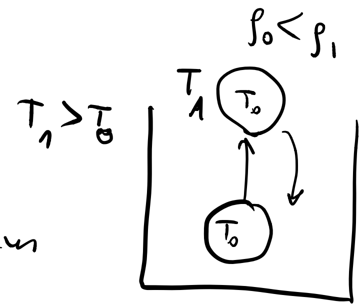
# Convective Instability

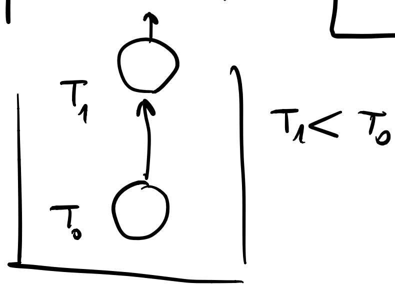
Fluid with a vertical temperature distribution. When is it stable?

- $T = T(z)$
- incompressible fluid
- Boussinesq approximation

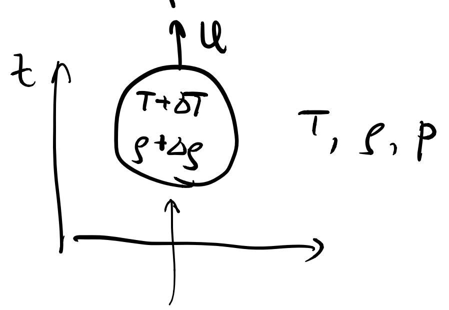



$\frac{dT}{dz} > 0$ stable hydrodynamic equilibrium

$\frac{dT}{dz} < 0$ potentially unstable




A displaced blob may rise further




Drag force $\sim (\text{velocity})^\alpha R^2$

(conductive heat loss) lowers the buoyancy

buoyancy force $\propto R^3$

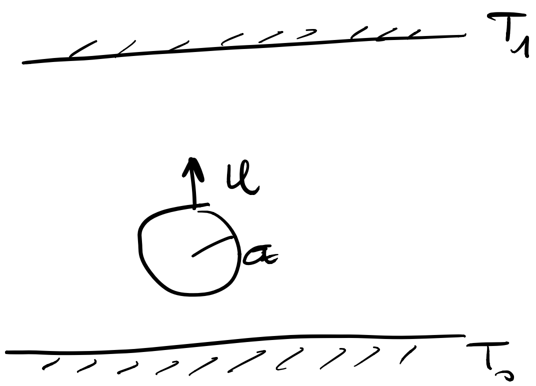
Large blobs will be more unstable than small ones.

## Stability estimate for a vertical blob of fluid

negative vertical temp. gradient

$g \downarrow$

$$\frac{dT}{dz} = -G \qquad T(z) = T_0 - Gz$$




Blob trasfers its excess heat over a diffusion time

$$t \sim \frac{a^2}{4\kappa}$$

During this time, the bubble moves upwards by

$$\Delta z \approx \mathcal{U} t \sim \frac{\mathcal{U} a^2}{4\kappa}$$

The environment cools down by $\Delta T \sim G \Delta z \sim \dfrac{G \mathcal{U} a^2}{4\kappa}$

Steady situation : competition between the environment getting cooler and bubble losing heat leads to $\Delta T$ being time-independent.

Péclet number $\quad Pe = \dfrac{2a\mathcal{U}}{\kappa} \longrightarrow 0$ for $\mathcal{U} \to 0$

$\uparrow$

$\dfrac{\text{advection}}{\text{diffusion}}$

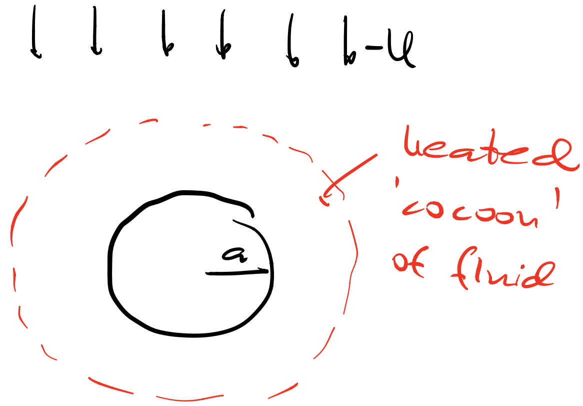
The blob generates a distribution of temp. around it, which decreases the buoyancy force. (for small $\mathcal{U}$ particularly).

Let's estimate this temp. distribution:

We move to the rest frame of the blob.
The true temperature field inside the blob must be

$$T' = T_0 - G(z + \mathcal{U}t) + \Delta T$$

by assumption time-independent




$T'$ obeys the Fourier's heat eq.

$$-GU = \kappa \nabla^2(\Delta T)$$

$$\cancel{\frac{\partial T}{\partial t}} + \underbrace{\underline{v}\cdot\nabla T}_{-U\cdot G} = \kappa \nabla^2 T$$

in spherical coordinates

$$-\frac{GU}{\kappa} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}(\Delta T)\right)$$

$$C_0 - \frac{GU r^3}{3\kappa} = r^2 (\Delta T)'$$

$$\Delta T = C_1 - \cancel{\frac{2C_0}{r^3}}^{\,0} - \frac{GU r^2}{6\kappa}$$

The temperature difference between the inside and outside of the blob becomes

$$\delta T = \Delta T - \Delta T(r=a) = \frac{GU}{6\kappa}\left(a^2 - r^2\right)$$

Upwards buoyancy force

density change $\delta\rho = -\alpha\, \delta T \rho_0$ inside

$$F_B = \int_V \delta\rho\,(-g)\, dV = \rho_0 g \alpha \int_0^a \delta T(r)\, 4\pi r^2\, dr = \frac{4\pi \rho_0 g \alpha G a^5 U}{45\kappa}$$

$\sim a^5$ because volume $\propto a^3$, diffusion time $\propto a^2$

Viscous drag (Stokes law)

$$F_D = 6\pi\eta a U$$

Valid for $Re = \frac{2aU}{\nu} << 1$

The stability condition

$$\boxed{\frac{F_B}{F_D} = \frac{2}{135}\,\frac{g\alpha G a^4}{\kappa\nu} < 1}$$

Puts an upper limit on the size of stable blobs

$d = 2a$

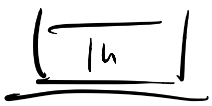
Rayleigh number (critical)

$$Ra_c \equiv \frac{g\alpha G d^4}{\kappa\nu} < 1080$$

<u>Example</u> Pot of water

$\Theta \approx 30\,K$ depth $h = 10\,cm$

$T = 20°C$ $\searrow$ $G = \dfrac{\Theta}{h} = 300\ K/m$




$T = 50°C$ The stability limit $d \lesssim 3.7\ mm$

If the fluid was heavy porridge with $\nu = 1\ m^2/s$ then $d \approx 12\ cm$

<u>Terminal blob speed</u>

For $Ra > Ra_c$ the blob will accelerate

For large $Re$, $F_D = \frac{1}{4}\rho\pi a^2 u^2$ will eventually balance buoyancy.

$F_D = F_B$ for $Ra > Ra_c$ gives

$$U \approx \frac{2}{45} \frac{g\alpha G d^3}{k}$$

Ex. water blob w. $d = 1\,cm$ will reach $U \approx 23$ cm/s.

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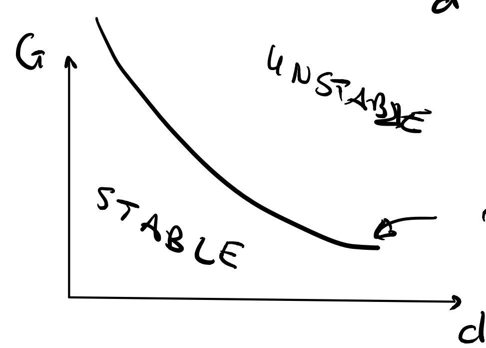
## LINEAR STABILITY ANALYSIS OF CONVECTION

→ linearise the eqs. of motion (book by Lautrup)

→ around a baseline state which may be stable or unstable.

Parameters: $G \equiv \nabla T$

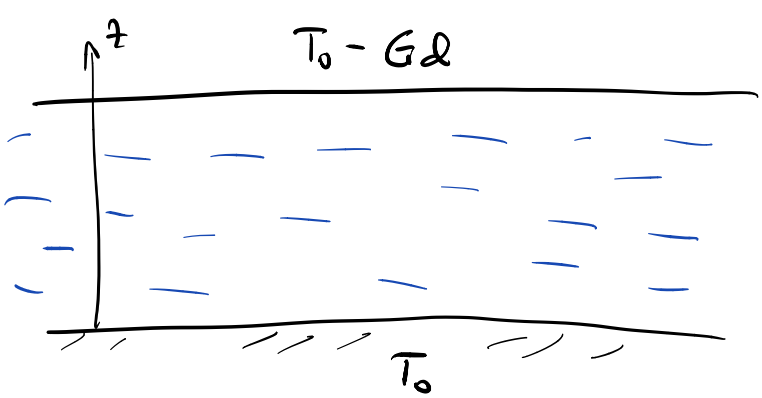
$d$




$\sim d^{-4}$ critical surface

The baseline state:

- incompressible fluid
- at rest $\underline{v} = 0$
- in hydrostatic equilibrium
- vertical negative temp. gradient $\boxed{T = T_0 - Gz}$



The pressure obeys the eqs. of hydrostatic equilibrium with the modified density

$$\rho = \rho_0 \left( 1 - \alpha (T - T_0) \right)$$

Equilibrium pressure

$$p = p_0 - \rho_0 g z - \frac{1}{2} \rho_0 g \alpha G z^2$$

$p(z=0)$ gravity thermal effects

Is this state stable with respect to perturbations?

Perturbation: $\underline{v}$ , $\Delta T$ , $\Delta p$

Now, true $T$ and $p$:

$$T = T_0 - G z + \Delta T$$

$$p = p_0 - \rho g z - \frac{1}{2} \rho g \alpha G z^2 + \Delta p$$

We derived before:

$$\begin{cases} \dfrac{\partial \Delta T}{\partial t} + (\underline{v} \cdot \nabla) \Delta T = \kappa \nabla^2 \Delta T \\ \dfrac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla) \underline{v} = - \dfrac{\nabla \Delta p}{\rho_0} + \nu \nabla^2 \underline{v} - \alpha \Delta T \underline{g} \\ \nabla \cdot \underline{v} = 0 \end{cases}$$

We want to linearise them:

$$\left. \begin{array}{l} \Delta p \\ \Delta T \\ \underline{v} \end{array} \right\} \text{small}$$ , we keep only linear terms

The heat eq. becomes:

$$\frac{\partial \Delta T}{\partial t} - G\,v_z = \kappa \nabla^2 \Delta T$$

$$(\underline{v}\cdot\nabla)T = \underline{v}\cdot(0,0,-G) = -G v_z$$

The N-S equation:

$$\frac{\partial \underline{v}}{\partial t} = -\frac{\nabla(\Delta p)}{\rho_0} + \nu \nabla^2 \underline{v} + \alpha\,\Delta T\, g\,\hat{\underline{e}}_z$$

and incompressibility

$$\nabla \cdot \underline{v} = 0$$

Five coupled PDEs for five fluctuation fields $\underline{v}, \Delta p, \Delta T$.
BCs are specified by the geometry.

<u>Fourier transformation</u>

Best for homogeneous, linear PDEs with constant coefficients.

We assume our fluctuation fields to be combinations of elementary harmonic waves

$$\sim \exp(\underset{\in\mathbb{C}}{\lambda} t + i\,\underset{\in\mathbb{R}}{\underline{k}}\cdot\underline{r})$$

The amplitudes: $\breve{\Delta T}, \breve{\Delta p}, \breve{\underline{v}}$

$$\nabla a \to i\underline{k}\,\breve{a}$$

$$\begin{cases} \lambda \, \Delta\tilde{T} - G\,\tilde{v}_z = -\kappa\, k^2\, \Delta\tilde{T} & \text{(a)} \\ \lambda\, \tilde{\underline{v}} = -\dfrac{i\underline{k}}{\rho_0}\, \Delta\tilde{p} - \nu k^2\, \tilde{\underline{v}} + \alpha\, \Delta\tilde{T}\, g\, \hat{\underline{e}}_z & \text{(b)} \\ \underline{k}\cdot\tilde{\underline{v}} = 0 & \text{(c)} \end{cases}$$

Linear eqs! We can solve them.

From (a)

$$\tilde{v}_z = \frac{\lambda + \kappa k^2}{G}\, \Delta\tilde{T}$$

From $\underline{k}\cdot$(b) and (c)

$$\frac{\Delta\tilde{p}}{\rho_0} = -\alpha\, \Delta\tilde{T}\, g\, \frac{ik_z}{k^2}$$

Inserting this into z-component of (b), we find

$$\lambda\, \tilde{v}_z = -\frac{ik_z}{\rho_0}\, \Delta\tilde{p} - \nu k^2\, \tilde{v}_z + \alpha\, \Delta\tilde{T}\, g$$

we get:

$$(\lambda + \nu k^2)(\lambda + \kappa k^2)\, \Delta\tilde{T} = \alpha g G \left(1 - \frac{k_z^2}{k^2}\right) \Delta\tilde{T}$$

Non-trivial solution:

$$(\lambda + \nu k^2)(\lambda + \kappa k^2) = \alpha g G \left(1 - \frac{k_z^2}{k^2}\right)$$

The eq. for the vanishing determinant of the linear system.

This eq has two roots:

$$\lambda = -\frac{1}{2}(\nu+\kappa)k^2 \pm \frac{1}{2}\sqrt{(\nu-\kappa)^2 (k^2)^2 + 4\alpha g G\left(1-\frac{k_z^2}{k^2}\right)}$$

The "−" root is negative (stable)

The "+" root can be unstable. When?

Negative "+" root means that

$$(\nu+\kappa)k^2 > \sqrt{(\nu-\kappa)k^4 + 4\alpha g G\left(1-\frac{k_z^2}{k^2}\right)}$$

which is

$$\frac{\alpha G g}{\kappa \nu} < \frac{k^6}{k^2-k_z^2} \sim d^{-4}$$

geometry of the container

Typical size $d$ $\rightarrow$ $|k| \sim \frac{1}{d}$, so $|k|^4 \sim d^{-4}$

For finite $d$, we write

$$Ra \equiv \frac{g\alpha G d^4}{\kappa \nu} < \frac{(k_x^2+k_y^2+k_z^2)^3}{k_x^2+k_y^2} d^4$$

The minimum of the RHS defines the critical Ra.