# INVISCID VORTEX DYNAMICS

Vorticity equation $\underline{\omega} = \nabla \times \underline{u}$

$$\frac{D\underline{\omega}}{Dt} = \boxed{\frac{\partial \underline{\omega}}{\partial t} + \underbrace{(\underline{u}\cdot\nabla)\underline{\omega}}_{\text{advection}} = \underbrace{(\underline{\omega}\cdot\underline{\nabla})\underline{u}}_{\text{stretching}} + \underbrace{\nu \nabla^2 \underline{\omega}}_{\text{diffusion}}}$$

$\nu = \frac{\mu}{\rho}$

- incompressible $\rho$ = const.
- $\mu$ = const.
- conservative body forces

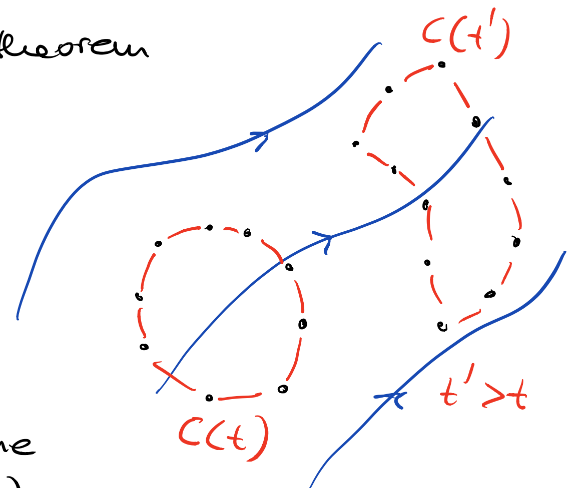
3 competing mechanisms

What happens when the flow is inviscid?

$$\mu = \nu = 0.$$

## ① Kelvin's circulation theorem

- inviscid fluid
- $\rho$ = const
- conservative forces.




**Theorem**

Let $C(t)$ denote a closed circuit that consists of the same fluid particles as time proceeds (material curve)

Then the circulation

$$\Gamma = \oint_{C(t)} \underline{u}\cdot d\underline{x}$$

is independent of time.

Proof 1) For an inviscid fluid → Euler's equation

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nabla\psi$$

body force potential

Take:

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_{C(t)} \underline{u}\cdot d\underline{x} = \underbrace{\oint_C \frac{D\underline{u}}{Dt}\cdot d\underline{x}}_{1^\circ\ =0} + \underbrace{\oint \underline{u}\cdot\frac{D\,d\underline{x}}{Dt}}_{2^\circ}$$

$1^\circ$ $$\oint_C \frac{D\underline{u}}{Dt}\cdot d\underline{x} = \oint_C\left(-\frac{1}{\rho}\nabla p + \nabla\psi\right)\cdot d\underline{x} = (*)$$

$$= \oint_C \nabla\left(-\frac{p}{\rho}+\psi\right)\cdot d\underline{x} = 0$$

F.T.C. $$\int_A^B \nabla(\ldots) = F(B) - F(A)$$

if $\rho \neq$ const.

but $\rho = \rho(p)$ (barotropic fluid)

$$(*) = \{\text{Stokes theorem}\} = \int_S \nabla\times\left(-\frac{1}{\rho}\nabla p + \nabla\psi\right)\cdot\underline{n}\,dS = 0$$

$$\nabla\times\nabla f = 0$$

$2^\circ$ evolution of a material line

we derived previously: $$\frac{D(d\underline{x})}{Dt} = (d\underline{x}\cdot\underline{\nabla})\underline{u}$$

use this

$$= \oint_C \underline{u}\cdot(d\underline{x}\cdot\underline{\nabla})\underline{u} = \frac{1}{2}\oint_C \nabla(|u|^2)\cdot d\underline{x} \overset{\text{FTC}}{=} 0$$

so $$\frac{D\Gamma}{Dt} = 0$$

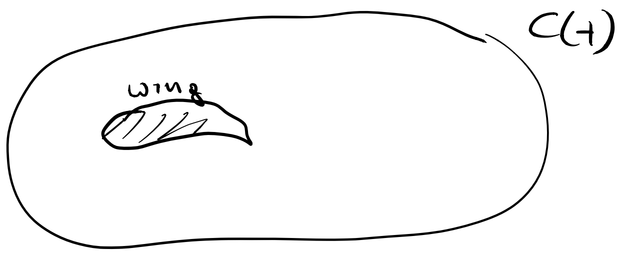
QED.

Notes

(a) $C$ is a material curve ('dyed'); the result is not true for any curve fixed in space.

(b) The theorem does not require the fluid to be a simply connected region.

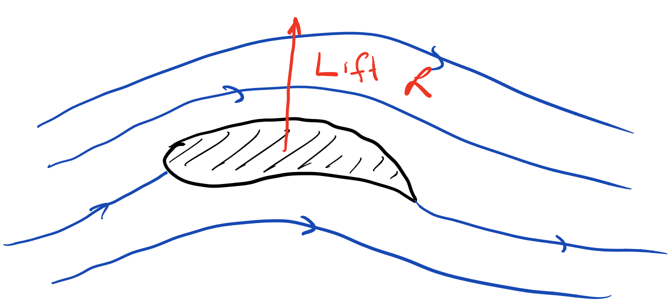
e.g.




is okay.

(c) The inviscid eqs. of motion only enter in evaluating the integral around $C$, so if viscous forces happen to be important elsewhere in the flow (off the curve $C$), these would not affect the conclusion.

## Lift generation on an airfoil




Kutta - Joukowski Lift Theorem (book by Acheson)

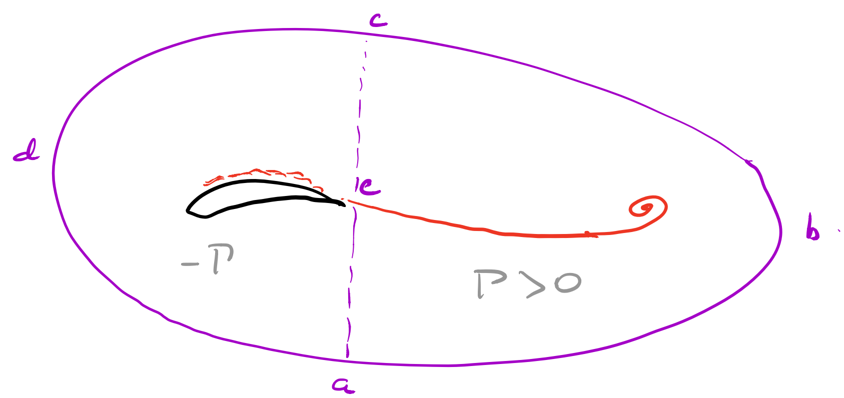
$$\boxed{\mathcal{L} = -\rho U \Gamma}$$

how is $\Gamma$ established? (H & E)

{ A simple Kelvin th. argument

- Choose a large circuit around.
- originally we start at $\Gamma = 0$ at $t = 0$.

$\Rightarrow \Gamma(abcd) = 0$ at all times.

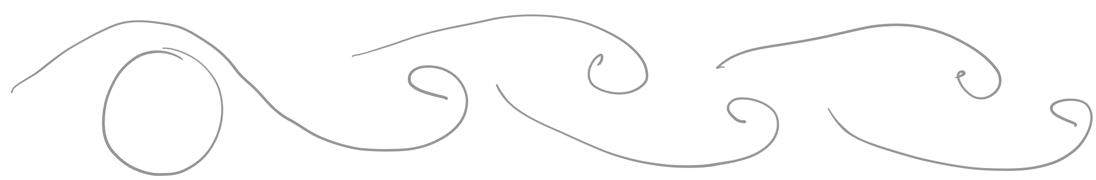


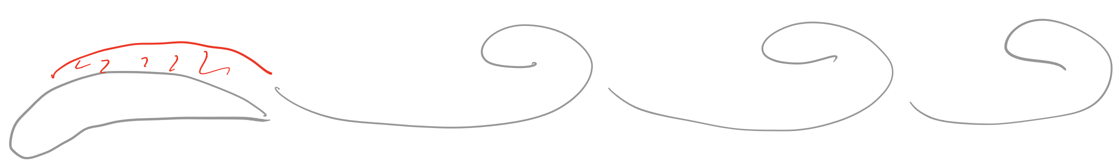

so $\Gamma(abcea) = -\Gamma(aecda)$

In abcea, a vortex of positive vorticity induces circulation.

(Stokes' theorem) $\int_S \overset{(\nabla\times\underline{u})}{\underline{\omega}} \cdot \underline{n}\, dS = \int_C \underline{u} \cdot d\underline{x} = \Gamma > 0$

. so Γ is established.





# The persistence of irrotational flow

For an inviscid incompressible fluid in a conservative body force field if a portion of the fluid is in an irrotational motion, it will always remain in irrotational motion. (Cauchy - Lagrange theorem)

Proof

Suppose that $\underline{\omega} = \underline{\nabla} \times \underline{u}$ is not identically 0 at a certain moment. Then (Stokes)

$$\int_C \underline{u} \cdot d\underline{x} = \int_S \underline{\omega} \cdot \underline{n}\, dS \quad \text{would be nonzero}$$

around a small selected circuit. But $\Gamma \neq 0$ would violate Kelvin's theorem, since $\Gamma = 0$ at $t = 0$.

For a 2D flow (ideal fluid) the vorticity eq is:

$$\frac{D\underline{\omega}}{Dt} = 0$$

so the result above is obvious, but in 3D it is not.

(Viscosity is the source of vorticity)

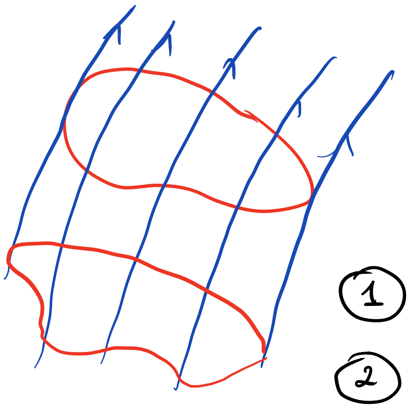
## Helmholtz vortex theorems

A vortex line at any time $t$ is a line which has the same direction as $\underline{\omega} = \nabla \times \underline{u}$

A vortex line $\underline{x} = \underline{x}(s)$ is obtained by solving

$x = x(s)$
$y = y(s)$
$z = z(s)$

$$\frac{\frac{dx}{ds}}{\omega_x} = \frac{\frac{dy}{ds}}{\omega_y} = \frac{\frac{dz}{ds}}{\omega_z}$$

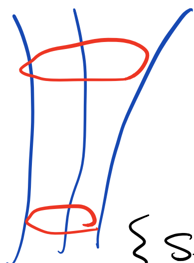


Vortex lines passing through a single closed curve bound a <u>vortex tube</u>

<u>Helmholtz vortex theorems</u>

(1) Vortex lines move with the fluid.

(2) $\Gamma = \int_S \underline{\omega} \cdot \underline{n} \, dS$ is the same for all cross-sections of a vortex tube.



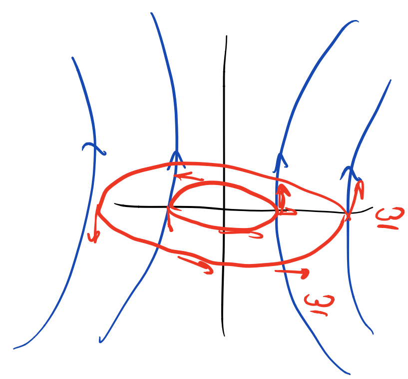
$\Gamma$ is independent of time and is called 'strength' of the tube.

{ Special case : axisymmetric flow (3D)

$$\underline{u} = u_R(R, z, t)\, \hat{\underline{e}}_R + u_z(R, z, t)\, \hat{\underline{e}}_z$$

$$\underline{\omega} = \underline{\nabla} \times \underline{u} = \omega\, \hat{\underline{e}}_\varphi$$

$$\omega = \frac{\partial u_R}{\partial z} - \frac{\partial u_z}{\partial R}$$




In axisymmetric flow, vortex tubes are ring-shaped. Because they are material lines, they might expand or contract about the axis.

Cross-sectional area $\delta S$ of a thin tube will be proportional to $\frac{1}{2\pi R}$.

But $\omega \delta S$ = constant. By (2) $\omega \propto 2\pi R$

One can show that $\frac{D}{Dt}\left(\frac{\omega}{R}\right) = 0$.

When a vortex tube is surrounded by irrotational motion, we call it a <u>vortex ring</u>.

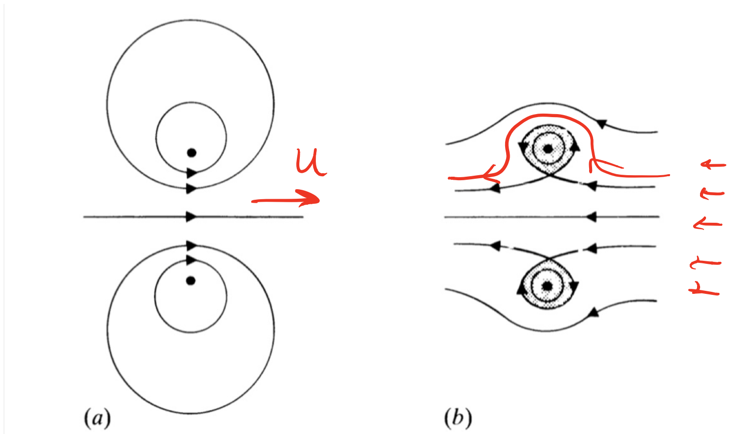
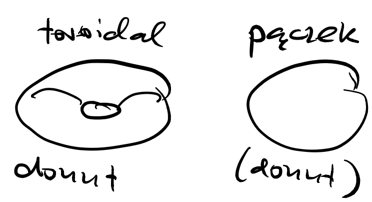



**Fig. 5.7.** Flow due to a vortex ring (*a*) relative to a fixed frame and (*b*) relative to a frame moving with the vortex core. Shading denotes smoke, in the case of a smoke ring, while the vortex core is indicated by the black dots.

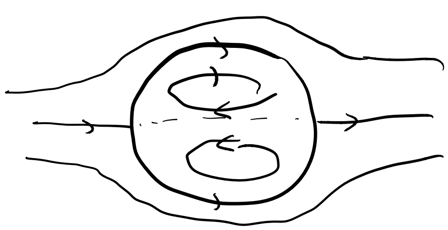
https://www.youtube.com/watch?v=yjgACB7urOo&t=210s

Look up YT videos from the Royal Institution!

We will now analyse the motion of an isolated patch of vorticity (axisymmetric)




<u>Hill's vortex</u>



In order to find the flow field, we introduce 2 concepts.

# The Stokes stream function

For 2D incompressible flow, the flow field can be represented by a stream function $\psi$ such that

$$u_x = \frac{\partial \psi}{\partial y} \qquad u_y = -\frac{\partial \psi}{\partial x} \qquad \underline{u} = \nabla \times (\psi \hat{\underline{e}}_z)$$

Why? Because then:

$$\underline{\nabla} \cdot \underline{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

automatically.

$\psi$ is constant on streamlines, because

$$(u \cdot \nabla)\psi = 0 \iff \underline{u} \cdot \nabla \psi = 0$$

For an axisymmetric incompressible flow in 3D, we have an analogous representation.

$$\underline{u} = \nabla \times (\psi' \hat{\underline{e}}_\varphi)$$

We define $\Psi$ to be constant on streamlines and it turns out that $\psi' = \frac{\Psi}{R}$, so

$$\underline{u} = \nabla \times \left( \frac{\Psi}{R} \hat{\underline{e}}_\varphi \right)$$

in spherical polar coordinates

$$\underline{u} = \nabla \times \left( \frac{\Psi}{r \sin\theta} \hat{\underline{e}}_\varphi \right)$$

whence

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \Psi}{\partial \theta} \qquad u_\theta = -\frac{1}{r \sin\theta} \frac{\partial \Psi}{\partial r}$$

We also have

$$(\underline{u}\cdot\underline{\nabla})\Psi = u_r \frac{\partial \Psi}{\partial r} + \frac{u_\theta}{r}\frac{\partial \Psi}{\partial \theta} = 0$$

$\Psi$ is constant along streamlines.

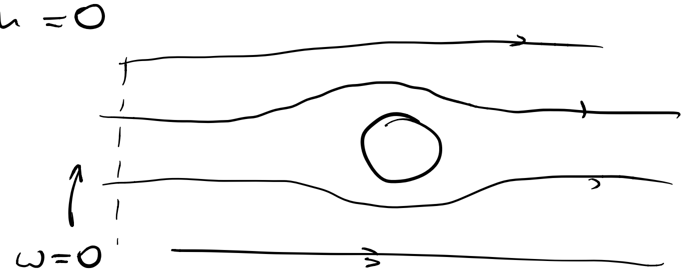
<u>Irrotational flow past a sphere</u>

<u>Steady</u> flow, axisymmetric

$$\frac{D}{Dt}\left(\frac{\omega}{R}\right) = 0 \Rightarrow (\underline{u}\cdot\underline{\nabla})\left(\frac{\omega}{R}\right) = 0 \Leftrightarrow (\underline{u}\cdot\underline{\nabla})\left(\frac{\omega}{r\sin\theta}\right) = 0$$

$\frac{\omega}{r\sin\theta}$ is const. along streamlines.

- Consider a uniform flow past a rigid sphere $r=a$
- $\mu = 0$




so $\omega = 0$ everywhere in $r > a$ $\Rightarrow$ Irrotational flow

$\omega = \omega\,\hat{\underline{e}}_\varphi$, where

$$\omega = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

so in terms of $\Psi$:

$$\omega = -\frac{1}{r\sin\theta}\underbrace{\left[\frac{\partial^2 \Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial\theta}\right]\right]}_{=0} = 0$$

need to solve

B.C.

- on $r=a$ $\quad \Psi = 0$ $\quad$ (streamline)
- at $\infty$ we have $\quad u_r \sim U\cos\theta$

$$u_\theta \sim -U\sin\theta$$

$$\Rightarrow \quad \Psi = \frac{1}{2} U r^2 \sin^2\theta \quad \text{as } r \to \infty$$

we try a solution of the form $\quad \Psi(r,\theta) = f(r)\sin^2\theta$

(solve)

We got $\Psi = \dfrac{1}{2} U \left( r^2 - \dfrac{a^3}{r} \right) \sin^2\theta$

There is a slip velocity on the surface

$$u_\theta = -\frac{3}{2} U \sin\theta \quad \text{on } r=a$$

Hill's spherical vortex $\quad$ suppose that $r<a$ is also filled with fluid.

The flow inside is rotational.

- $\Psi$ can vary between streamlines

we claim

$$\frac{\omega}{r\sin\theta} = c(\Psi) \quad \text{in } r \leq a$$

So

$$\frac{\partial \Psi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta} \left( \frac{1}{\sin\theta} \frac{\partial \Psi}{\partial \theta} \right) = -c(\Psi)\, r^2 \sin^2\theta$$

with B.C. $\quad \dfrac{\partial \Psi}{\partial r} = \dfrac{3}{2} U a \sin^2\theta \quad \text{on } r=a$

We try $\Psi = g(r)\sin^2\theta$

We find:

$$\Psi = -\frac{3}{4} U r^2 \left(1 - \frac{r^3}{a^2}\right) \sin^2\theta$$

Circulation around any hemispherical cross-section is (by Stokes')

$$\Gamma_{max} = \int_0^{\pi}\int_0^{a} \omega r \, dr \, d\theta = c \int_0^{\pi}\int_0^{a} r^2 \sin\theta \, dr \, d\theta = -5Ua$$

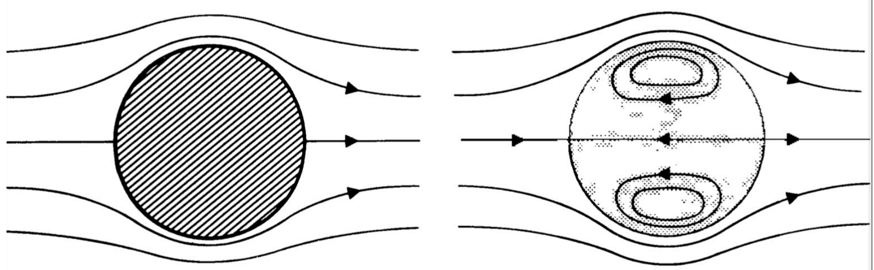
Equivalently, the Hill's spherical vortex moves at

$$U = \frac{\Gamma_{max}}{5a}$$

### Hill's spherical vortex

Let us now suppose instead that the region $r < a$ is also filled with fluid. Remarkably, it is possible to find a closed-streamline inviscid flow in $r < a$ which matches on to eqn (5.21) in the sense that (i) $\Psi$ is zero on $r = a$ and (ii) the tangential component of velocity $u_\theta$ matches with eqn (5.22) on $r = a$.

In this closed-streamline region (5.16) tells us only that $\omega/r \sin\theta$ is constant along each streamline; there is no reason to suppose it is the same constant along each one, let alone zero.



(*a*) (*b*)

**Fig. 5.12.** (*a*) Irrotational flow past a sphere. (*b*) Hill's spherical vortex.

The most we can claim, then, is that

$$\frac{\omega}{r \sin\theta} = c(\Psi) \qquad \text{in } r \leqslant a,$$

where the function $c(\Psi)$ is at this stage unknown. Using eqn (5.18) this implies that

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) = -c(\Psi) r^2 \sin^2\theta \qquad (5.23)$$

is $r \leqslant a$, and $c(\Psi)$ is to be determined as part of the solution (if, indeed, such a solution exists).

Now, in order that $u_\theta$ matches with eqn (5.22) on $r = a$ we need

$$\frac{\partial\Psi}{\partial r} = \tfrac{3}{2}Ua \sin^2\theta \qquad \text{on } r = a, \qquad (5.24)$$

and this suggests trying $\Psi = g(r)\sin^2\theta$ in eqn (5.23). The left-hand side is then a function of $r$ times $\sin^2\theta$, and the form of the right-hand side then shows that $c(\Psi)$ will need to be a constant, $c$, if eqn (5.23) is to reduce to an ordinary differential equation for $g(r)$. The function $g(r)$ then emerges as

$$g(r) = Ar^2 + \frac{B}{r} - \tfrac{1}{10}cr^4.$$

We must choose $B = 0$ to keep $\boldsymbol{u}$ finite at $r = 0$, and $A$ must then be chosen so that $\Psi = 0$ on $r = a$. Finally, eqn (5.24) implies that $c = -15U/2a^2$, so

$$\Psi = -\tfrac{3}{4}Ur^2\left(1 - \frac{r^2}{a^2}\right)\sin^2\theta \qquad \text{in } r \leqslant a. \tag{5.25}$$

The corresponding streamlines are sketched in Fig. 5.12($b$).

The circulation round these streamlines varies from one to the other, of course, because the flow in $r \leqslant a$ has vorticity, but the circulation round the perimeter of a full hemispherical cross-section is, by Stokes's theorem,

$$\Gamma_{\max} = \int_0^\pi \int_0^a \omega r \, \mathrm{d}r \, \mathrm{d}\theta = c \int_0^\pi \int_0^a r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta = -5Ua.$$

Equivalently, a Hill spherical vortex will travel through stationary fluid with uniform speed $\Gamma_{\max}/5a$, distinguished from an ordinary smoke ring by the absence of a hole and by the way in which the vorticity is spread throughout the whole of the closed streamline region (cf. Fig. 5.7($b$)).

https://www.youtube.com/watch?v=yjgACB7urOo&t=210s