Stokes flows "creeping flow"

Flows with $Re = \frac{LU}{\nu} \ll 1$

for instance
* U, L small : colloids, bacteria, cells $L \sim 10\,\mu m$, $U \sim 1\,\mu m/s$
* U small, L large : oil reservoir $U \sim 1\,m/day$
* ν large : lava, glacier
  $\nu = 10^8 \frac{m^2}{s}$ (molten glass)
  $L \sim 1\,m$ $U \sim 1\,m/s$ $Re \approx 10^{-8}$
* droplet of water falling under g
  $L \sim 1\,\mu m$ $u \sim 0.1\,mm/s$
  $Re \approx 10^{-5}$

So when $Re \to 0$, the Navier-Stokes eq. becomes Stokes equation

$$\begin{cases} 0 = -\underline{\nabla} p + \mu \nabla^2 \underline{u} \quad (+ \underline{F}) \\ 0 = \underline{\nabla} \cdot \underline{u} \end{cases}$$

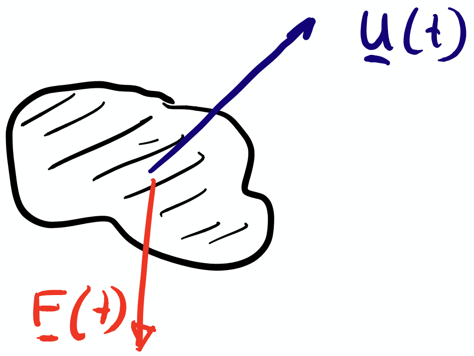
Typically, $\underline{u}$ is given on the boundary.
→ solve for $\underline{u}(\underline{x}, t)$ everywhere
→ calculate stresses $\sigma_{ij} n_j$
forces $\underline{F} = \int_S \underline{\underline{\sigma}} \cdot \underline{n}\, dS$

Simple properties

(a) **Instantaneity**: no $\frac{\partial \underline{u}}{\partial t}$ term → instantaneous response of the flow to a change in BCs

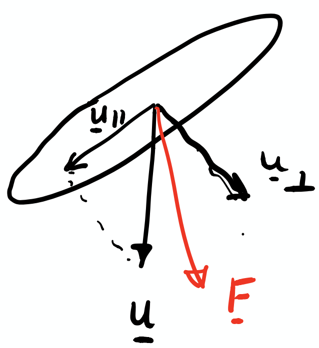
'quasi-steady' flows because any change in time only through $\underline{F}(t)$ or BCs as fcn of time.

(b) **Linearity** no $(\underline{u}\cdot\underline{\nabla})\underline{u}$ term, so $\underline{u}$ (and $p, \underline{\underline{\sigma}}, \underline{\underline{e}}$) is linear in $F(t)$ and in BCs.



$$\underline{F}(t) = \underline{\underline{A}} \cdot \underline{u}(t)$$

↑ depends on the size, orientation, surface properties



$$\underline{F} = \alpha\, \underline{u}_{\parallel}(t) + \beta\, \underline{u}_{\perp}(t)$$

(c) **Time reversibility** · apply a force $\underline{F}(t)$ on $0 \leq t \leq t_1$
· reverse the force;
apply $-\underline{F}(2t_1 - t)$ on $t_1 \leq t \leq 2t_1$

⇒ the flow reverses its path

⇒ all fluid particles return to their initial positions

<u>YT video:</u> G.I. Taylor (Cambridge)
Low Reynolds Number Flows
(National Committee for Fluid Mechanics Films)

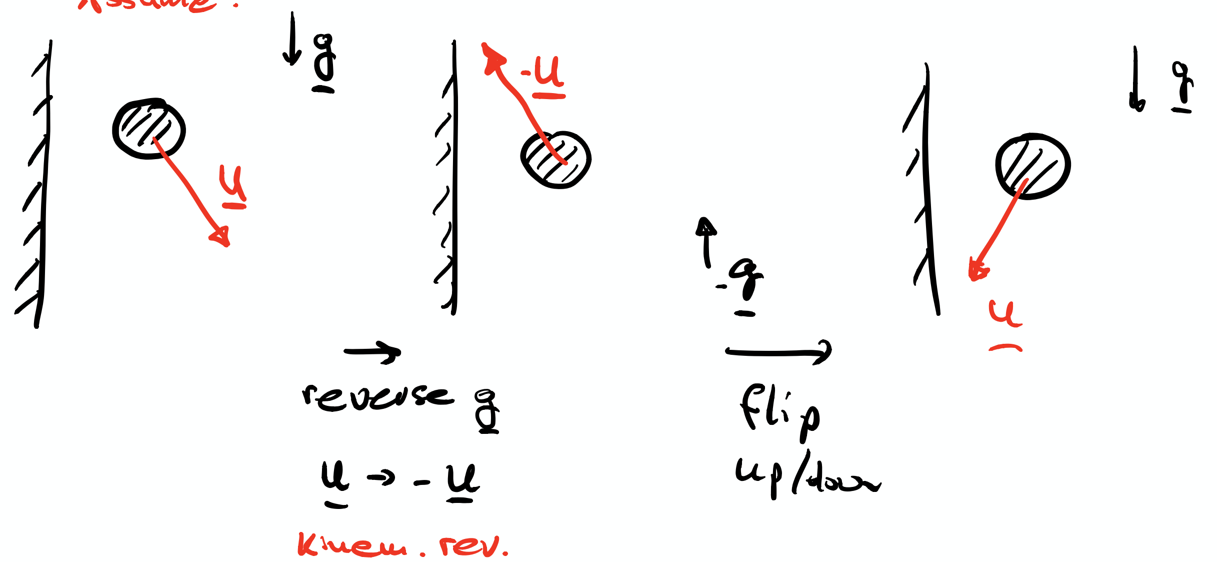
R.R. → important for swimming

④ <u>Spatial reversibility</u>

Linearity + geometric symmetry require certain components of $\underline{u}$ to be 0.

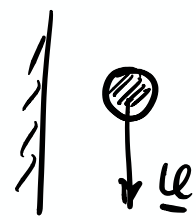
<u>Ex 1</u> Sphere sedimenting near a wall

Assume:

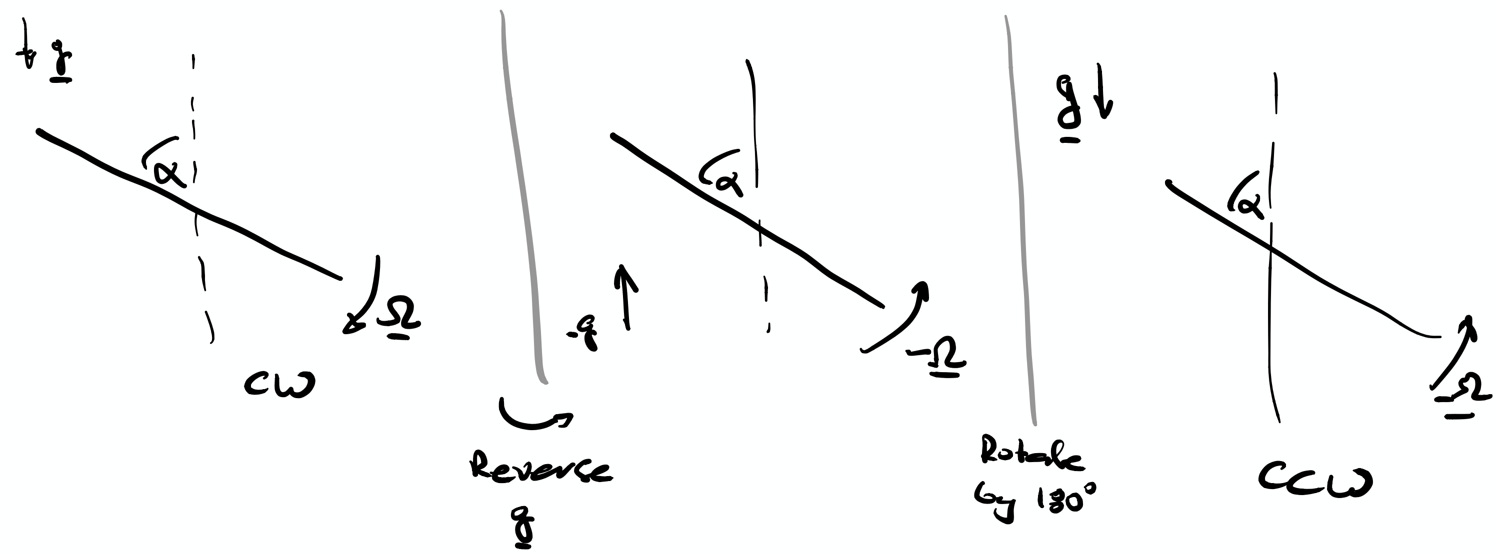


There cannot be a component ⊥ to the wall.

A sphere sediments at a constant distance.



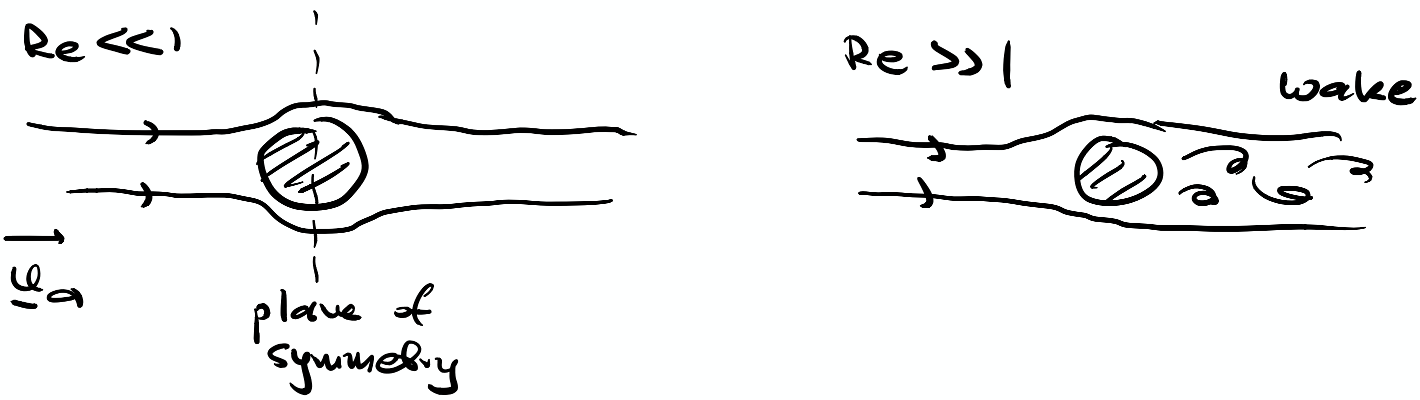
Ex 2 A sedimenting rod does not rotate



$$\underline{\Omega} = 0$$

Ex 3 Two equal spheres fall without separating.

Ex 4 Bodies that are front-back symmetric have front-back symmetric streamlines



(5) Harmonic functions

$$\begin{cases} \underline{\nabla} p = \mu \nabla^2 \underline{u} & (*) \\ \underline{\nabla} \cdot \underline{u} = 0 \end{cases}$$

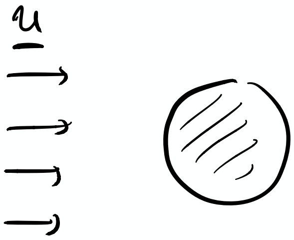
$$\nabla \cdot [*] \rightarrow \nabla^2 p = 0$$

$$\nabla \times [*] \rightarrow \nabla^2 \omega = 0$$

$$\nabla^2 [*] \rightarrow \nabla^2 \nabla^2 \underline{u} = 0$$

$\underline{p}$, ω are harmonic ; $\underline{u}$ is biharmonic.

Flow past a sphere



Rigid sphere, radius a, steady flow $\underline{U}$

The solution

$$\underline{u} = \underline{U}\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) + \underline{x}\,(\underline{U}\cdot\underline{x})\left(-\frac{3a}{4r^3} + \frac{3a^3}{4r^5}\right)$$

$$p = -\frac{3a\mu}{2r^3}\,\underline{U}\cdot\underline{x}$$

Calculate the force

$$\underline{\underline{\sigma}}\cdot\underline{n}\Big|_{r=a} = \frac{3\mu}{2a}\underline{U}$$ independent of the position

4 from p, 2 comes from tangential viscous friction.

$$\underline{F} = \int_S \underline{\underline{\sigma}}\cdot\underline{n}\,dS = \frac{3\mu}{2a}\underline{U}\cdot 4\pi a^2 = 6\pi\mu a\,\underline{U}$$

Stokes' drag

Stokes' law

On a spherical bubble we have different BCs:

(i) $\underline{u}\cdot\underline{n} = 0$ on bubble

(ii) $\underline{t}\cdot\underline{\underline{\sigma}}\cdot\underline{n} = 0$ on bubble ($\underline{t}$ is the tangent vector)

(iii) $u(\infty) = \underline{U}$

$$\underline{F}_B = 4\pi\mu a\,\underline{U}$$

For a translating sphere

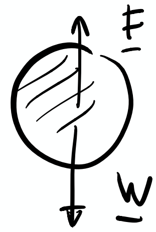
$$\underline{F} = -6\pi\mu a \underline{U}$$

force on the sphere

speed of sphere relative to flow at ∞

Sedimentation of a sphere

After any transient, $\sum_i F = 0$



$$\underline{0} = \frac{4\pi}{3} a^3 \Delta\rho \, \underline{g} - 6\pi\mu a \underline{U}$$

no inertia weight + buoyancy Stokes drag

$$\underline{U} = \frac{2}{9} \frac{a^2 \Delta\rho}{\mu} \underline{g}$$

Ex. $a \sim 1\mu m$

$\mu \sim 10^{-3}$ Pa.s

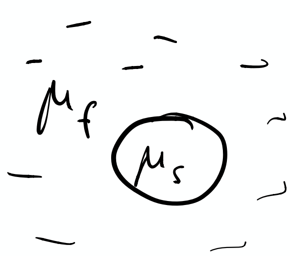
$\Delta\rho \sim \rho_{water} = 10^3 \frac{kg}{m^3}$

$\Rightarrow u \sim 10^{-6} \frac{m}{s} \sim 1 \frac{\mu m}{s}$

Drag force on a fluid sphere

Hadamard - Rybczyński (1911)

$$\underline{F} = -2\pi \frac{2\mu_f + 3\mu_s}{\mu_f + \mu_s} \mu_f a \underline{U}$$



Flow in the far field

Disturbance flow $\underline{u} = \underline{U}\Big(\underbrace{-\frac{3a}{4r}}_{\sim \frac{1}{r}} - \underbrace{\frac{a^3}{4r^3}}_{\sim \frac{1}{r^3}}\Big) + \underbrace{(\underline{r}\cdot\underline{U})\underline{r}}_{r^2}\Big(\underbrace{-\frac{3a}{4r^3}}_{\frac{1}{r^3}} + \underbrace{\frac{3a^3}{4r^5}}_{\sim\frac{1}{r^5}}\Big)$

Consider $|r| \gg a$, the dominant terms are

$$\underline{u} \approx \underline{U}\left(-\frac{3a}{4r}\right) + (\underline{r}\cdot\underline{U})\underline{r}\left(-\frac{3a}{4r^3}\right)$$

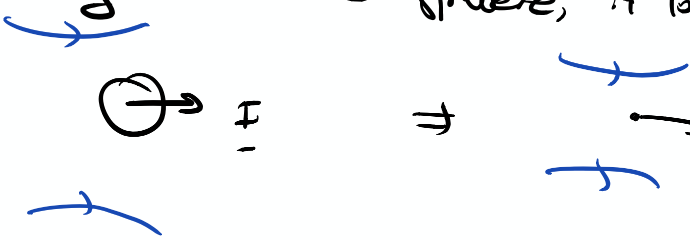
use $\underline{F} = -6\pi\mu a \underline{U}$ to rewrite it:

$$\underline{u} = \frac{1}{8\pi\mu}\left[\frac{\underline{F}}{r} + \frac{(\underline{r}\cdot\underline{F})\underline{r}}{r^3}\right] = \underbrace{\frac{1}{8\pi\mu r}\left[\mathbb{1} + \frac{\underline{r}\underline{r}}{r^2}\right]}_{G(\underline{r})}\cdot\underline{F}$$

Stokeslet

Green's function for Stokes eqns.

Far away from the sphere, it looks like a point force



Mathematically, the flow is the solution to

$$\begin{cases} 0 = -\nabla p + \mu\nabla^2\underline{u} + \underline{F}\,\delta(\underline{r}) \\ \nabla\cdot\underline{u} = 0 \end{cases}$$

Far field flow is independent of the shape.

More properties of Stokes flow

A useful lemma

Consider two flows:

- $\underline{u}^S(\underline{x}), p^S(\underline{x})$ – solution to Stokes eq. in volume V with $\underline{F}=0$
- $\underline{u}(\underline{x})$ – any incompressible flow field (admissible flow)

Lemma

$$\boxed{\int_V 2\mu\, e^S_{ij} e_{ij}\, dV = \int_S \sigma^S_{ij} u_i n_j\, dS}$$

mixed dissipation rate

mixed rate of working

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Proof By definition

$$2\mu e^S_{ij} = \sigma^S_{ij} + p^S \delta_{ij}$$

so

$$2\mu e^S_{ij} e_{ij} = \underbrace{\sigma^S_{ij} e_{ij}}_{(*)} + p^S \underbrace{\delta_{ij} e_{ij}}$$

$$e_{jj} = \nabla\cdot\underline{u} = 0 \quad \text{incompressible}$$

$$e_{ij} = \frac{\partial u_i}{\partial x_j} - \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)} =$$

$$= \frac{\partial u_i}{\partial x_j} - \Omega_{ij}$$

antisymmetric

$$(*) \quad \sigma^S_{ij}\frac{\partial u_i}{\partial x_j} - \cancel{\Omega_{ij}\sigma^S_{ij}} \to 0$$

antis. sym.

$$\int_V 2\mu\, e_{ij}^S e_{ij}\, dV = \int_V \underbrace{\left(\sigma_{ij}^S \frac{\partial u_i}{\partial x_j}\right)}\, dV = (**)$$

$$\frac{\partial}{\partial x_j}\left(\sigma_{ij}^S u_i\right) - u_i \cancelto{0}{\frac{\partial \sigma_{ij}}{\partial x_j}} \quad \text{Stokes eq.}$$

$$(**) = \int_V \frac{\partial}{\partial x_j}\left(\sigma_{ij}^S u_i\right) dS = \int_S \sigma_{ij}^S n_j u_i\, dS \qquad \square$$

With this, we can prove the following theorems.

1. <u>Minimum dissipation theorem</u> says that Stokes flow has the minimum dissipation out of all admissible flows with the same BCs.

$$\int_V 2\mu\, e_{ij}^S e_{ij}^S \leq \int_V 2\mu\, e_{ij} e_{ij}\, dV$$

2. <u>Uniqueness</u> Stokes flows are unique.