CONVECTION

1) Heat - major driving factor
- hurricanes, cyclones, tornadoes
- continental drift
- stellar transport
- central heating

2) Convection is a combination of gravity & material properties — most fluids expand when heated (density decreases with T)

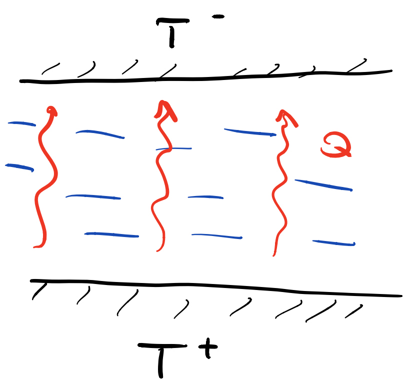
3) Buoyancy forces set the fluid in motion.

Examples:

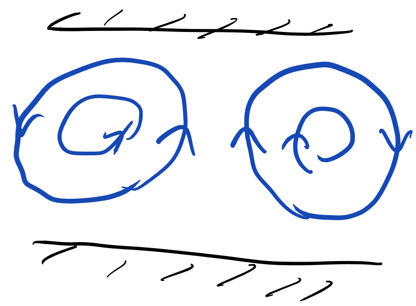
(A) steady laminar convection flow

(B) thermal instability - Rayleigh-Bénard convection

$T^+ > T^-$



$T^+ \gg T^-$



Note: Concentration gradients in mixed fluids can also cause convective flows (sometimes called advective flows) (ocean)

We need to incorporate new variables into the eqs. of motion.

Heat & motion are coupled PDE.
(Typically v. hard to solve analytically)
→ approximations needed.

Thermal expansion coefficient

Isobaric th. exp. coeff.

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

- material 'constant'
- for a gas $\qquad pV = nRT$

$$\Rightarrow \quad \rho \sim \frac{1}{v} \propto \frac{p}{T} \qquad \rho \sim \frac{1}{T}$$

$$\Rightarrow \quad \alpha = \frac{1}{T}$$

air $\quad \alpha \simeq 3 \cdot 10^{-3} \frac{1}{K}$ @ room T.

- for liquids similarly

water $\quad \alpha \simeq 2.5 \cdot 10^{-4} \frac{1}{K}$ @ $25°C$

The resulting change in density

$$\Delta\rho = -\alpha\rho\,\Delta T \qquad \text{for } |\alpha \Delta T| \ll 1$$

→ extra buoyancy term $\Delta\rho\, g$ in N-S eqs.

For a flow with U, L, temperature variation Θ, the ratio of buoyancy to advective terms

$$Ri = \frac{|\Delta\rho g|}{|\rho(\underline{v}\cdot\underline{\nabla})\underline{v}|} = \frac{\alpha\rho\Theta g}{\rho\frac{U^2}{L}} = \frac{\alpha\Theta g}{U^2}L$$

Richardson number

Ri small - advection dominates

$Ri \sim 1$ $\Rightarrow$ flow driven by convection with typical speed $U \sim \sqrt{\alpha\Theta}\sqrt{gL}$

small

free-fall velocity from $\frac{L}{2}$.

HEAT IN FLUIDS

Heat is transported in a fluid $\rightarrow$ law of thermodynamics

First law

$$\underset{\text{heat}}{\Delta E = Q} + \underset{\text{work}}{W}$$

(interaction with environment)

Evolution of energy of a fluid particle.

$$\frac{DE}{Dt} = \dot{Q} + \dot{W}$$

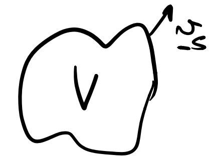
heat transfer rate from the environment

rate of work performed on the fluid element.

$$E = \int_V \rho\epsilon \, dV \qquad \rho\epsilon = \frac{dE}{dV}$$

energy density

Reynolds theorem



$$\frac{D\mathcal{E}}{Dt} = \frac{d\mathcal{E}}{dt} + \oint_{S=\partial V} \rho\epsilon\,(\underline{v} - \underline{v}_s)\cdot d\underline{S}$$

energy production

net outflow from the control volume.

Rate of work

(see Lautrup for details)

through contact forces on the surface of control volume

$$\dot{W} = \oint_S \underline{v}\cdot\underline{\underline{\sigma}}\cdot d\underline{S}$$

Rate of heat transfer

→ conduction (diffusion)
→ advection (convection)
→ radiation

Current density $\underline{q}(\underline{x}, t)$ such that the amount of heat through $d\underline{S}$ in a time δt is $\delta Q = \underline{q}\,\delta t \cdot d\underline{S}$

The total rate of heat transfer into the system becomes

$$\dot{Q} = -\oint_S \underline{q}\cdot d\underline{S} + \int_V h\,dV \qquad (*)$$

internal heat production rate (nuclear, chemical or other)

Internal energy and heat equation

$$\mathcal{E} = \mathcal{T} + \mathcal{V} + \mathcal{U}$$

kinetic

potential

internal energy

$$\mathcal{T} = \int_V \tfrac{1}{2}\rho v^2\,dV$$

$$\mathcal{V} = \int_V \rho\,\Phi\,dV$$

$$\mathcal{U} = \int_V \rho\,u\,dV$$

$$\epsilon = \frac{1}{2}v^2 + \Phi + u$$

$$\frac{DU}{Dt} = \frac{DE}{Dt} - \frac{DT}{Dt} - \frac{DV}{Dt} = \dot{W} + \dot{Q} - \tilde{P}$$

total power without gravity contribution

$$\frac{DU}{Dt} = \dot{Q} + \dot{W}_{int}$$

(work against internal stresses (dissipation))

(*) transferred into a volume integral:

$$\underbrace{\int \rho \frac{Du}{Dt}\, dV}_{=\ \text{advection}} = \int \Big(\underbrace{-\nabla \cdot \underline{q}}_{\text{heat conduction}} + \underbrace{h}_{\text{heat production}} + \underbrace{\sum_{ij} \sigma_{ij} \nabla_j v_i}_{\sum_{ij} e_{ij} e_{ij}} \Big)\, dV$$

rate of dissipation (density)

HEAT EQUATION

<u>Heat flux</u> Second Law implies that heat flows from high T to low T regions.

The law by which this happens is <u>empirical</u>.

For an isotropic medium — Fourier's law of heat conduction

$$\underline{q} = -k \nabla T$$

thermal conductivity

water $k \sim 0.6$ W/Km

air $k \sim 0.025$ W/Km

Fourier's equation

thermodynamic variable $T(\underline{x}, t)$

specific energy $u = u(T)$ is a local fcn of T.

We assume $u = c_0 T$

c_0 ← specific heat capacity

for a fluid at rest $(\underline{v} = 0)$ the heat eq. becomes

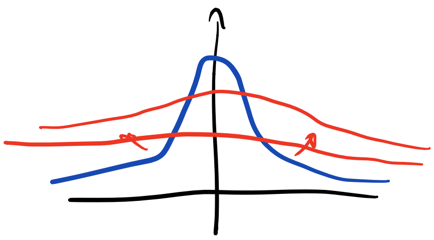
$$\boxed{\rho_0 c_0 \frac{\partial T}{\partial t} = \underbrace{k \nabla^2 T}_{\text{conduction}} + \underbrace{h}_{\text{source}}}$$

If $h = 0$

$$\boxed{\frac{\partial T}{\partial t} = \kappa \nabla^2 T}$$

Fourier's diffusion equation

$$\kappa = \frac{k}{\rho_0 c_0}$$ heat diffusivity



water

$$\kappa = 1.4 \cdot 10^{-7} \frac{m^2}{s}$$

$$\kappa \sim \frac{\nu}{6} \quad \left(\frac{1}{6} \text{ of momentum diffusivity}\right)$$

For a moving fluid we have (approximately)

$$\frac{DT}{Dt} = \boxed{\frac{\partial T}{\partial t} + \underbrace{(\underline{v} \cdot \nabla) T}_{\text{advection}} = \underbrace{\kappa \nabla^2 T}_{\text{diffusion}}}$$

advection time $\quad t_{adv} \sim \dfrac{L}{U}$

diffusion time $\quad t_{diff} \sim \dfrac{L^2}{\kappa}$

$\left\{ \begin{array}{ll} Pe \to \infty & \text{advection-dominated} \\ Pe \to 0 & \text{diffusion-dominated} \end{array} \right.$

Péclet number $\quad Pe = \dfrac{t_{adv}}{t_{diff}} \qquad Pe = \dfrac{|(\underline{v} \cdot \nabla) T|}{|\kappa \nabla^2 T|} \simeq \dfrac{UL}{\kappa}$

Note Full heat eq.

shear viscosity

bulk viscosity

$$\rho c_p \left(\frac{\partial T}{\partial t} + (\underline{v}\cdot\nabla)T \right) = \underbrace{\kappa \nabla^2 T}_{\text{temp. diffusion}} + h + \underbrace{2\eta\, \underline{\underline{e}} : \underline{\underline{e}} + \zeta (\nabla\cdot\underline{v})^2}_{\text{viscous dissipation}}$$

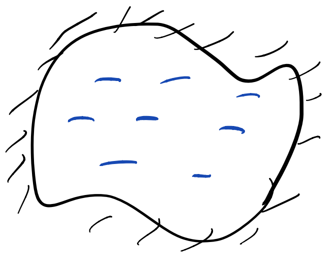
advection

heat production

(only for compressible)

$\frac{\eta}{\rho c_p}\, \underline{\underline{e}} : \underline{\underline{e}}$ is typically small.

The Boussinesq approximation



$\underline{g} = (0, 0, -g)$

- incompressible fluid ρ_0
- temperature T_0
- hydrostatic pressure $p = p_0 - \rho_0 g z$

At a certain time boundary temperatures are changed and a flow $\underline{v}$ is created.

Main assumption: thermal variations are small

i.e. $|\alpha \Delta T| \ll 1$ $\qquad \Delta T = T - T_0$

so we can write

$$\Delta \rho = -\underset{\substack{\uparrow\\ \text{const.}}}{\rho_0} \alpha \Delta T$$

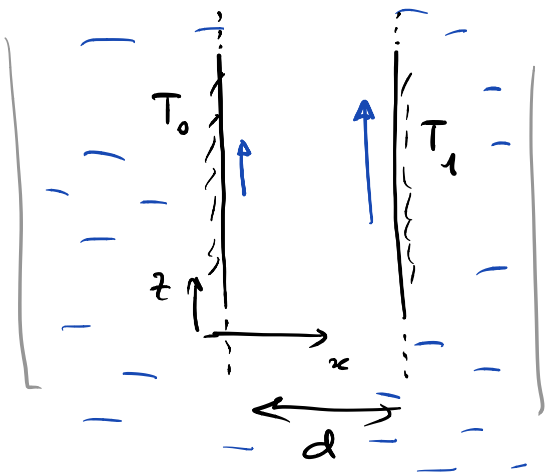
We add to N-S a term $\Delta\rho\, \underline{g} = -\rho_0 \alpha \Delta T\, \underline{g}$.

We modify $p = p_0 - \rho_0 g z + \Delta p$ to get:

Our final eqs:

$$
\begin{cases}
\dfrac{\partial \Delta T}{\partial t} + (\underline{v}\cdot\nabla)\Delta T = \kappa \nabla^2 \Delta T \\[2ex]
\dfrac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v} = -\dfrac{\nabla \Delta p}{\rho_0} + \nu \nabla^2 \underline{v} - \alpha \Delta T\, \underline{g} \\[2ex]
\nabla\cdot\underline{v} = 0
\end{cases}
$$

Steady convection in an open vertical slot heated on one side



- $T_1 > T_0$
- Steady flow of heat & mass

  $\underline{v} = (0, 0, v_z(x))$

  $T = T(x)$

  (Long plates)
- all in a reservoir of fluid

Equations of motion

Heat eq. $\quad \dfrac{\partial}{\partial t} = 0 \qquad ((\underline{v}\cdot\nabla) = 0$ unidirectional flow)

$$\hookrightarrow \quad \nabla^2 T = 0 \iff \frac{d^2}{dx^2} T(x) = 0$$

$$T = T_0 + \Delta T = T_0 + \Theta \frac{x}{d} \qquad \Theta = T_1 - T_0$$

stable profile

N-S $\quad \nabla_x \Delta p = \nabla_y \Delta p = 0$

$$\Delta p = p(z)$$

The z-component:

$$\underbrace{\frac{1}{\rho_0}\frac{\partial}{\partial z}\Delta p(z)}_{\text{dep. on } z} = \underbrace{\nu \frac{\partial^2}{\partial x^2} v_z(x) + \alpha \Theta \frac{x}{d} g}_{\text{dep. on } x}$$

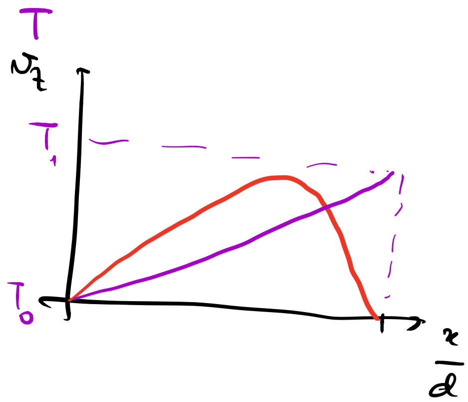
both have to be const.

1) $\Delta p = 0$

2) $\nu\, v_z''(x) + \alpha \Theta \frac{x}{d} g = 0$

+ b.c. $\quad v_z(0) = 0$

$v_z(d) = 0$



we find

$$v_z = \frac{\alpha \Theta g d^2}{6\nu} \frac{x}{d}\left(1 - \frac{x^2}{d^2}\right)$$

. max v at $x = \frac{d}{\sqrt{3}}$

Avg. velocity $\quad U = \frac{1}{d}\int_0^d v_z(x)\,dx = \frac{\alpha \Theta g d^2}{24\nu}$

Reynolds number

$$Re = \frac{Ud}{\nu} = \frac{1}{24}\frac{\alpha \Theta d^3 g}{\nu^2}$$

Ex. • water $\quad d = 1\,cm \quad \Theta = 10\,K \to U \sim 12\,\frac{cm}{s}$

$Re = 1400$ (laminar)

• air $\quad U \sim 8\,cm/s \quad Re \approx 50.$

Entrance length for heat

The rate at which heat is transported to the reservoir is the entire internal energy of the fluid at exit:

$$\dot{Q} = \int_0^d dx\, (\rho c \Delta T) v_z = \frac{\rho c \alpha \Theta^2 g d^3 L}{45\nu}$$

Mass conservation forces the fluid to enter at avg. velocity U. Heat takes $t \sim \frac{d^2}{4\kappa}$ to diffuse across the slot.

By that time the fluid will have moved by

$$l = Ut = \frac{\alpha \Theta g d^4}{96 \kappa \nu} = \frac{Ra}{96} d$$

$$\boxed{Ra = \frac{\alpha \Theta g d^3}{\nu \kappa}}$$ Rayleigh number

thermal transport $\frac{\text{diffusion}}{\text{convection}}$

Ex. $Ra = 2 \cdot 10^5$ $\quad l = 21\,m$ (water)

$Ra = 900$ $\quad l = 10\,cm$ (air)