VORTICITY GENERATION & CONFINEMENT

- back to high Re $= \dfrac{\text{"inertia"}}{\text{"viscosity"}}$

- inertia dominated, but viscosity has a role to play in thin layers close to boundaries (boundary layers)

Vorticity is the local angular velocity of the fluid

Def. $\underline{\omega} = \underline{\nabla} \times \underline{v}$

Consider the flow field :

$$u_i(\underline{x}) = \underbrace{u_i(0) + x_j \left.\frac{\partial u_i}{\partial x_j}\right|_0} + \frac{1}{2} x_j x_k \left.\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right|_0 + \dots$$

we write

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{e_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\Omega_{ij}}$$

symmetric — strain-rate tensor

antisymmetric — vorticity tensor

$$\underline{\omega} = \underline{\nabla} \times \underline{u} = \left(\epsilon_{ijk} \frac{\partial u_k}{\partial x_j}\right)\underline{e}_i = -2\,\underline{\Omega}$$

$$u_i(x) = u_i(0) + \underbrace{e_{ij} x_j}_{\text{strain}} + \underbrace{\frac{1}{2}(\underline{\omega} \times \underline{x})_i}_{\text{local solid body rotation}}$$

strain (flow, true deformation)

local solid body rotation

We want to know how $\underline{\omega}$ evolves in a fluid?

Take N-S eqs. for constant ρ, μ

$$\rho \frac{D\underline{u}}{Dt} = \underline{f} - \nabla p + \mu \nabla^2 \underline{u}$$

ext force   pressure   viscous forces

- we assume the force to be conservative $\underline{f} = -\nabla \phi$

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla}) \underline{u} = \frac{\partial \underline{u}}{\partial t} + \nabla\left[\frac{1}{2}u^2\right] - \underline{u} \times \underline{\omega}$$

$$LHS = (\underline{u} \cdot \underline{\nabla}) \underline{u} = u_j \partial_j u_i$$

$$RHS = \partial_i \left(\tfrac{1}{2} u_j u_j\right) - \epsilon_{ijk} u_j \omega_k = \left\{ \underline{\omega} = \nabla \times \underline{u} \right\}$$

$$= u_j \partial_i u_j - \epsilon_{ijk} u_j \epsilon_{kpq} \partial_p u_q =$$

$$= u_j \partial_i u_j - \epsilon_{kij} \epsilon_{kpq} u_j \partial_p u_q =$$

$$= u_j \partial_i u_j - (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) u_j \partial_p u_q =$$

$$= \cancel{u_j \partial_i u_j} - \cancel{u_j \partial_i u_j} + u_j \partial_j u_i = u_j \partial_j u_i = LHS$$

$$(\underline{u} \cdot \underline{\nabla}) \underline{u} = \nabla\left(\frac{1}{2}u^2\right) - \underline{u} \times \underline{\omega}$$

We have

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \nabla\left[\frac{1}{2}u^2\right] - \underline{u} \times \underline{\omega}\right) = -\nabla \phi - \nabla p + \mu \nabla^2 \underline{u} \qquad (*)$$

Take $\nabla \times (*)$

$$\rho\left(\frac{\partial \underline{\omega}}{\partial t} + \nabla \times \cancelto{0}{\nabla}\left(\tfrac{1}{2}u^2\right) - \nabla \times (\underline{u} \times \underline{\omega})\right) =$$

$$= -\nabla \cancelto{0}{\times} \nabla \phi - \nabla \cancelto{0}{\times} \nabla p + \mu \nabla^2 \underline{\omega}$$

$$(\nabla \times \nabla f)_i = \underbrace{\epsilon_{ijk}}\,\underbrace{\partial_j \partial_k} f = 0$$

$$\rho\left(\frac{\partial \underline{\omega}}{\partial t} - \nabla \times (\underline{u} \times \underline{\omega})\right) = \mu \nabla^2 \underline{\omega}$$

$$\nabla \times (\underline{u} \times \underline{\omega}) = \underline{A}$$

Now

$$A_i = \epsilon_{ijk}\, \partial_j\, (\underline{u} \times \underline{\omega})_k = \underbrace{\epsilon_{ijk}}_{=\epsilon_{kij}} \partial_j\, \epsilon_{kpq}\, u_p\, \omega_q =$$

$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\, \partial_j (u_p \omega_q) =$$

$$= \partial_j (u_i \omega_j) - \partial_j (u_j \omega_i) =$$

$$= \omega_j (\partial_j u_i) + u_i (\partial_j \omega_j) - u_j (\partial_j \omega_i) - \omega_i (\partial_j u_j)$$

$$\underline{A} = (\underline{\omega} \cdot \underline{\nabla})\underline{u} + \underline{u}\,(\cancelto{0}{\underline{\nabla} \cdot \underline{\omega}}) - (\underline{u} \cdot \underline{\nabla})\underline{\omega} - \underline{\omega}\,(\cancelto{0}{\nabla \cdot \underline{u}})$$

kinematic viscosity $\nu = \frac{\mu}{\rho}$

$$\frac{D(*)}{Dt} = \frac{\partial (*)}{\partial t} + (\underline{u} \cdot \underline{\nabla})(*)$$

Finally, we get the vorticity equation

$$\frac{\partial \underline{\omega}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{\omega} = (\underline{\omega} \cdot \underline{\nabla})\underline{u} + \nu \nabla^2 \underline{\omega}$$

vorticity changing in time

vorticity advection by the flow

stretching (amplification) of vorticity

diffusion of vorticity with diffusivity ν

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2} \quad \text{diffusion equation}$$

1) No source of vorticity in the interior of the fluid if ρ, ν = constant and $\underline{f}$ - conservative. Vorticity can only be generated at the boundaries.

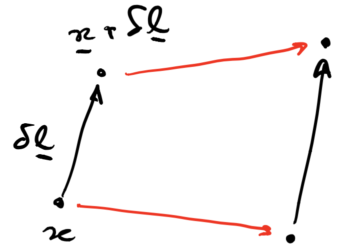
2) If 2D flow $(u_x(x,y), u_y(x,y)) \Rightarrow \underline{\omega} = \omega \underline{\hat{e}}_z$

$\Rightarrow (\underline{\omega} \cdot \underline{\nabla})\underline{u} \equiv 0$ no stretching in 2D.

Vorticity stretching (ballerina effect)

What is the meaning of $(\underline{\omega} \cdot \underline{\nabla})\underline{u}$ term?

Consider a line element $\delta \underline{l}$ moving with the fluid



$\underline{x} + \delta\underline{l}$

$\underline{x} + \delta\underline{l} + \delta t\, \underline{u}(\underline{x} + \delta\underline{l}, t) = \underline{x} + \delta\underline{l} + \delta t [\underline{u}(x,t) + \delta\underline{l} \cdot \nabla \underline{u}(x,t)]$

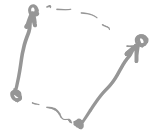
$\delta\underline{l}$

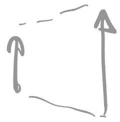
$\underline{x}$

$\underline{x} + \delta t\, \underline{u}(x,t)$

for small $\delta\underline{l}$, $\quad \delta\underline{l} \longrightarrow \delta\underline{l} + \delta t (\delta\underline{l} \cdot \underline{\nabla})\underline{u}$

$$\frac{d}{dt}\,\delta\underline{l} = (\delta\underline{l}\cdot\underline{\nabla})\,\underline{u}$$

→ rotation 

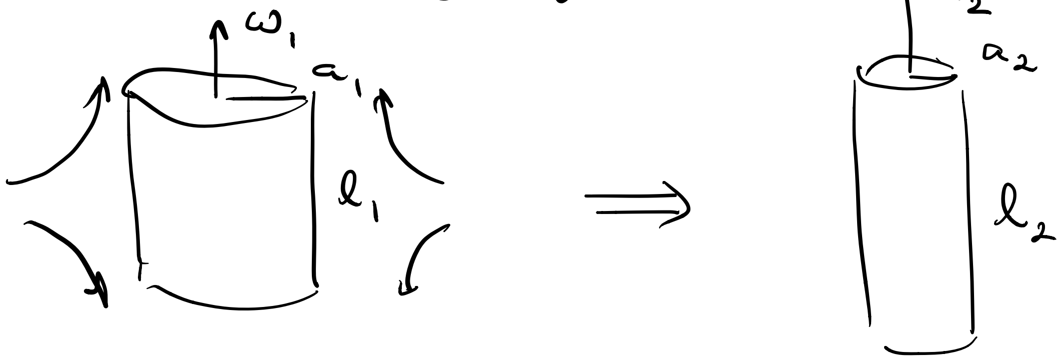
→ stretched 

$(\underline{\omega}\cdot\underline{\nabla})\,\underline{u}$ is analogous

1) Vorticity can be rotated or stretched just like a material line element.

2) Stretching is a result of conservation of angular momentum.

Consider a rotating cylinder



Mass conservation $\quad \rho\, a_1^2 l_1 = \rho\, a_2^2 l_2$

Angular momentum conservation $\quad$ "m" "r" "v"

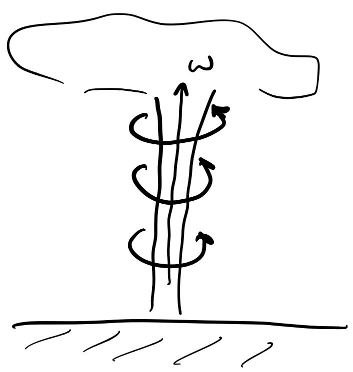
$$\rho\, a_1^2 l_1 \cdot a_1^2 \omega_1 = \rho\, a_2^2 l_2 \cdot a_2^2 \omega_2$$

$$\Rightarrow \boxed{\omega_2 = \frac{l_2}{l_1}\,\omega_1}$$

ω varies linearly with l.

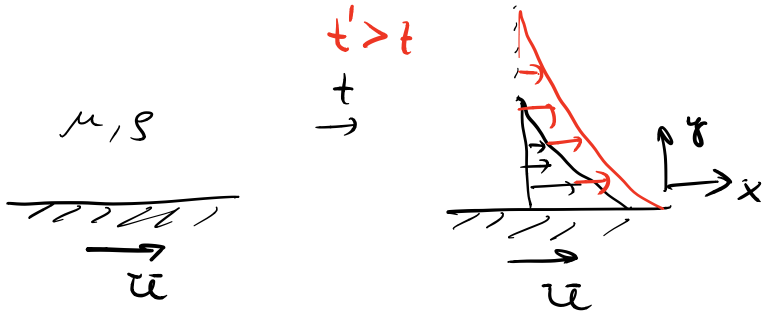
- Stretching vortex tubes intensifies the local vorticity

Ex 1 Tornado



strong thermal updraughts in the thunderclouds stretched the vortex tube, enhancing rotational motion.

Diffusion of vorticity



$\underline{u} = u(y)\underline{e}_x$

When we consider a plate impulsively moving in a viscous fluid, the velocity in the N-S eqs. satisfies the diffusion eq.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

so momentum diffuses from the plate influencing ($u \gtrsim \frac{1}{5} U$) a region next to the plate $y \lesssim \sqrt{\nu t}$.

$$\omega = -\frac{\partial u}{\partial y}$$

One can view the problem alternatively:

- $u(y)$ discontinuous at $y=0$

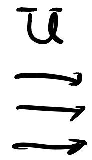
$\Rightarrow$ the flow has $\omega \sim \delta(0)$

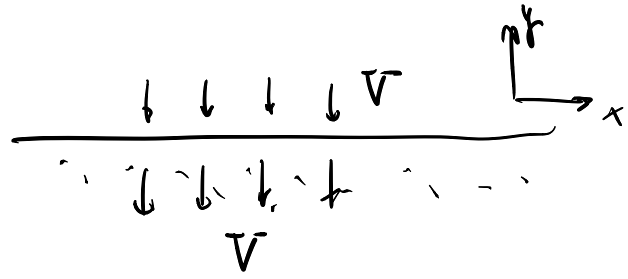
So vorticity was generated at the boundary at $t = 0^+$ and then diffused into the interior of the fluid

$$\partial_t \omega = \nu \partial_y^2 \omega$$

Confinement of vorticity (wall with suction)

- porous boundary
- cross-flow u
- suction flow V
- 2D flow (no stretching)
- steady





- Far away, only suction V and cross-flow U
- take $\nabla p = 0$

Try a solution $\underline{u} = (u(y), -V, 0)$

with b.c. as $y \to \infty$ $u \to U$

as $y = 0$ $u = 0$

- mass conservation automatically satisfied
- momentum eq. (N-S) in the x-direction

$$-\rho V \frac{\partial u}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2}$$

solution

$$u(y) = U\left[1 - \exp\left(-\frac{Vy}{\nu}\right)\right] \qquad \nu = \frac{\mu}{\rho}$$

vorticity $\omega = -\dfrac{\partial u}{\partial y} = -\dfrac{uV}{\nu} e^{-\frac{Vy}{\nu}}$

is confined to a region next to the wall of thickness

$$\delta = \frac{\nu}{V}$$

- Vorticity diffuses away in time t to a distance $\sqrt{\nu t}$ but it is advected back by the suction flow, moving a distance Vt.

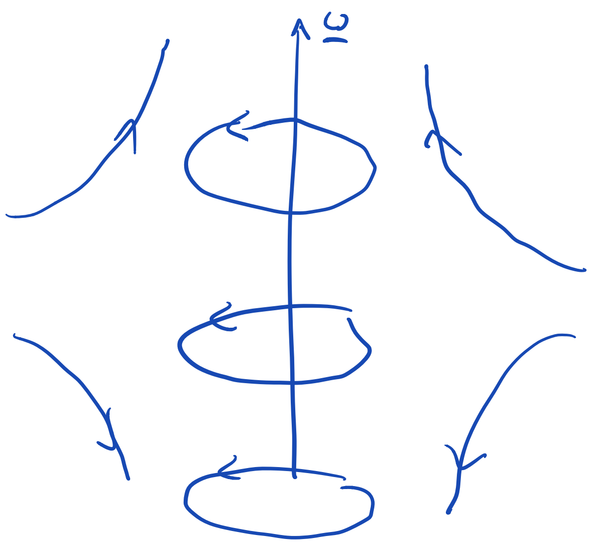
Balancing these, we get $\delta = \frac{\nu}{V}$

$$\left.\begin{aligned} -\rho V \frac{\partial u}{\partial y} &\sim \frac{\rho V U}{\delta} \\ \mu \frac{\partial^2 u}{\partial y^2} &\sim \mu \frac{U}{\delta^2} \end{aligned}\right\} \quad \delta \sim \frac{\nu}{V}$$

then the terms are comparable (diffusion balances advection)

Burgers vortex

- exact solution of N-S eqs.
- steady flow
- balance of advection, diffusion & stretching



Combination of:

a) axisymmetric swirl

$$\underline{u} = v(r)\, \underline{e}_\theta$$

b) axisymmetric stretching

$$\underline{u} = -\alpha r\, \hat{\underline{e}}_r + 2\alpha z\, \hat{\underline{e}}_z$$

This flow field satisfies mass conservation:

$$\nabla \cdot \underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = -2\alpha + 0 + 2\alpha = 0$$

Vorticity comes only from swirling motion:

$$\underline{\omega} = \underline{\nabla} \times \underline{u} = (0, 0, \omega(r)) \qquad \omega(r) = \frac{1}{r}\frac{\partial}{\partial r}(r v)$$

Now we take the vorticity eq.

$$\cancel{\frac{\partial \omega}{\partial t}} + (\underline{u} \cdot \underline{\nabla})\, \underline{\omega} = (\omega \cdot \underline{\nabla})\, \underline{u} + \nu \nabla^2 \underline{\omega}$$

0 only non-zero component in the z-direction:

$$(\underline{u} \cdot \underline{\nabla})\, \underline{\omega} = -\alpha r \frac{\partial \omega}{\partial r}$$

$$(\underline{\omega} \cdot \nabla)\, \underline{u} = \omega(r) \frac{\partial}{\partial z}(2\alpha z)$$

$$\nu \nabla^2 \underline{\omega} = \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \omega}{\partial r}\right)$$

So

$$\boxed{-\alpha r \frac{d\omega}{dr} = \omega \frac{d}{dz}(2\alpha z) + \frac{\nu}{r}\frac{d}{dr}(r\omega')}$$

$$\frac{d}{dr}\left(\alpha r^2 \omega + \nu r \frac{d\omega}{dr}\right) = 0$$

Integrate: with $\omega \to 0$ as $r \to \infty$

$$\omega = \omega_0 e^{-\frac{\alpha r^2}{2\nu}} = \frac{1}{r}\frac{d}{dr}(\theta(r))$$

we find

$$v(r) = \frac{\nu \omega_0}{\alpha r}\left[1 - e^{-\frac{\alpha r^2}{2\nu}}\right]$$

Near the axis $r \ll \delta = \sqrt{\frac{\alpha}{\nu}}$

$$v(r) \sim \frac{1}{2}\omega_0 r \qquad \text{(solid body rotation)}$$

Far away

$$v(r) \sim \frac{1}{r}\frac{\nu \omega_0}{\alpha} \qquad \text{(line vortex)}$$