Conservation laws

rate of change of — flux — source

$$\frac{\partial}{\partial t}A + \nabla\cdot(A\underline{v}) = (***)$$

Conservation of mass (ρ)

$$\underbrace{\frac{\partial \rho}{\partial t}}_{\text{rate of change}} + \underbrace{\nabla\cdot(\rho\underline{v})}_{\text{flux of mass}} = \underbrace{0}_{\text{source}}$$

or

$$\frac{D\rho}{Dt} + \rho(\underline{\nabla}\cdot\underline{v}) = 0$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_j}(\rho v_j)$$

Conservation of momentum

Volume density of momentum $\rho\underline{v}$

$$\frac{\partial}{\partial t}(\rho\underline{v}) = \underbrace{\underline{v}\frac{\partial \rho}{\partial t}}_{\text{cons. of mass}} + \underbrace{\rho\frac{\partial \underline{v}}{\partial t}}_{\text{N-S eq.}}$$

$$\rho\frac{D\underline{v}}{Dt} = \rho\left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}\right] = -\nabla p + \mu\nabla^2\underline{v} + \rho\underbrace{\underline{f}}_{\text{body force density}}$$

more generally: $\nabla\cdot\underline{\underline{\sigma}}'$ ← viscous stress tensor

Components:

$$\frac{\partial}{\partial t}(\rho v_i) = -v_i\frac{\partial(\rho v_j)}{\partial x_j} - \rho v_j\frac{\partial v_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{\partial \sigma'_{ij}}{\partial x_j} + \rho f_i \quad (*)$$

This eq. is valid for all fluids – Newtonian or not, compressible or not.

Incompressible Newtonian fluid

$$\sigma'_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad \rightarrow \quad \nabla \cdot \underline{\underline{\sigma}}' = \mu \nabla^2 \underline{v}$$

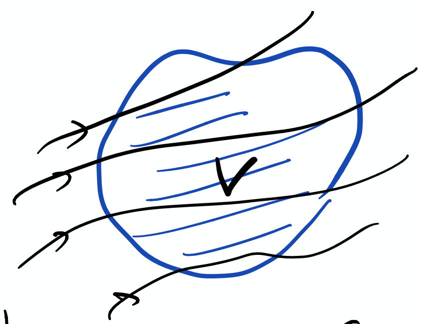
Let us integrate eq. (*) over a fixed volume V

$$\int_V \frac{\partial}{\partial t}(\rho v_i)\, dV = -\int_V \frac{\partial}{\partial x_j}\left(\rho v_i v_j + p\delta_{ij} - \sigma'_{ij}\right) dV + \int_V \rho f_i \, dV$$

Gauss' theorem:

$$\int_V \frac{\partial}{\partial t}(\rho v_i)\, dV = -\int_S (\rho v_i v_j + p\delta_{ij} - \sigma'_{ij})\, n_j \, dS + \int_V \rho f_i \, dV$$

V is fixed, so



$$\frac{d}{dt}\int_V (\rho v_i)\, dV = \ldots$$

In the vector form: (transport) (pressure) (viscous stresses)

$$\underbrace{\frac{d}{dt}\underbrace{\int_V (\rho \underline{v})\, dV}_{\text{Total momentum}}}_{\substack{\text{Rate of change} \\ \text{of total} \\ \text{momentum}}} = \underbrace{-\int_S \left(\rho \underline{v}(\underline{v}\cdot\underline{n}) + p\underline{n} - \underline{\underline{\sigma}}'\cdot\underline{n}\right) dS}_{\text{flux of momentum through } S = \partial V} + \underbrace{\int_V \rho \underline{f}\, dV}_{\substack{\text{Total force} \\ \text{(source term)}}}$$

For a stationary flow, we get

$$\int_S \rho \underline{v}(\underline{v}\cdot\underline{n})\,dS + \int_S p\underline{n}\,dS - \int_S \underline{\underline{\sigma}}'\cdot\underline{n}\,dS - \int_V \rho \underline{f}\,dV = 0$$

We can use this to determine forces on objects in flow

Back to ideal fluids $(\eta = 0\,;\ \underline{\underline{\sigma}}' = 0)$

Bernoulli's equation

Start from Euler eq.

$\underline{f} = \nabla\chi$
potential force

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v} = -\frac{1}{\rho}\nabla p - \nabla\chi$$

Now we use

$$(\underline{v}\cdot\underline{\nabla})\underline{v} = \nabla\left(\frac{1}{2}v^2\right) + (\underline{\nabla}\times\underline{v})\times\underline{v}$$

Proof

$$\text{RHS} = \partial_i\left(\frac{1}{2}v_j v_j\right) + \epsilon_{ijk}(\nabla\times\underline{v})_j v_k =$$

$$= \frac{1}{2}\partial_i(v_j v_j) + \epsilon_{ijk}\epsilon_{jlm}(\partial_l v_m)v_k = \left\{\epsilon_{ijk} = -\epsilon_{jik}\right\}$$

$$= v_j\frac{\partial v_j}{\partial x_i} - \underbrace{\epsilon_{jik}\epsilon_{jlm}}_{=(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})}(\partial_l v_m)v_k =$$

$$= \cancel{v_j\frac{\partial v_j}{\partial x_i}} - \cancel{v_m\frac{\partial v_m}{\partial x_i}} + v_k\frac{\partial v_i}{\partial x_k} = v_j\frac{\partial v_i}{\partial x_j} = \text{LHS}$$

Using this

$$\frac{\partial \underline{v}}{\partial t} + (\nabla \times \underline{v}) \times \underline{v} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}v^2 + \chi\right) \qquad (**)$$

__Before__ we found for potential & irrotational flow

$\underline{v} = \nabla\phi \qquad \underline{\nabla} \times \underline{v} = 0$

$$H = \frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}v^2 + \chi = \text{constant everywhere}$$

Now we do __not__ assume these, but let's assume the flow is stationary $\frac{\partial \underline{v}}{\partial t} = 0$

Then we find: $(**) \cdot \underline{v}$

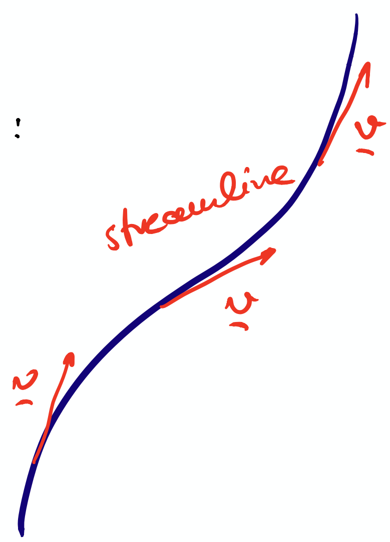
$$\underline{v} \cdot \nabla \underbrace{\left(\frac{p}{\rho} + \frac{1}{2}v^2 + \chi\right)}_{\text{Bernoulli function } H} = 0$$

What does H mean physically that:

$\underline{v} \cdot \nabla H = 0$

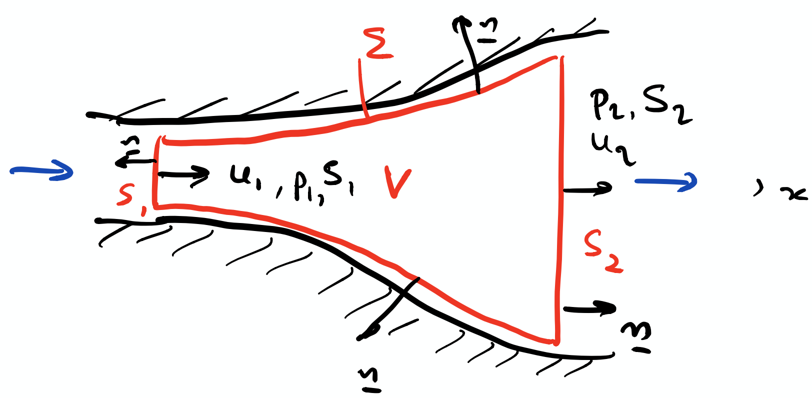
so $\nabla H \perp \underline{v}$, so

H is __constant on streamlines.__



$$\frac{p}{\rho} + \frac{1}{2}v^2 + \chi = \text{const on streamlines.}$$

Example The force on the walls of an axisymmetric channel with a variable cross-section



What is F_x ?

Conservation of momentum in the x-direction

$$\int_S \rho v_x (v_j n_j) + \int_S p\, n_x \, dS = 0 \qquad \left(\begin{array}{l}\text{no viscosity}\\ \text{no body force}\end{array}\right)$$

$$F_{p x \Sigma} - \int_{S_1} p\, n_x \, dS + \int_{S_2} p\, n_x \, dS - \int_{S_1} \rho v_x^2 dS + \int_{S_2} \rho v_x^2 dS = 0$$

force on lateral walls in the x-dir

$$\Rightarrow F_\Sigma = \int_{S_1} (p + \rho v_x^2) dS - \int_{S_2} (p + \rho v_x^2) dS$$

Assume uniform velocity

$$F_\Sigma = (p_1 S_1 + \rho u_1^2 S_1) - (p_2 S_2 + \rho u_2^2 S_2)$$

(•) Bernoulli's theorem (along a streamline)

$$p_1 + \frac{\rho u_1^2}{2} = p_2 + \frac{\rho u_2^2}{2}$$

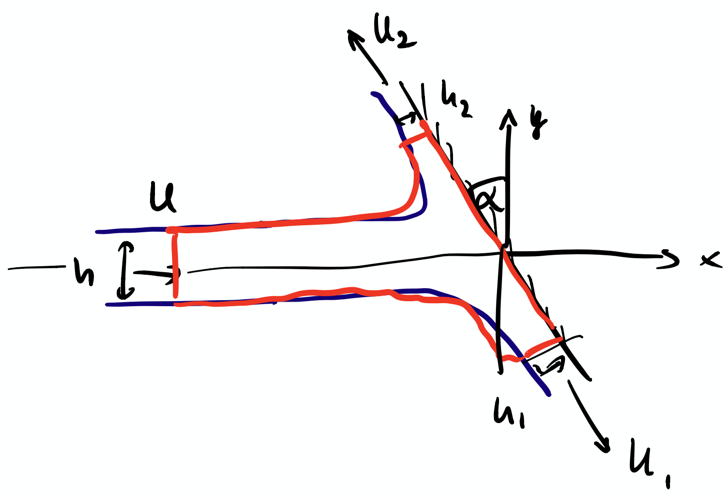
Constant flow rate $\quad u_1 S_1 = u_2 S_2 \quad$ (mass conservation)

We find

$$F = p_1 (S_1 - S_2) - \frac{1}{2} \rho u_1^2 S_1 \left(\sqrt{\frac{S_1}{S_2}} - \sqrt{\frac{S_2}{S_1}} \right)^2$$

Example 2 A jet incident on a plane

2D



We neglect gravity and viscosity.
(assume potential flow)

1) Mass conservation :

$$u h = h_1 u_1 + h_2 u_2$$

2) Bernoulli (everywhere)

$$p + \frac{1}{2} \rho u^2 = p_1 + \frac{1}{2} \rho u_1^2 = p_2 + \frac{1}{2} \rho u_2^2$$

$p = p_1 = p_2 = p_{atm}$ (atmospheric pressure)

We find : $u = u_1 = u_2$,

and : $h = h_1 + h_2$.

To calculate the force, we will use the conservation of momentum.

Choose control volume V, we find

x direction

$$\rho\left(u_1^2 h_1 \sin\alpha - u_2^2 h_2 \sin\alpha - u^2 h\right) + \overbrace{\int_{wall} (\delta p \, n_x)\, dS}^{F_x} = 0$$

y direction

$$\rho\left(- u_1^2 h_1 \cos\alpha + u_2^2 h_2 \cos\alpha\right) + \overbrace{\int_{wall} (\partial p \, n_y)\, dS}^{F_y} = 0$$

Thus:

$$F_x = \rho u^2 \left(h - (h_1 - h_2)\sin\alpha \right)$$

$$F_y = \rho u^2 (h_1 - h_2)\cos\alpha$$

But we know that $F_\parallel = 0$ (no viscous stresses)

$$F_\parallel = F_x \sin\alpha - F_y \cos\alpha = 0$$

$$\Rightarrow \quad h_1 - h_2 = h\sin\alpha$$

$$h_1 = \frac{h}{2}(1 + \sin\alpha)$$

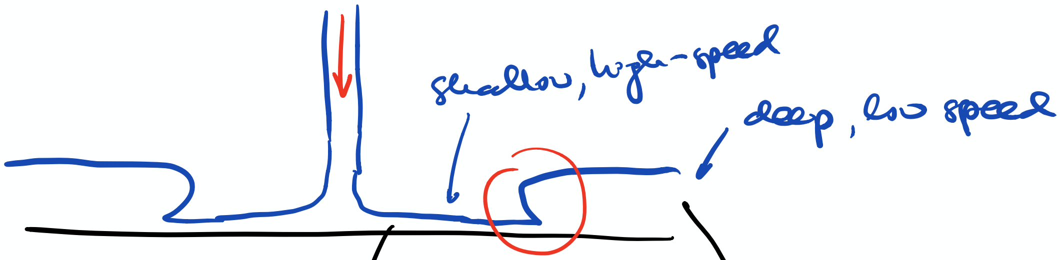
$$h_2 = \frac{h}{2}(1 - \sin\alpha)$$

Normal force:

Inertial characteristic $F \sim u^2$

$$\underline{F_\perp = \rho u^2 h \cos\alpha}$$

The hydraulic jump



transition region

subcritical

$U(x) < c(x)$

$U(x) > c(x)$

local flow speed

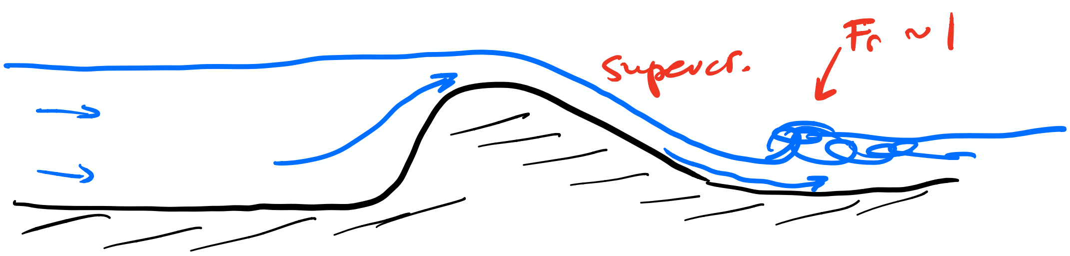
surface waves velocity

supercritical

The ratio defines the Froude number

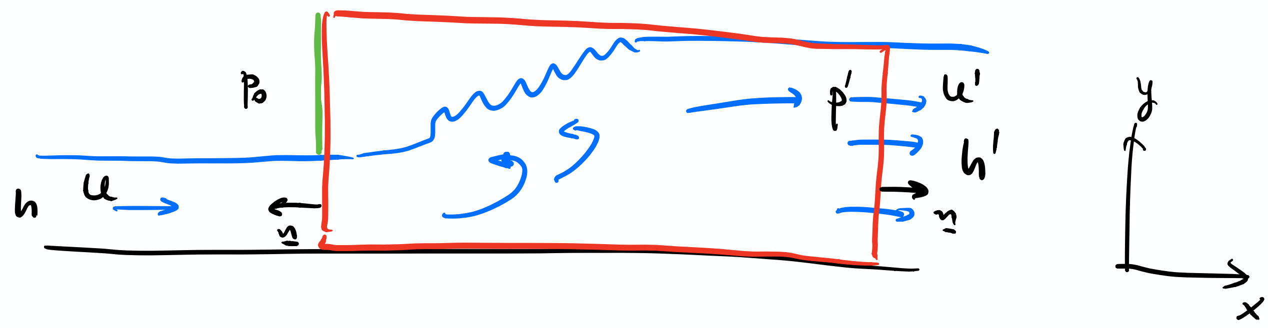
$$Fr = \frac{U(x)}{c(h)}$$

① We observe hydraulic jumps on spillways of water dams – they are associated with gravity waves $\left(c(h) = \sqrt{gh} \right)$



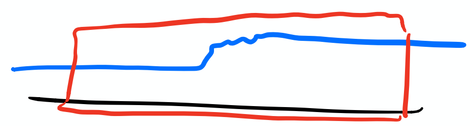
② Breaking waves

Hydraulic jump



$$\int_S \rho v_x (v_j n_j)\, dS = \rho\left(u'^2 h' - u^2 h\right)$$

The respective pressures



$$p = p_0 + \rho g(h-y)$$
$$p' = p_0 + \rho g(h'-y)$$

The control volume encloses some air, so we need to add a term $-p_0(h'-h)$

We obtain (momentum conservation in the x-direction)

$$\int_S p n_x\, dS = p_0(h-h') - \int_0^h (p_0 + \rho g(h-y))\,dy +$$
$$+ \int_0^{h'} (p_0 + \rho g(h'-y))\,dy = \frac{\rho g}{2}\left(h'^2 - h^2\right)$$

Conservation of momentum:

$$\left(u^2 h - u'^2 h'\right) + g\left(\frac{h^2}{2} - \frac{h'^2}{2}\right) = 0$$

Continuity requires $u'h' = uh$

We find:

$$u' = \sqrt{g h' \left(\frac{h}{h'} \frac{1}{2} \left(1 + \frac{h}{h'} \right) \right)}$$

$$u = \sqrt{g h \left(\frac{h'}{h} \frac{1}{2} \left(1 + \frac{h'}{h} \right) \right)}$$

Since $h < h'$, we find:

$$u' < \sqrt{g h'}$$

$$u > \sqrt{g h}$$

The flow is supercritical in the shallow section ($Fr > 1$)

and subcritical in the shallow section ($Fr' < 1$)