Length scales

- depth h
- wave length $\lambda = \dfrac{2\pi}{k}$
- capillary length $\ell_c = \sqrt{\dfrac{\gamma}{\rho g}}$

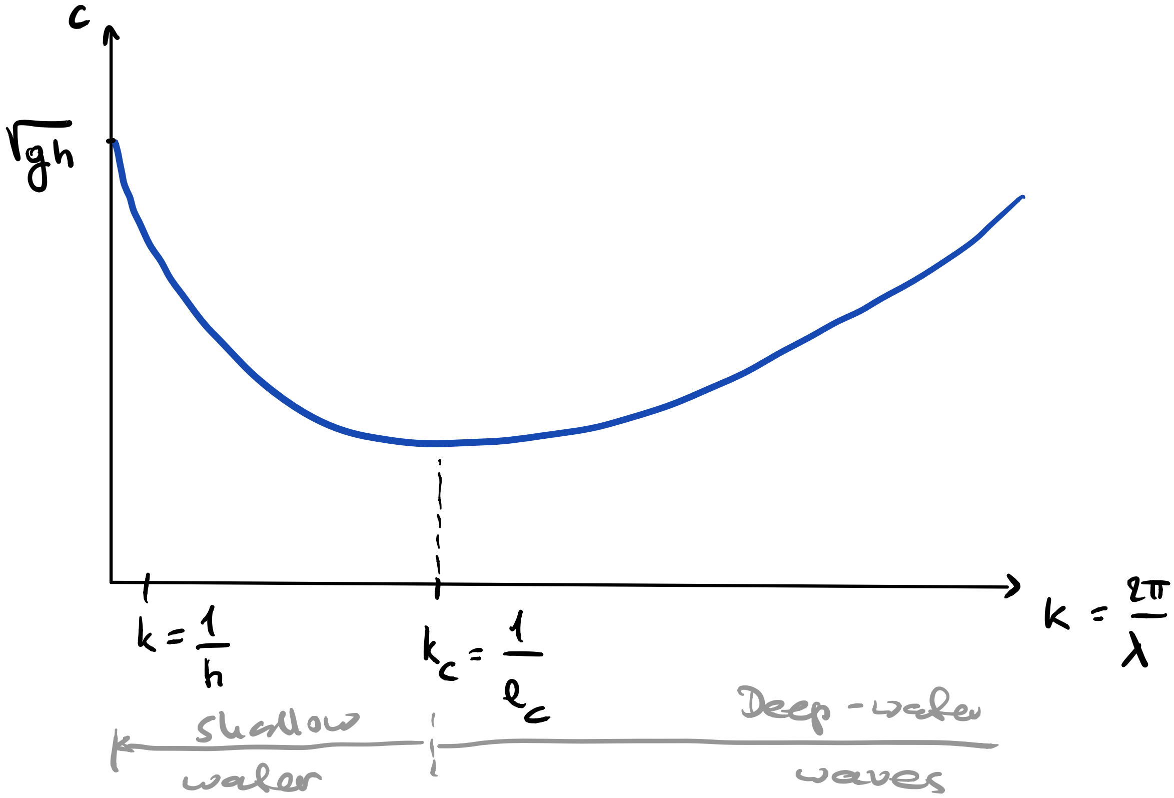
- amplitude A (small)

$\gamma \sim kg \cdot \frac{1}{s^2}$ $\quad \rho \sim \frac{kg}{m^3}$

$g = \quad \frac{m}{s^2}$

$$c^2 = \frac{g}{k}\tanh(kh)\left(1 + k^2 l_c^2\right)$$

we consider here $h \gg l_c$ — for water $l_c \sim 3\,mm$



(A) If $h \gg \lambda$, $\tanh(kh) \approx 1$

$$c^2 = \frac{g}{k}\left(1 + k^2 l_c^2\right)$$

deep-water waves

(B) If also $k l_c \ll 1$ (λ large compared to l_c)

we have gravity waves

$$c = \sqrt{g/k}$$

(no external length scale)

ex. ocean swells

(C) for short wave lengths

$k l_c \gg 1$

$$c = \sqrt{g k}\, l_c = \sqrt{\frac{\gamma k}{\rho}}$$

ex. cat's paws (wind-caused)

(D) Minimum velocity corresponds to comparable gravity & capillary effects.

Critical wavelength $\lambda_c = 2\pi l_c = 2\pi\sqrt{\frac{\gamma}{\rho g}}$

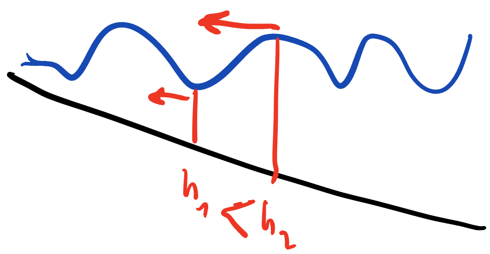
$\lambda_c \sim 17$ mm

$c(\lambda_c) \sim 23 \frac{cm}{s}$

(E) When both $kh \ll 1$, $kl_c \ll 1$

$c = \sqrt{gh}$ shallow-water waves

Ex. Breaking waves



Ex. Turning waves