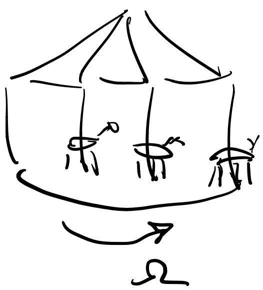
# ROTATING FLOWS

In a rotating frame of reference



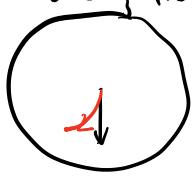

A carousel $R = 5m$, $T = 6s$

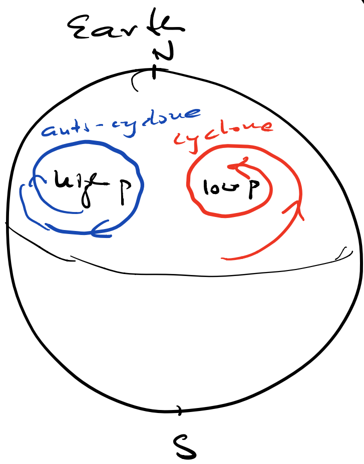
centrifugal force at the rim $\sim 50\%$ gravity

walking across at $1\,m/s$

$\to$ Coriolis force (*) $\sim 20\%$ gravity

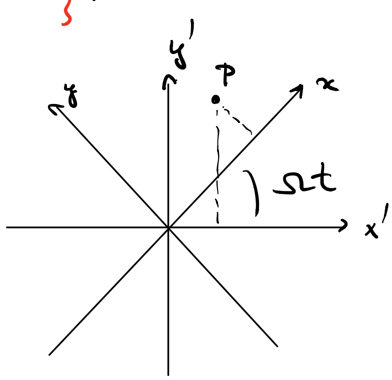
looking from above






Coriolis important!

{ Inertial forces in a rotating frame of reference




- in a rotating system $(x, y, z)$
- LAB frame (inertial) $(x', y', z')$

with an $\Omega$, so

$$x = x' \cos \Omega t + y' \sin \Omega t \qquad (*)$$
$$y = -x' \sin \Omega t + y' \cos \Omega t$$
$$z = z'$$

Velocity : $\underline{v} = \frac{d\underline{x}}{dt}$ in terms of $\underline{v}' = \frac{d\underline{x}'}{dt}$

$\frac{d}{dt}(*)$

$$\begin{cases} v_x = v_x' \cos\Omega t + v_y' \sin\Omega t \underbrace{- \Omega x' \sin\Omega t + \Omega y' \cos\Omega t}_{+\Omega y} \\ v_y = -v_x' \sin\Omega t + v_y' \cos\Omega t - \Omega x \\ v_z = v_z' \end{cases}$$

Acceleration $\underline{a} = \frac{d\underline{v}}{dt}$ in terms of $\underline{a}' = \frac{d\underline{v}'}{dt}$.

Introduce a rotation matrix $\underline{\underline{A}}$ with angle $\phi = \Omega t$

Then :

$$\begin{cases} \underline{x} = \underline{\underline{A}} \cdot \underline{x}' \\ \underline{v} = \underline{\underline{A}} \cdot \underline{v}' - \underline{\Omega} \times \underline{x} \\ \underline{a} = \underline{\underline{A}} \cdot \underline{a}' - \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) - 2\underline{\Omega} \times \underline{v} \end{cases}$$

for a general $\underline{\Omega}$

So, Newton's 2nd law : $\underline{\Omega}$ = constant

$$m \frac{d^2\underline{x}}{dt^2} = \underline{f} \underbrace{- m\,\Omega \times (\Omega \times \underline{x})}_{\text{centrifugal force}} - 2m \underbrace{\Omega \times \frac{d\underline{x}}{dt}}_{\text{Coriolis force}}$$

both $\propto m$, like gravity.

§ Estimates for Earth

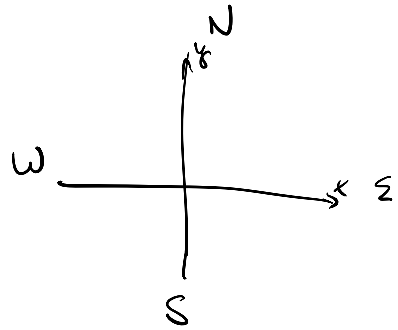
- scale for centrifugal force $\Omega^2 R / g \sim \frac{1}{291}$ small!

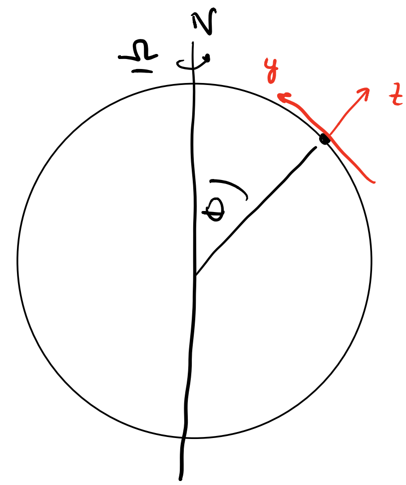
  typically unimportant

  Earth's radius 6371 km(?)

Coriolis force is different!

- $x$ - to the East
- $y$ → to the North
- $z$ → locally vertical




In this system $\underline{\Omega} = \Omega(0, \sin\theta, \cos\theta)$.

The Coriolis acceleration $\underline{g}^c = -2\,\underline{\Omega} \times \underline{v}$

$$\underline{g}^c = 2\Omega \begin{pmatrix} v_y \cos\theta - v_z \sin\theta \\ -v_x \cos\theta \\ v_x \sin\theta \end{pmatrix}$$

Horizontal at poles; at equator - vertical for horizontal motion.

$g^c_z$ is typically very small compared to gravity.
We can neglect it (as well as centrifugal force)

In normal circumstances, we approximate:

$$\underline{g}^c = 2\,\Omega_\perp \begin{pmatrix} +v_y \\ -v_x \\ 0 \end{pmatrix} \qquad \Omega_\perp = \Omega \cos\theta$$

local angular velocity

For all practical purposes, the Coriolis force in a local flat-Earth coordinate system looks as if Earth was flat, rotating with $\Omega_\perp$ about the local vertical.

At $\Theta = 45^\circ$, we have $\Omega_\perp = 10^{-4}\ \frac{1}{s}$

Period of a Foucault pendulum $\frac{2\pi}{\Omega_\perp} \sim 34\,h$

## Flow in a rotating system

- Steady rotation.
- Incompressible flow.

Navier - Stokes eqs.

$$\begin{cases} \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v} + 2\underline{\Omega}\times\underline{v} - \Omega\times(\Omega\times\underline{r}) = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{v} + \underline{g} \\ \nabla\cdot\underline{v} = 0 \end{cases}$$

Formally, the gravitational and centrifugal fields can be included in the effective pressure:

$$p^* = p + \rho\phi - \frac{1}{2}\rho(\underline{\Omega}\times\underline{r})^2$$

Then N-S become

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v} = -2\,\underline{\Omega}\times\underline{v} - \nabla\left(\frac{p^*}{\rho}\right) + \nu\nabla^2\underline{v}$$

+ At Earth's surface use $\underline{\Omega}_\perp$ for Coriolis.

### Dimensionless numbers

length scale $L$, velocity scale $\mathcal{U}$

Reynolds no. $\qquad Re = \frac{\mathcal{U}L}{\nu}$

Typically $Re \gg 1$ advection $\gg$ viscosity

$$\frac{\text{advective acc.}}{\text{Coriolis acc.}} = \text{Rossby no.}$$

$$Ro = \frac{|(\underline{v}\cdot\nabla)\underline{v}|}{|2\underline{\Omega}\times\underline{v}|} \approx \frac{u^2/L}{2\Omega u} = \frac{u}{2\Omega L}.$$

For nearly ideal flow, the Coriolis force is significant only if $Ro \lesssim 1$, $u \lesssim 2\Omega L$

Ex

Ocean currents and weather cyclones

$u \sim 1\,m/s$ $\qquad$ $u \sim 10\,m/s$

$L \sim 1000\,km$

$2\Omega \sim 10^{-4}\,s^{-1}$ $\Rightarrow$ $Ro \approx 0.01$ ocean currents

$0.1$ cyclones

both dominated by Coriolis force

Swimming $L \sim 1m$ $\quad u \sim 1\,m/s$ $\quad Ro \approx 10^4$

The ratio

$$\frac{\text{viscous forces}}{\text{Coriolis forces}} = \text{Ekman no}$$

$$Ek = \frac{|\nu\nabla^2\underline{v}|}{|2\underline{\Omega}\times\underline{v}|} \approx \frac{\nu\frac{u}{L^2}}{2\Omega u} = \frac{\nu}{2\Omega L^2} = \frac{Ro}{Re}$$

Small Ek – viscosity negligible

large Re + moderate Ro = Ek small.

Ek large close to boundaries when viscosity dominates over advection. → Ekman layer

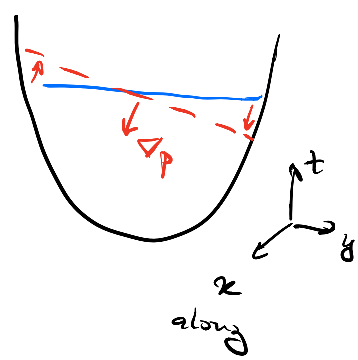
## § Geostrophic flow

- Natural large-scale systems : Ro, Ek $\ll 1$
- ignore advection, viscosity
- only Coriolis
- steady

N-S takes the form: steady geostrophic flow

$$\boxed{-2(\underline{\Omega} \times \underline{v}) - \frac{1}{\rho}\nabla p^* = 0} \qquad (**)$$

Water level in an open canal




flow $u$ $\quad \underline{\Omega} \parallel \hat{\underline{e}}_z$

width $d$

we have

$$\nabla_x p^* = \nabla_z p^* = 0$$

and $$\frac{1}{\rho}\frac{\partial p^*}{\partial y} = -2\Omega u \quad \rightarrow \quad p^* = -2\rho u \Omega y$$

$$p = p^* - \rho g z = -\rho(2\Omega u y + g z)$$

$$h(y) = -\frac{2\Omega u}{g}\, y$$

In the N hemisphere water level highest on the RHS bank of the stream

Ex. For a canal $d \sim 10\,\text{km}$, $U \sim 1\,\text{m/s}$

$h_{max} \sim 10\,\text{cm}$

$Ro = U/2\Omega d \sim 1$, $Re \sim 10^{10}$

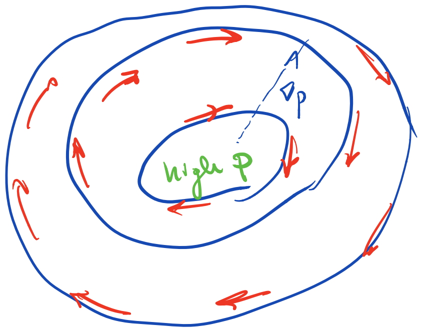
## Isobaric flow and weather maps

$\underline{v} \cdot (**)$

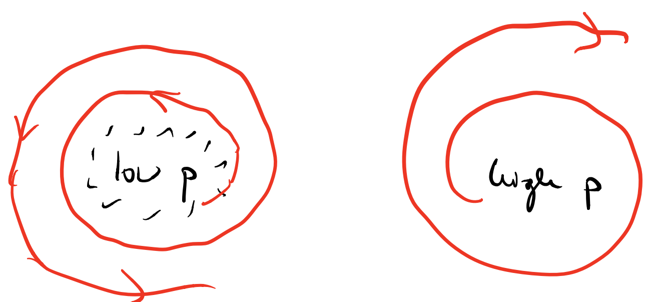
$$\underline{v} \cdot 2\cancelto{0}{(\underline{\Omega} \times \underline{v})} - \frac{1}{\rho}\, \underline{v} \cdot \nabla p^* = 0$$

$$\underline{\underline{v} \cdot \nabla p^* = 0}$$

Streamlines and isobars coincide in geostrophic flow




In N hemisphere




cyclone    anti-cyclone

We can invert (* *) to find $\left\{ \Omega \times (\Omega \times \underline{v}) = -\Omega^2 \underline{v} \right\}$

$\underline{\Omega} \times (**)$

$$\underline{v} = \frac{\Omega \times \nabla p^*}{2\Omega^2 \rho}$$

wind velocity from the pressure gradient.

## § Two-dimensionality of geostrophic flows

$\underline{\Omega} \cdot (**)$

$$(\underline{\Omega} \cdot \nabla) p^* = 0$$

$$\Omega \frac{\partial p^*}{\partial z} = 0 \qquad p^* \text{ is constant along the axis of rotation.}$$

Hydrostatic pressure

$$p = p^*(x, y) - \rho g z$$

Take $\nabla \times (**)$ and

$$0 = \nabla \times (\underline{\Omega} \times \underline{v}) + \cancel{\nabla \times \tfrac{1}{\rho} \nabla p^*}^{\,0}$$

$$0 = \underline{\Omega} (\cancel{\underline{\nabla} \cdot \underline{v}}^{\,0}) - (\underline{\Omega} \cdot \nabla) \underline{v} = -\Omega \frac{\partial \underline{v}}{\partial z}$$

$$\frac{\partial \underline{v}}{\partial z} = 0 \qquad \text{Taylor-Proudman theorem}$$

The flow field is a fcn of $x$ & $y$ only

## § Taylor columns

if one disturbs the flow of a rotating fluid at $z=0$, the disturbance is copied to all values of $z$.

Taylor column
Proudman pillar