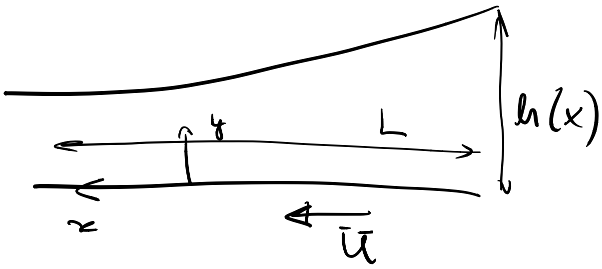
FLOWS IN THIN LAYERS (cont'd)



① Geometry → gap $h(x)$

② Flow

- unidirectional
- $u_x(x,y)$
- $p = p(x)$ $\quad (\partial_y p = 0)$

Integrating Stokes eqs.

$$u(x,y) = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y\,(h(x) - y) - U\,\frac{h(x) - y}{y}$$

from integrating

$$\begin{cases} \dfrac{\partial p}{\partial x} = \mu \dfrac{\partial^2 u}{\partial y^2} \\ u(h) = 0 \\ u(0) = U \end{cases}$$

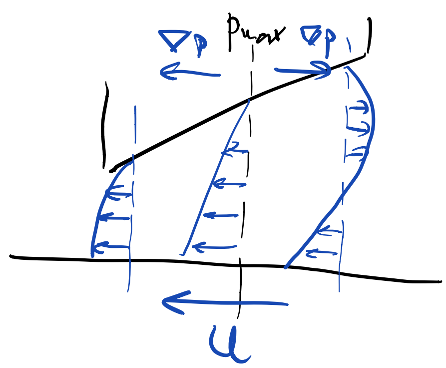
③ Q: What is $\dfrac{\partial p}{\partial x}$?

A: Use mass conservation.

Flow rate : $Q = \int_0^h u\, dy$ is constant.

$$Q = \text{const.} \Rightarrow \frac{\partial p}{\partial x} = -\frac{12\mu}{h^3(x)}\,Q - \frac{6\mu U}{h^2(x)}$$

Flow has p_{max}



④ Forces on the plates

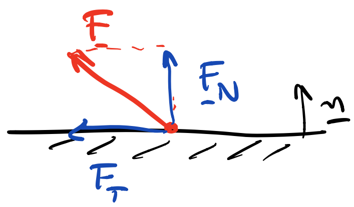
$$\underline{F} = \int \underline{\underline{\sigma}} \cdot \underline{n} \, dS$$

<u>Bottom plate</u>

- need surface tractions $\sigma_{ij} n_j$

$$\underline{n} = (0, 1, 0)$$

$$\sigma_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad i,j = \{x, y, z\}$$



Normal force on the bottom plate:

$$F_N = \int_0^L \left[\underline{\underline{\sigma}} \cdot \underline{n}\right] dx = \int_0^L \sigma_{yy} \, dx =$$

def: y component of the force acting on a surface with a normal vector in the y-direction

$$= \int_0^L \left(-p + 2\mu \frac{\partial u_y}{\partial y}\right) dx$$

Scalings:

$$p \sim \mu \frac{UL}{h^2} \qquad \#1$$

$$\mu \frac{\partial u_y}{\partial y} \sim \mu \frac{u_y}{h} \sim \mu \frac{U \frac{h}{L}}{h} \sim \mu \frac{U}{L} \qquad \#2$$

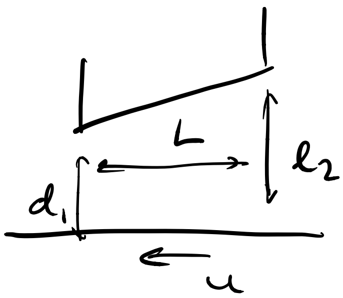
$$\frac{\#1}{\#2} \sim \left(\frac{L}{h}\right)^2 \gg 1$$

$\Rightarrow$ we can neglect <u>normal</u> viscous forces $\left(\mu \frac{\partial u_y}{\partial y}\right)$

we get:

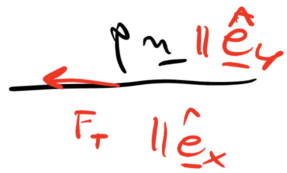
$$|F_N| = \int_0^L p \, dx = \frac{6\mu U}{\alpha^2}\left(\ln\frac{d_2}{d_1} - 2\,\frac{d_2-d_1}{d_2+d_1}\right)$$

we know $p(x)$ [see last lecture]



$$\alpha = \frac{d_2-d_1}{L}$$ slope of the channel

Tangential force



$$F_T = \int_0^L \sigma_{xy}\, dx$$

$$= \mu\left(\frac{\partial u_x}{\partial y} + \cancel{\frac{\partial u_y}{\partial x}}^{\,0}\right)$$

$$\frac{h}{L} \ll 1$$

$$\underbrace{\mu\frac{U}{h}} \qquad \mu\frac{U_y}{L} \sim \mu\frac{Uh}{L^2}$$

$$F_T = \int_0^L \mu\frac{\partial u_x}{\partial y}\, dx = \frac{4\mu U}{\alpha}\left(\ln\frac{d_2}{d_1} - \frac{3}{2}\,\frac{d_2-d_1}{d_2+d_1}\right)$$

we know

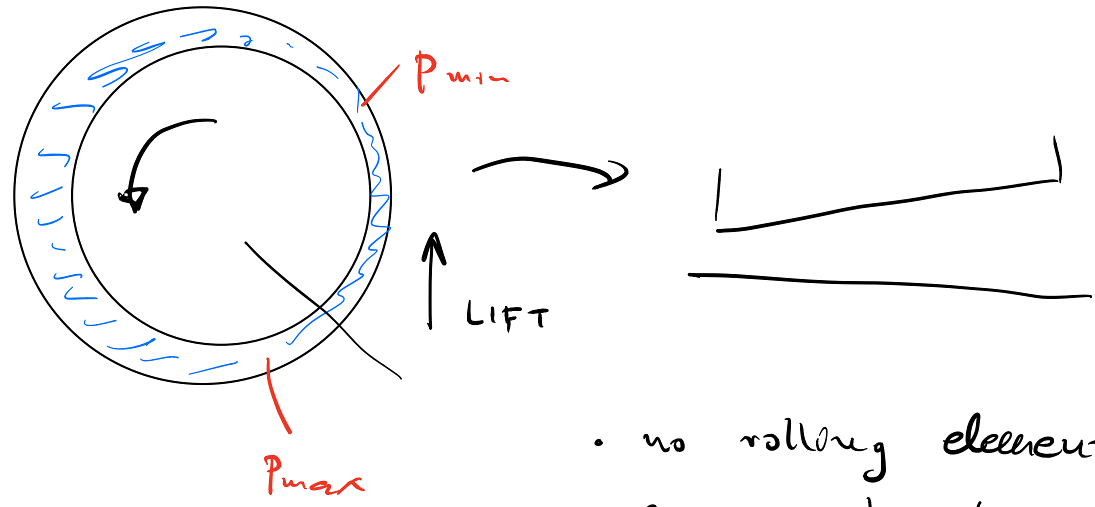
As $\alpha \to 0$ (slowly converging channel)

$$\begin{cases} F_N \sim \dfrac{1}{\alpha^2} \\ F_T \sim \dfrac{1}{\alpha} \end{cases}$$

Then $F_N \gg F_T$ → basis for "lubrication bearings"
(thin layers of fluid allow to build up large pressures)

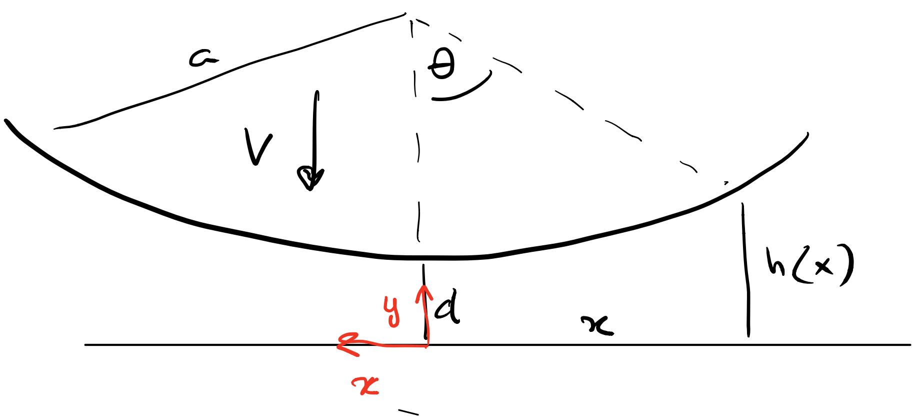
Journal bearing

[PL] "Łożysko ślizgowe"



- no rolling elements
- can support a large load at low friction.

CYLINDER APPROACHING A WALL



If no fluid
$t_{contact} = \frac{d}{V}$
What if there is a fluid layer?

Instantaneous separation $d \ll a$

① Geometry (gap size)

$$h = d + a(1 - \cos\theta)$$

$$\cos\theta \sim 1 - \frac{\theta^2}{2}$$

$\left.\begin{array}{l} d \text{ small} \\ \theta \text{ small} \end{array}\right\}$ $h - d \sim \frac{a}{2}\theta^2$ but $\theta \approx \frac{x}{a}$

$$h(x) = d\left(1 + \frac{x^2}{2ad}\right)$$

We will see that all the action is where $h \lesssim 3d$, where

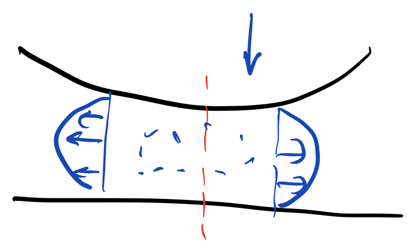
$$x \sim \sqrt{ad} \begin{cases} \ll a & \text{so } \theta \text{ is small, so approximation OK} \\ \gg d & \text{so approximately unidirectional flow} \end{cases}$$

$$h(x) = d\left(1 + \frac{x^2}{2L^2}\right)$$

(2) Flow

$$\begin{cases} \frac{\partial p}{\partial y} = 0 \ ; \quad \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u_x}{\partial y^2} \\ u(y=0) = 0 \\ u(y=h) = 0 \end{cases}$$

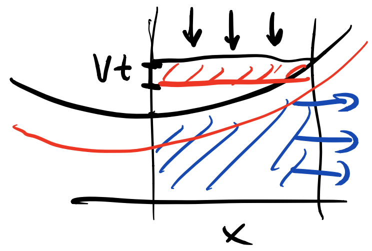
$$\Rightarrow \quad u(y) = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\, y\,(h(x) - y)$$



(3) Mass conservation

volume flux $\quad Q = \int_0^h u_x\, dy = -\frac{h^3}{2\mu}\frac{dp}{dx} = Q(x)$

The top boundary is moving down at a velocity V, so the total volume flux



$$Q(x) = Vx$$

Hence the pressure gradient

$$\frac{dp}{dx} = -\frac{12\mu V x}{h^3(x)} = -\frac{12\mu V x}{d^3\left(1+\frac{x^2}{2ad}\right)^3}$$

so that:

$$p = p_0 + \frac{6\mu V a}{d^2\left[1+\frac{x^2}{2ad}\right]^2}$$

- p_{max} at $x=0$
- decays rapidly $\sim \frac{1}{x^4}$

Most of the pressure is where $h \leq 3d$ so we chose L correctly.

(4) Normal force on the cylinder =

= normal force on the bottom plate

$$F_N = \int_{-\infty}^{+\infty} (p-p_0)\,dx = \frac{6\mu V a}{d^2}\sqrt{2ad}\underbrace{\int_{-\infty}^{+\infty}\frac{d\xi}{(1+\xi^2)^2}}_{=\pi/2} = 3\sqrt{2}\pi\left(\frac{a}{d}\right)^{3/2}\mu V$$

"∞" means $h \leq 3d$

(*) $F_N \sim V$ linear

(*) $F_N \sim d^{-3/2} \to \infty$ as $d \to 0$

<u>Note</u> Consider sedimentation due to gravity

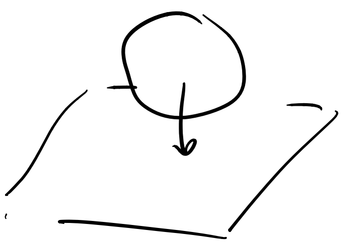
$$F = \text{const} \sim \frac{V}{d^{3/2}}$$

but $V = -\dot{d}$, so $\underline{\dot{d}\, d^{-3/2} = \text{const} = -A}$

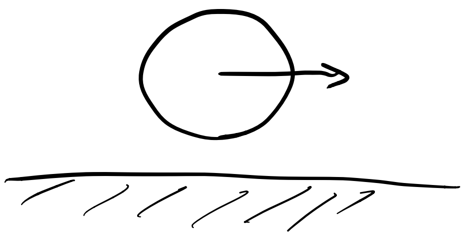
$$d^{-1/2} \sim A(t-t_0)$$

$$d \sim \frac{1}{[A(t-t_0)]^2} \sim t^{-2}$$

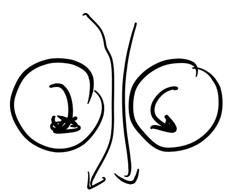
Other classical problems:



sphere approaching a wall



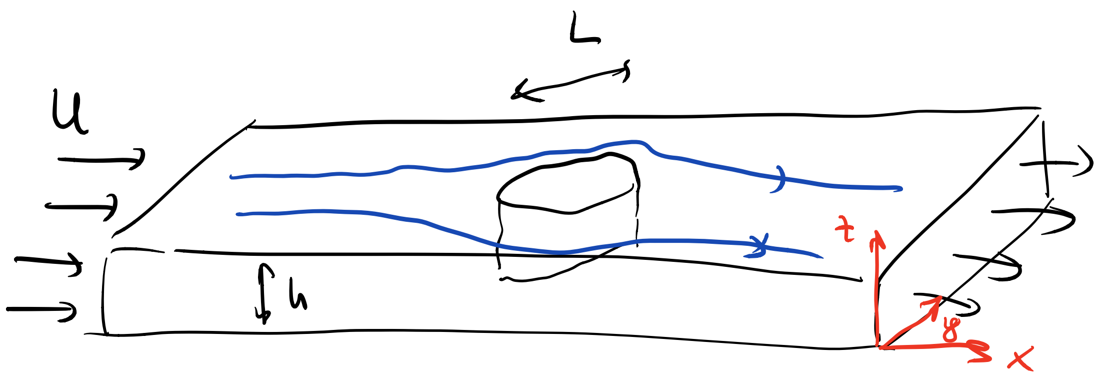
parallel motion



HELE - SHAW CELL

The device invented to visualise 2D potential flow and flow through porous media. (Same governing eqs.)

[Henry Selby Hele-Shaw (1898)]



Flow in a thin gap between parallel plates

- gap $h \ll L$ (obstacle)
- flow varies on the scale of
  - h in the z-direction
  - L in the xy direction.

$\frac{h}{L} \ll 1$ lubrication flow

Lubrication flow in 3D

$$\begin{cases} \frac{\partial p}{\partial z} = 0 \\ \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u_x}{\partial z^2} \\ \frac{\partial p}{\partial y} = \mu \frac{\partial^2 u_y}{\partial z^2} \end{cases}$$

∇p is in the xy plane

Define $\underline{u}_{\parallel} = (u_x, u_y)$

Rewrite $$\frac{\partial^2}{\partial z^2} \underline{u}_{\parallel} = \frac{1}{\mu} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) p = \frac{1}{\mu} \nabla_{\parallel} p$$

We can integrate this eq:

$$\underline{u}_{\parallel} = \frac{1}{2\mu} (\nabla_{\parallel} p)\, z(h-z)$$

Integrated flow rate:

$$\underline{q}_{\parallel} = \int_0^h \underline{u}_{\parallel}\, dz = -\frac{h^3}{12\mu} \nabla_{\parallel} p$$

Mass conservation:

$$0 = \nabla \cdot \underline{u} = \frac{\partial}{\partial x} u_x + \frac{\partial}{\partial y} u_y + \frac{\partial}{\partial z} u_z$$

Integrate from $z=0$ to $z=h$

$$\frac{\partial}{\partial x}\underbrace{\left[\int_0^h u_x\, dz \right]}_{\underline{q}_{\parallel} \cdot \underline{\hat{e}}_x} + \frac{\partial}{\partial y}\underbrace{\left[\int_0^h u_y\, dz \right]}_{\underline{q}_{\parallel} \cdot \underline{\hat{e}}_y} + \underbrace{\cancel{u_z(h)}}_{0} - \underbrace{\cancel{u_z(0)}}_{0} = 0$$

so $$\underline{\nabla}_{\parallel} \cdot \underline{q}_{\parallel} = 0$$

but $$\underline{q}_{\parallel} = -\frac{h^3}{12\mu} \nabla_{\parallel} p$$

together we have

$$\underline{\nabla}_{\parallel} \cdot \left[-\frac{h^3}{12\mu} \underline{\nabla}_{\parallel} p \right] = 0 \quad \Rightarrow \quad \nabla_{\parallel}^2 p = 0$$

<u>Reminder</u>: 2D irrotational potential flow we had:

$$\underline{\omega} = \nabla \times \underline{u} = 0 \quad \Rightarrow \quad \exists \phi \text{ such that } \nabla \phi = \underline{u}, \quad \nabla^2 \phi = 0$$

Here, the mean flow

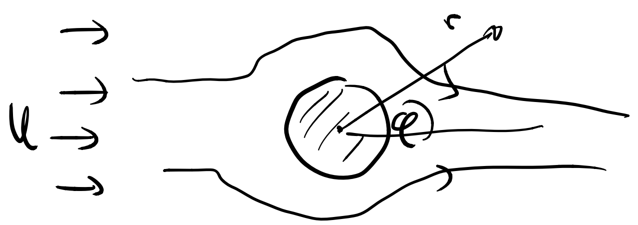
$$\langle \underline{u}_{\parallel} \rangle = \frac{1}{h}\,\underline{q}_{\parallel} = \nabla_{\parallel} \underbrace{\left[-\frac{h^3}{12\mu}\, p \right]}_{\phi}$$

we have $\nabla_{\parallel}^2 \phi = 0$

$\langle \underline{u}_{\parallel} \rangle$ is potential.

So we can visualise flows!

Flow past a circular cylinder



$$\underline{q}_{\parallel} = -\frac{h^3}{12\mu} \nabla_{\parallel} p \qquad (*)$$

$$\underline{\nabla}_{\parallel} \cdot \underline{q}_{\parallel} = 0 \Rightarrow \nabla_{\parallel}^2 p = 0$$

for $r \geq a$

The flow is uniform at large distances, so

$$\underline{q}_{\parallel} \to (Q, 0) \quad \text{in } (x, y) \text{ as } r \to \infty$$

so that $(*)$

$$p \to -\frac{12\mu Q}{h^3}\, r\cos\theta$$

On the surface of a cylinder $r = a$

$$\underline{q}_{\parallel} \cdot \underline{n} = 0, \text{ so } q_r = 0, \text{ so } \frac{\partial p}{\partial r} = 0$$

So

$$p = -\frac{12\mu Q}{h^3}\cos\theta\left(r + \frac{a^2}{r}\right)$$

$$\partial_r p = -\frac{12\mu Q}{h^3}(\cos\theta)\left(1 - \frac{a^2}{r^2}\right) = 0 \quad \text{at } r = a$$

Identical to the velocity potential for a 2D ideal flow around a cylinder.