Stokes Flows (part 2)

Reminder

In the last episode, we proved

$\underline{u}^S(\underline{x})$ Stokes flow

$\underline{u}(\underline{x})$ admissible flow

Lemma:
$$\int_V 2\mu e_{ij}^S e_{ij}\, dV = \int_S \sigma_{ij}^S u_i n_j\, dS$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Lorentz Reciprocal Theorem

Let $\underline{u}^1(\underline{x})$ and $\underline{u}^2(\underline{x})$ be two Stokes flows inside the same volume but with different BCs.

Apply lemma

$$\int_V 2\mu\, e_{ij}^1 e_{ij}^2\, dV \;\to\; \int_S \sigma_{ij}^1 n_j u_i^2\, dS \quad \left(\text{because } u^1 \text{ Stokes}\right)$$

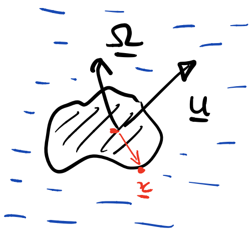
$$\int_V 2\mu\, e_{ij}^1 e_{ij}^2\, dV \;\to\; \int_S \sigma_{ij}^2 n_j u_i^1\, dS \quad \left(\text{because } u^2 \text{ Stokes}\right)$$

Thus:
$$\int_S \sigma_{ij}^1 n_j u_i^2\, dS = \int_S \sigma_{ij}^2 n_j u_i^1\, dS$$

$$\boxed{\int_S \underline{u}_2 \cdot \underline{\underline{\sigma}}_1 \cdot \underline{n}\, dS = \int_S \underline{u}_1 \cdot \underline{\underline{\sigma}}_2 \cdot \underline{n}\, dS}$$

RECIPROCAL THEOREM

Application Rigid particle moving in a fluid.



Boundary conditions on the surface:

$$\underline{u}(\underline{x}) = \underline{U} + \underline{\Omega} \times \underline{x}$$

Consider two motions $(\underline{U}_1, \underline{\Omega}_1)$ & $(\underline{U}_2, \underline{\Omega}_2)$

Apply LRT:

$$LHS = \int_S (\underline{u}_2 \cdot \underline{\underline{\sigma}}_1 \cdot \underline{n}) dS = \int_S \left(\underline{U}_2 + \underline{\Omega}_2 \times \underline{x}\right) \cdot \underline{\underline{\sigma}}_1 \cdot \underline{n}\, dS =$$

$$= \underbrace{\int_S \underline{U}_2 \cdot \underline{\underline{\sigma}}_1 \cdot \underline{n}\, dS}_{\underline{U}_2 \cdot \underbrace{\int_S \underline{\underline{\sigma}}_1 \cdot \underline{n}\, dS}_{-F_1 \text{ force on the particle}} = -\underline{U}_2 \cdot \underline{F}_1} + \underbrace{\int (\underline{\Omega}_2 \times \underline{x}) \cdot \underline{\underline{\sigma}}_1 \cdot \underline{n}\, dS}_{\underline{\Omega}_2 \cdot \int_S \underline{x} \times (\underline{\underline{\sigma}}_1 \cdot \underline{n}) dS = \text{(check!)} = \underline{\Omega}_2 \cdot (-\underline{T}_1) \text{ torque on solid 1}}$$

$$(\sigma_1)_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$

$$LHS = -\underline{U}_2 \cdot \underline{F}_1 - \underline{\Omega}_2 \cdot \underline{T}_1$$

$RHS =$ (same $1 \leftrightarrow 2$)

By LRT:

$$\underline{U}_1 \cdot \underline{F}_2 + \underline{\Omega}_1 \cdot \underline{T}_2 = \underline{U}_2 \cdot \underline{F}_1 + \underline{\Omega}_2 \cdot \underline{T}_1$$

Special case $\underline{\Omega}_1 = \underline{\Omega}_2 = 0$ (translation)

$$\underline{U}_1 \cdot \underline{F}_2 = \underline{U}_2 \cdot \underline{F}_1$$

but by linearity $\quad \underline{F}_1 = -\underline{\underline{A}}\cdot\underline{u}_1 \qquad \underline{\underline{A}}$ is a shape property

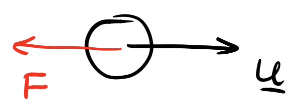
$$\underline{F}_2 = -\underline{\underline{A}}\cdot\underline{u}_2$$

$$\underline{u}_1\cdot\underline{\underline{A}}\cdot\underline{u}_2 = \underline{u}_2\cdot\underline{\underline{A}}\cdot\underline{u}_1$$

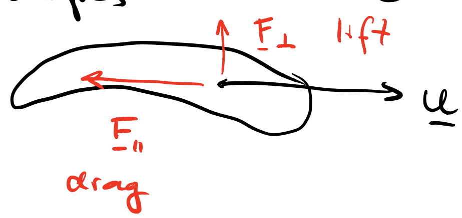
$\Rightarrow$ $\underline{\underline{A}}$ must be symmetric.

Pure rotation $\Rightarrow$ similar result.

<u>Note</u> For most symmetric shapes $\underline{F} \parallel \underline{u}$



Weird shapes can have $\underline{F}\times\underline{u} \neq 0$



For the most general rigid body motion $(\underline{u}, \underline{\Omega})$

force $\quad \underline{F} = -\underline{\underline{A}}\cdot\underline{u} + \underline{\underline{B}}\cdot\underline{\Omega}$

torque $\quad \underline{T} = -\underline{\underline{C}}\cdot\underline{u} + \underline{\underline{D}}\cdot\underline{\Omega}$

$$\begin{pmatrix}\underline{F}\\ \underline{T}\end{pmatrix} = -\underbrace{\begin{pmatrix}\underline{\underline{A}} & \underline{\underline{B}}\\ \underline{\underline{C}} & \underline{\underline{D}}\end{pmatrix}}_{\text{grand resistance matrix (GRM)}}\begin{pmatrix}\underline{u}\\ \underline{\Omega}\end{pmatrix}$$

$\underline{\underline{\xi}}$

$$\begin{pmatrix} \underline{F} \\ \underline{T} \end{pmatrix} = - \underline{\underline{\zeta}} \cdot \begin{pmatrix} \underline{u} \\ \underline{\Omega} \end{pmatrix} \quad \xrightarrow{\text{invert}} \quad \begin{pmatrix} \underline{u} \\ \underline{\Omega} \end{pmatrix} = - \underline{\underline{\mu}} \cdot \begin{pmatrix} \underline{F} \\ \underline{T} \end{pmatrix}$$

Grand mobility matrix

$$\underline{\underline{\mu}} = \underline{\underline{\zeta}}^{-1}$$

From LRT:

$$\underline{\underline{A}}^T = \underline{\underline{A}}$$

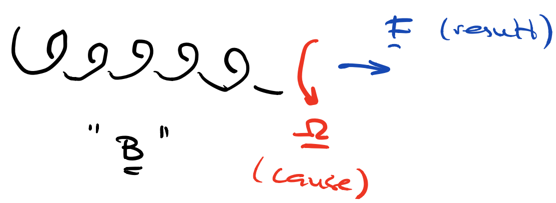
$$\underline{\underline{D}}^T = \underline{\underline{D}}$$

$$\underline{\underline{B}}^T = \underline{\underline{C}}$$ force due to rotation = torque due to translation

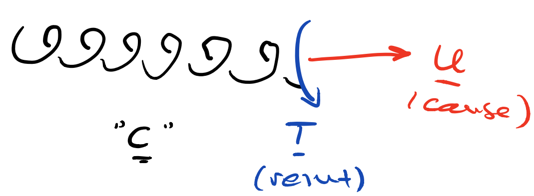
Remarks

- GRM shares the symmetries of the shape
- $\underline{\underline{A}}, \underline{\underline{D}}$ are positive definite
- for $\underline{\underline{B}}, \underline{\underline{C}} \neq 0$ need reflectional asymmetry.

example: a helix



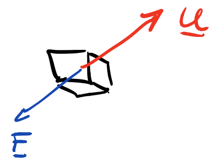
"$\underline{\underline{B}}$"



"$\underline{\underline{C}}$"

- sphere, cube, Platonic solids

$\underline{\underline{B}} = \underline{\underline{C}} = 0$ $\quad \underline{\underline{A}} = \alpha \mathbb{1}$ $\quad \underline{\underline{D}} = \beta \mathbb{1}$ isotropic



sphere $\alpha = 6\pi\mu a$ (Stokes law)

$\beta = 8\pi\mu a^3$

- Exact solutions : ellipsoids
- Dimensions

$$[\underline{\underline{A}}] = \mu L$$

$$[\underline{\underline{B}}] = [\underline{\underline{C}}] = \mu L^2$$

$$[\underline{\underline{D}}] = \mu L^3$$

FLOWS IN THIN LAYERS

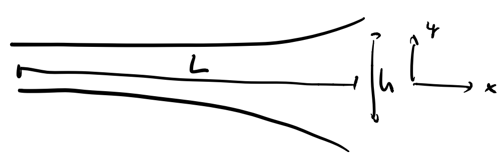
Lubrication theory

slowly-varying flows

long-wavelength approximation

Essentially : unidirectional (Poiseuille) flows at low Re.

Hydrodynamics in long, narrow geometries (2D)



Assume $h \ll L$. What happens to N-S?

- From $\underline{\nabla} \cdot \underline{u} = 0 \rightarrow \dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} = 0$

$$\sim \frac{u_x}{L} \qquad \sim \frac{u_y}{h} \qquad \Rightarrow \quad u_y = \underbrace{\frac{h}{L}} u_x \ll u_x$$

• The flow is almost unidirectional.

$$\underline{u} \sim (u(y), 0, 0) \qquad \underline{u} = (u_x , \underset{\text{small}}{u_y} , 0)$$

The non-linear term $(\underline{u} \cdot \underline{\nabla})\underline{u} = u(y) \frac{\partial}{\partial x} u(y) = 0$

Stokes eqs. describe the flow.

$$0 = -\nabla p + \mu \nabla^2 \underline{u}$$

$$\nabla^2 \underline{u} = \mu \left(\cancel{\frac{\partial^2}{\partial x^2}} + \frac{\partial^2}{\partial y^2} \right) \underline{u} \sim \mu \left(\frac{1}{L^2} + \frac{1}{h^2} \right) \underline{u} \approx \mu \frac{u}{h^2}$$

$$\approx \mu \frac{\partial^2}{\partial y^2}$$

Our eqs. become:

$$x : \quad 0 = - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u_x}{\partial y^2}$$

$$y : \quad 0 = - \frac{\partial p}{\partial y} + \mu \frac{\partial^2 u_y}{\partial y^2}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

x -balance

$$0 = - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u_x}{\partial y^2}$$

$$\sim \frac{P}{L} \qquad \sim \mu \frac{u_x}{h^2} \qquad \Rightarrow \quad P \sim \mu \frac{u_x}{h^2} L$$

Y-balance

$$-\frac{\partial p}{\partial y} + \mu \frac{\partial^2 u_y}{\partial y^2} = 0$$

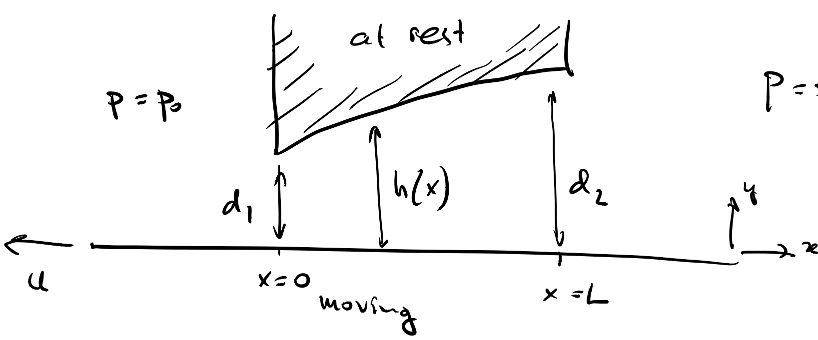
$$\sim \frac{P}{h} = \mu \frac{u_x L}{h^3} \qquad \sim \mu \frac{u_y}{h^2} = \mu \frac{\frac{h}{L} u_x}{h^2} = \frac{\mu u_x}{hL}$$

#1   #2

$$\frac{\#2}{\#1} = \frac{\frac{\mu u_x}{hL}}{\frac{\mu u_x L}{h^3}} = \left(\frac{h}{L}\right)^2 \longrightarrow 0$$

Balance in y-direction is $\frac{\partial p}{\partial y} = 0 \Rightarrow p = p(x)$.

Thrust bearing



We are moving with the lower boundary:

1. Geometry
2. Flow (unidirectional)
3. Mass conservation (Reynolds' approach)
4. Forces

① Geometry The gap is:

$$h(x) = d_1 + \alpha x \qquad \alpha = \frac{d_2 - d_1}{L}$$

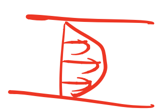
Thin film requires $h \ll L$, $h' \ll 1$

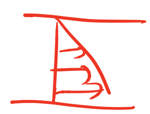
② Flow

$$\begin{cases} \dfrac{dp}{dx} = \mu \dfrac{\partial^2 u}{\partial y^2} \\ u(y=0) = -U \\ u(y = h(x)) = 0 \end{cases} \qquad \underline{u} = (u(y), 0, 0)$$

Integrate to get:

$$u = \underbrace{-\frac{1}{2\mu}\frac{dp}{dx}\, y(h-y)}_{\text{parabolic flow}} - \underbrace{U\left(\frac{h-y}{h}\right)}$$





What is $\frac{dp}{dx}$?

③ Mass conservation:

Flow rate $\quad Q = \int_0^{h(x)} u\, dy = -\frac{h^3}{12\mu}\frac{dp}{dx} - \frac{1}{2} U h.$

Mass conservation requires Q = const., independent of x.

The fact Q = const gives:

$$\frac{dp}{dx} = -\frac{12\mu Q}{h^3(x)} - \frac{6\mu U}{h^2(x)}$$

We want to find the pressure. Since $p = p_0$ at $x = 0$

$$p - p_0 = \int_0^L \frac{dp}{dx}\,dx$$

with $h(x) = d_1 + \alpha x$

we get

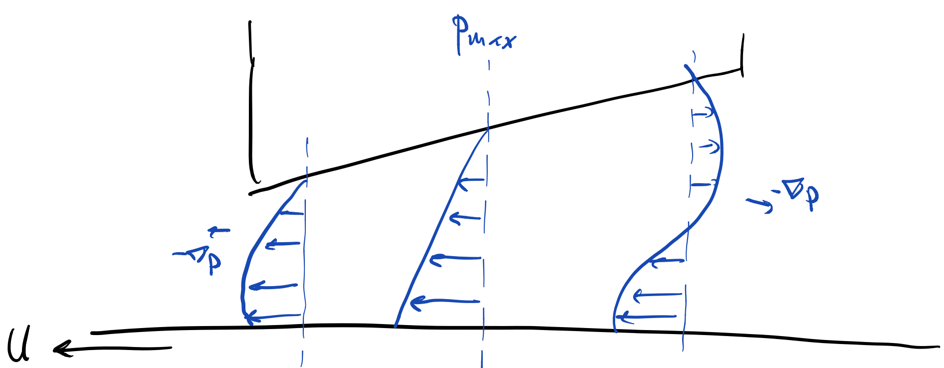
$$p(x) = p_0 + \frac{6\mu Q}{\alpha}\left(\frac{1}{h^2(x)} - \frac{1}{d_1^2}\right) + \frac{6\mu U}{\alpha}\left(\frac{1}{h(x)} - \frac{1}{d_1}\right).$$

But we know that $p = p_0$ also at $x = L$

$$p(L) = p_0$$

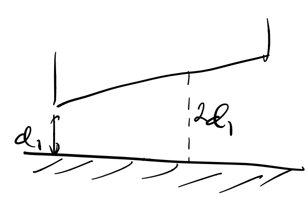
we find
$$Q = -\frac{U d_1 d_2}{d_1 + d_2}$$

<u>Note</u> $\frac{dp}{dx} = 0$ when $h(x) = \frac{2 d_1 d_2}{d_1 + d_2}$



$$u = -\frac{1}{2\mu}\frac{dp}{dx}\,y(h-y) - U\left(\frac{h-y}{h}\right)$$

When $d_2 \gg d_1$, $Q = -U d_1$
maximum pressure is at $h(x) = 2d_1$



4 Forces on the plates