UNITARY REPRESENTATIONS OF QUANTUM LORENTZ GROUP

W. PUSZ and S.L.WORONOWICZ
Department of Mathematical Methods in Physics
Faculty of Physics, University of Warsaw
Hoza 74, PL 00-682 Warsaw
Poland

ABSTRACT.Recent results concerning representation theory of quantum Lorentz group are presented.

1 Introduction

In this talk we describe recent results of the representation theory of quantum Lorentz group. This review is based on [6] and [5]. It was inspired by the classsical results for the Lorentz group as presented by M.A.Najmark [3] and I.M.Gelfand, M.I.Graev, N.J.Vilenkin [2]

A quantum Lorentz group considered here means the quantum deformation of the Lorentz group described in [4] corresponding to a fixed value of the deformation parameter $\mu = q \in]0, 1[$. The group will be denoted by *QLG*.

In the first part we describe all irreducible unitary representations of *QLG*. They split into principal and two complementary series. Beside that we have two 1-dimensional representations including the trivial one.

In the second part we investigate a large family of (not necessarily unitary) representations of *QLG* induced by 1-dimensional representations of the parabolic subgroup $P \subset QLG$ consisting of all upper-triangular matrices. The representations act on the space of smooth sections of (quantum) line boundles over the homogeneous space $P \setminus QLG$. This spaces denoted by D_χ (where χ runs over the set of 1-dimensional representations of P) play the fundamental role in [2]. We call them Gelfand spaces.

A deeper investigation (with the technique of invariant bilinear forms) of D_χ shows that in principle all results concerning the classical Lorentz group contained in Chapter 3 of [2] remains in power in the quantum case. In particular the conditions distinguishing unitary representations are of the same form and lead in a natural way to principal and complementary series.

The difference between the classical and quantum case consists in a slightly different topological structure of the set of 1-dimensional representations of the group P. In the classical case this representations are labeled by pairs (n_1, n_2) where $n_1, n_2 \in \mathbf{C}$ with $n_1 - n_2 \in \mathbf{Z}$ and the different pairs correspond to the different representations.

In the quantum case the correspondence between the pairs (n_1, n_2) and the representations of P is no longer one-to-one: the pairs (n_1, n_2) and $(n_1 + \frac{2\pi i}{\log q}, n_2 + \frac{2\pi i}{\log q})$ (where q is the deformation parameter) gives rise to the same representations of P.

As we shall see the above difference between the classical and the quantum case explains in a simple way all the suprising features of the theory of unitary representations of quantum Lorentz group such as the new topological structure of the principal series, the existence of two (instead of one) complementary series and the existence of non-trivial 1-dimensional representation.

2 Quantum groups and their representations

To introduce the basic notions of the representation theory of quantum groups we consider at first the classical case.

Let G be a compact group, $A = C(G)$ be the C^*-algebra of continuous functions on G. The group structure of G is encoded in the comultiplication $\Delta \in \mathrm{Mor}(A, A \otimes A)$ introduced by

$$(\Delta a)(g, g') \;=\; a(gg')$$

where a is a continuous function on G and g, g' runs over G.

In the compact case it is sufficient to consider finite dimensional representations. Let H be a finite dimensional Hilbert space. We have natural bijections:

①		②
$C(G, B(H))$ (continuous families of operators on H labeled by G)	$\longleftrightarrow$	$B(H) \otimes A$

③ The set of all linear mappings $H \to H \otimes A$

In what follows for any linear $v:\ H \to H \otimes A$ we shall use the same letter to denote the corresponding elements of $B(H) \otimes A$ and $C(G, B(H))$.

One can easily verify that

$$\boxed{\begin{array}{c} \text{①} \\ v \in C(G, B(H)) \\ v(gg') \;=\; v(g)v(g') \end{array}} \quad \Longleftrightarrow \quad \boxed{\begin{array}{c} \text{②} \\ v \in B(H) \otimes A \\ (\mathrm{id} \otimes \Delta)v \;=\; v_{12}v_{13} \end{array}}$$

③

The diagram

$$\begin{array}{ccc} H & \xrightarrow{\;\;v\;\;} & H \otimes A \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id} \otimes \Delta} \\ H \otimes A & \xrightarrow{\;v \otimes \mathrm{id}\;} & H \otimes A \otimes A \end{array}$$

is commutative

In ② we used leg numbering notation [4]: if $v \;=\; \sum m_i \otimes a_i$ then

$$\begin{aligned} v_{12} &= \sum m_i \otimes a_i \otimes I \\ v_{13} &= \sum m_i \otimes I \otimes a_i \\ v_{23} &= \sum I \otimes m_i \otimes a_i \end{aligned}$$

We refer to ② saying that v is a representation of G in H and to ③ saying that v is an action of G on H. For example one may consider Δ as an action of G on A. More generally if D is a finite dimensional vector subspace of A invariant under the right shifts, then:

$$\Delta : \; D \longrightarrow D \otimes A$$

and $v = \Delta \mid_D$ is an action of G on D. This kind of action we meet in the theory of induced representations.

The non-compact case is more complicated. Let G be a non-compact group. In this case we can associate with G different algebras which are the same in the compact case: $C_\infty(G)$ - the (non-unital) C^* - algebra of continuous functions on G tending to 0 at infinity, $C_{\text{bounded}}(G)$ - the (unital) C^* - algebra of bounded continuous functions on G and $C(G)$ - the *-algebra of continuous functions on G.

As the basic algebra related to G we take $A = C_\infty(G)$. Due to the famous Gelfand-Naimark theorem A contains the full information of G as a topological locally compact space. In particular the other algebras such as $C_{\text{bounded}}(G)$ and $C(G)$ can be reconstructed in a purely algebraic way once $C_\infty(G)$ is given. We have:

$$\begin{array}{rcl} C_{\text{bounded}}(G) & = & M(C_\infty(G)) \\ C(G) & = & C_\infty(G)^\eta \end{array}$$

where for any C^* - algebra A, $M(A)$ is the multiplier algebra and A^η is the set of all elements affiliated with A.

Let us recall that for any (non-unital) C^* - algebra A

$$M(A) \;=\; \begin{array}{c}\text{The largest } C^*\text{-algebra that} \\ \text{contains } A \text{ as a separating ideal}\end{array}$$

It is always the unital C^* - algebra. If $A \subset B(H)$ (non-degenerated embeding) then

$$M(A) \;=\; \{b \in B(H) : ba \in A, ab \in A \text{ for all } a \in A\}$$

It is clear that for the unital C^* - algebra A : $M(A) = A$ and if $A = CB(H)$ is the C^* - algebra of compact operators on H then $M(CB(H)) = B(H)$. The reader should notice that $M(C_\infty(G)) = C_{\text{bounded}}(G) = C(\overline{G})$ where $\overline{G}$ is the Čech-Stone compactification of G. The multiplier functor M is an algebraic counterpart of the Čech-Stone compactification of locally compact spaces.

We have not enough time to explain the notion of affiliated elements. The affiliation relation is denoted by η :

$$T \eta A \Longleftrightarrow T \in A^\eta$$

In any case $A^\eta \supset M(A)$. Elements of A^η may be regarded as unbounded multipliers acting on A [7]. It turns out that $C_\infty(G)^\eta = C(G)$ and

$$CB(H)^\eta = \begin{array}{c}\text{The set of all closed} \\ \text{operators on } H\end{array}$$

The later example shows that in general A^η is not even a vector space. If A is unital then $A^\eta = A$.

In many cases we reconstruct an algebra A from a given sets of affiliated elements. One says that $A = C^*(\alpha, \beta, \gamma, \delta, \ldots)$ is generated by $\alpha, \beta, \gamma, \delta, ...$ if A is in a sense the smallest C^* - algebra such that $\alpha, \beta, \gamma, \delta, ... \;\eta A$.

For example if $H = L^2(R)$, $\hat{x} =$ multiplication by x and $\hat{p} = \frac{1}{i}\frac{d}{dx}$ then the C^* - algebra generated by $\hat{x}$ and $\hat{p}$ coincides with $A = CB(H)$.

For two C^* - algebras A, B we shall denote by $\text{Mor}(A, B)$ the space of morphisms from A to B : $\Phi \in \text{Mor}(A, B)$ means that Φ is a non-degenerated *-algebra homomorphism of A into $M(B)$. The nondegeneracy means that $\Phi(A)B$ is dense in B. Clearly $\text{Mor}(A, CB(H))$ $= \text{Rep}(A, H)$ where $\text{Rep}(A, H)$ is the set of all non-degenerate representations of A in H.

This notion of a morphism is a generalization of a morphism in the category of commutative C^* - algebras and corresponds to a continuous map between locally compact spaces. Any $\Phi \in \text{Mor}(A, B)$ can be extended in the canonical way to $M(A)$ and A^η :

$$\begin{array}{rcl} \Phi : M(A) & \longrightarrow & M(B) \\ \Phi : A^\eta & \longrightarrow & B^\eta \end{array}$$

In the case of non-compact groups one has to consider representations in infinite dimensional Hilbert spaces. Let H be a seperable Hilbert space. Then the correspondence ② $\Longleftrightarrow$ ③ becomes more complicated and reflects different continuity properties of infinite-dimensional representations:

v belongs to	$\Longleftrightarrow$ $v : G \longrightarrow B(H)$ is
$B(H) \otimes A$	norm continuous and $v(g) \to 0$ for $g \to \infty$
$B(H) \otimes M(A)$	norm continuous, bounded and $v(G)$ is almost finite-dimensional
$M(B(H) \otimes A)$	norm continuous and bounded
$M(CB(H) \otimes A)$	*-strong continuous and bounded

In the quantum group case the algebra $A = C_\infty(G)$ is no longer commutative. To introduce a quantum group G one has to fix a C^* - algebra A and a coassociative morphism $\Delta \in \mathrm{Mor}(A, A \otimes A)$. Elements of A may be considered as "continuous functions vanishing at infinity" on non-compact quantum space G whereas Δ encodes the group structure of G. In brief we write $G = (A, \Delta)$. This point of view is sufficient for the purposes of the present paper. We shall deal only with a few concrete quantum groups not entering the general theory so there is no necessity to present a formal definition of a quantum group.

Considerations presented above lead to the following notion of unitary strongly continuous representation of a quantum grup G.

Definition 1 *Let $G = (A, \Delta)$ be a quantum group and H be a separable Hilbert space. We say that v is a unitary representation of G in H if v is an unitary element of $M(CB(H) \otimes A)$ and*

$$(\mathrm{id} \otimes \Delta)v \;=\; v_{12}v_{13}$$

3 Quantum SU(2) group and its representation theory

The quantum Lorentz group contains the quantum $SU(2)$ group and its quantum Pontryagin dual group as subgroups. This fact gives a deeper insight into the strucure of the quantum Lorentz group so we consider these subgroups first.

Let q be a fixed number of the interval]0,1[and A_c be the C^* -algebra generated by two elements α_c, and γ_c satisfying the well known commutation relations

$$\left.\begin{array}{c} \alpha_c^*\alpha_c + \gamma_c^*\gamma_c = I, \qquad \alpha_c\alpha_c^* + q^2\gamma_c^*\gamma_c = I, \\ \alpha_c\gamma_c = q\gamma_c\alpha_c, \qquad \alpha_c\gamma_c^* = q\gamma_c^*\alpha_c, \qquad \gamma_c\gamma_c^* = \gamma_c^*\gamma_c \end{array}\right\} \tag{3.1}$$

The quantum $SU(2)$ group is by definition

$$S_qU(2) \;=\; (A_c, \Delta_c)$$

where $\Delta_c \in \mathrm{Mor}(A_c, A_c \otimes A_c)$ is uniquely defined by its values on the generators

$$\begin{array}{rcl} \Delta_c(\alpha_c) &=& \alpha_c \otimes \alpha_c \;-\; q\gamma_c^* \otimes \gamma_c \\ \Delta_c(\gamma_c) &=& \gamma_c \otimes \alpha_c \;+\; \alpha_c^* \otimes \gamma_c \end{array}$$

According to (3.1) the elements α_c and γ_c are bounded. Therefore A_c is unital and $S_qU(2)$ is compact.

Let

$$\mathcal{A}_c = \mathrm{Pol}(\alpha_c, \gamma_c, \alpha_c^*, \gamma_c^*)$$

be the smallest *-subalgebra of A_c containing α_c and γ_c. Then $\mathcal{A}_c$ is dense in A_c and the set $\{\alpha_{ck}\gamma_c^m\gamma_c^{*n} \ : \ k \in \mathbf{Z}, m, n = 0, 1, 2\ldots\}$ where

$$\alpha_{ck} = \begin{cases} \alpha_c^k & \text{for } k \geq 0 \\ (\alpha_c^*)^{-k} & \text{for } k \leq 0 \end{cases}$$

is a linear basis in $\mathcal{A}_c$.

We consider linear functionals

$$f_0, \ f_+, \ f_- : \mathcal{A}_c \longrightarrow \mathbf{C}$$

defined on the elemets of the basis in the following way:

$$f_0(\alpha_{ck}\gamma_c^m\gamma_c^{*n}) = \begin{cases} q^{-\frac{k}{2}} & \text{for } m = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_+(\alpha_{ck}\gamma_c^m\gamma_c^{*n}) = \begin{cases} q^{\frac{k}{2}} & \text{for } m = 1, n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_-(\alpha_{ck}\gamma_c^m\gamma_c^{*n}) = \begin{cases} -q^{\frac{k-2}{2}} & \text{for } m = 0, n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let v_c be an unitary representation of $S_qU(2)$ acting on a finite-dimensional Hilbert space H. Then $v_c \in B(H) \otimes \mathcal{A}_c$ and setting

$$\left.\begin{array}{rcl} q^{J_3} &=& (\mathrm{id} \otimes f_0)v_c \\ J_+ &=& (\mathrm{id} \otimes f_+)v_c \\ J_- &=& (\mathrm{id} \otimes f_-)v_c \end{array}\right\} \tag{3.2}$$

we introduce three operators q^{J_3}, J_+, J_- acting on H. They satisfy the commutation relations:

$$\left.\begin{array}{c} q^{J_3}J_+ = qJ_+q^{J_3}, \qquad q^{J_3}J_- = q^{-1}J_-q^{J_3}, \\ [J_+, J_-] = \dfrac{q^{-2J_3} - q^{2J_3}}{q^{-1} - q} \quad , \\ (J_+)^* = J_-, \qquad q^{J_3} > 0 \end{array}\right\} \tag{3.3}$$

Any strongly continuous unitary representation v_c of $S_qU(2)$ acting on a (infinite - dimensional) Hilbert space H is a direct sum of irreducible finite dimensional representations. In this case (3.2) are (in general unbounded) closed operators acting on H. They have a common invariant dense essential domain (a core) consisting of vectors belonging to finite

- dimensional v_c - invariant subspaces of H.

The set of irreducible representations of $S_qU(2)$ is labeled by spin parameter $s = 0, 1/2, 1, 3/2, \ldots$. Let s be one of this number. The corresponding unitary representation denoted by u^s acts on (2s+1)-dimensional Hilbert space H^s : $u^s \in B(H^s) \otimes \mathcal{A}_c$. In this case the operators (3.2) are denoted by $q^{J_3^s}, J_+^s, J_-^s \in B(H^s)$. For example:

$$u^{1/2} = \begin{pmatrix} \alpha_c & -q\gamma_c^* \\ \gamma_c & \alpha_c^* \end{pmatrix}$$

and

$$q^{J_3^{1/2}} = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \quad J_+^{1/2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad J_-^{1/2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

To introduce the Pontryagin dual of $S_qU(2)$ we consider the C^* - algebra

$$A_d = \sum_s^{\oplus} B(H^s).$$

Let π^s be the canonical projection $\pi^s \in \mathrm{Mor}(A_d, B(H^s))$. Any element $a\,\eta\,A_d$ is uniquely determined by sequence $(\pi^s(a))_{s=0,1/2,1,\ldots}$. Any sequence $(a_s)_{s=0,1/2,1,\ldots}$ where $a_s \in B(H^s)$ can be obtained in this way. An element a belongs to A_d ($M(A_d)$ respectively) if $\|\pi^s(a)\|$ goes to 0 for $s \to \infty$ (is bounded respectively). The reader should notice that in this case A_d^η carries a natural *-algebra structure.

Let

$$u = \sum_s^{\oplus} u^s$$

and

$$q^{J_{d3}} = \sum_s^{\oplus} q^{J_3^s}, \qquad J_{d+} = \sum_s^{\oplus} J_+^s, \qquad J_{d-} = \sum_s^{\oplus} J_-^s$$

Then $u \in M(A_d \otimes A_c)$ and $q^{J_{d3}}, J_{d+}, J_{d-}$ are unbounded elements affiliated with A_d satisfying relations (3.3). One can show that

$$A_d = C^*(q^{J_{d3}}, J_{d+}, J_{d-}).$$

Moreover there exists one and only one $\Delta \in \mathrm{Mor}(A_d, A_d \otimes A_d)$ such that

$$\Delta_d(J_{d\pm}) = q^{J_{d3}} \otimes J_{d\pm} + J_{d\pm} \otimes q^{-J_{d3}}$$

$$\Delta_d(q^{J_{d3}}) = q^{J_{d3}} \otimes q^{J_{d3}}$$

Δ_d is coassociative and the quantum Pontryagin dual $S_qU(2)$ group is:

$$\widehat{S_qU(2)} = (A_d, \Delta_d)$$

To explain why we refer to the Pontryagin duality let us notice that

$$\begin{aligned} (\mathrm{id} \otimes \Delta_c)u &= u_{12}u_{13} \\ (\Delta_d \otimes \mathrm{id})u &= u_{23}u_{13} \end{aligned}$$

This relation expresses the bicharacter property of u : u is a representation of $S_qU(2)$ and u^{-1} is a representation of $\widehat{S_qU(2)}$. The bicharacter u plays the same role in representation theories of $S_qU(2)$ and $\widehat{S_qU(2)}$ as a bicharacter e^{ipx} in the representation theories of $\mathbf{R}$ and its Pontryagin dual group $\widehat{\mathbf{R}} = \mathbf{R}$.

Using this property one can prove a duality theorem [5]:

Theorem 2

$$\left(\begin{array}{c} v_c \in M(CB(H)\otimes A_c) \\ \text{is a unitary representation} \\ \text{of } S_qU(2) \end{array}\right) \Longleftrightarrow \left(\begin{array}{c} v_c \ = \ (\psi_d \otimes \mathrm{id})u \\ \text{where} \\ \psi_d \in \mathrm{Rep}(A_d, H) \end{array}\right)$$

$$\left(\begin{array}{c} v_d \in M(CB(H)\otimes A_d) \\ \text{is a unitary representation} \\ \text{of } \widehat{S_qU(2)} \end{array}\right) \Longleftrightarrow \left(\begin{array}{c} v_d \ = \ (\psi_c \otimes \mathrm{id})\tau(u^{-1}) \\ \text{where } \psi_c \in \mathrm{Rep}(A_c, H) \\ \text{and } \tau \text{ is a flip :} \\ A_d \otimes A_c \rightarrow A_c \otimes A_d \end{array}\right)$$

It means that $S_qU(2)$ and $\widehat{S_qU(2)}$ are mutually Pontryagin dual groups to each other and there is one-to-one correspondence between strongly continuous unitary representations of the group and nondegenerate representations of "the algebra of functions" on the dual group. This dual approach in the representation theory of quantum groups is quite natural and corresponds to the Lie algebra approach in the Lie group representation theory. A description of a representation of an algebra is often simpler and more convenient then a description of a group action. For example to introduce $\psi_c \in$ Rep(A_c, H) it is enough to fix two operators $\alpha_c, \gamma_c \in B(H)$ satisfying (3.1) Similarly to given $\psi_d \in$ Rep(A_d, H) there correspond operators q^{J_3}, J_+, J_- acting in H and satisfying (3.3).

4 QLG and its irreducible unitary representations

To consider the quantum Lorentz group

$$QLG \ = \ (A, \Delta)$$

we have to describe C^* - algebra A and a comultiplication Δ.

We fix $q \in]0,1[$ and let A be a (non-unital) C^* - algebra generated by four unbounded elements α, β, γ and δ satisfying the following 17 relations proposed by Podleś:

$$
\begin{array}{lll}
\alpha\beta \ = q\beta\alpha, & \alpha\gamma = q\gamma\alpha, & \alpha\delta - q\beta\gamma = I, \\
\beta\delta \ = q\delta\beta, & \beta\gamma = \gamma\beta, & \delta\alpha - q^{-1}\beta\gamma = I, \\
\gamma\delta \ = q\delta\gamma, & & \\
\alpha\alpha^* = \alpha^*\alpha + (1-q^2)\gamma^*\gamma, & & \gamma\alpha^* = q\alpha^*\gamma, \\
\beta\alpha^* = q^{-1}\alpha^*\beta + q^{-1}(1-q^2)\gamma^*\delta, & & \delta\alpha^* = \alpha^*\delta, \\
\beta\beta^* = \beta^*\beta + (1-q^2)[\delta^*\delta - \alpha^*\alpha] - (1-q^2)^2\gamma^*\gamma, & & \\
\delta\beta^* = q\beta^*\delta - q(1-q^2)\alpha^*\gamma, & & \gamma\beta^* = \beta^*\gamma, \\
\delta\delta^* = \delta^*\delta - (1-q^2)\gamma^*\gamma, & & \gamma\gamma^* = \gamma^*\gamma, \\
\delta\gamma^* = q^{-1}\gamma^*\delta. & &
\end{array}
\tag{4.1}
$$

One can prove that there is the unique morphism $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that

$$
\begin{array}{cc}
\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, & \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta, \\
\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, & \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta.
\end{array}
$$

This morphism is coassociative and encodes a group structure in *QLG*.

The above commutation relations are complicated. Fortunately it was realized that any matrix $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ where $\alpha, \beta, \gamma, \delta$ are operators in a Hilbert space satisfying this relations is of the form

$$
\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} = \begin{pmatrix} \alpha_c, & -q\gamma_c^* \\ \gamma_c, & \alpha_c^* \end{pmatrix} \begin{pmatrix} q^{J_3}, & (1-q^2)q^{-1/2}J_+ \\ 0, & q^{-J_3} \end{pmatrix} \tag{4.2}
$$

(and a similar formula for adjoints) where operators $\alpha_c,\ \gamma_c$ satisfy (3.1) and $q^{J_3}, J_+,$ satisfy (3.3). Moreover any operator from the set $\{\alpha_c,\ \gamma_c\}$ commutes with any operator from the set $\{q^{J_3}, J_+, J_-\}$. Above formula is a quantum version of the Iwasawa decomposition of the classical Lorentz group. It shows that

$$
A \ = \ A_c \otimes A_d
$$

Let

$$
p_c = \mathrm{id} \otimes e_d, \qquad p_d = e_c \otimes \mathrm{id}
$$

where $e_c \in \mathrm{Mor}(A_c, \mathbf{C}),\ e_d \in \mathrm{Mor}(A_d, \mathbf{C})$ are counits of $S_qU(2)$ and $S_q\hat{U}(2)$ respectively ($e_c,\ e_d$ are the unique morphisms such that $e_c(\alpha) = 1,\ e_c(\gamma) = 0;\ e_d(q^{J_3}) = 1,\ e_d(J_\pm) = 0$). Then

$$
p_c \in \mathrm{Mor}(A, A_c) \qquad p_d \in \mathrm{Mor}(A, A_d)
$$

and they correspond to embeddings

$$
S_qU(2) \longrightarrow QLG, \qquad S_q\widehat{U(2)} \longrightarrow QLG
$$

One can check that

$$
\Delta_c p_c = (p_c \otimes p_c)\Delta \qquad \Delta_d p_d = (p_d \otimes p_d)\Delta
$$

This means that $S_qU(2)$ and $S_q\widehat{U(2)}$ are subgroups of *QLG* : $\Delta \mid_{S_qU(2)} = \Delta_c$ and $\Delta \mid_{S_q\widehat{U(2)}} = \Delta_d$.

The group structure of *QLG* can be reproduced from that of $S_qU(2)$ and $S_q\widehat{U}(2)$. Let $\sigma \in \mathrm{Mor}(A_c \otimes A_d, A_d \otimes A_c)$ be given by

$$\sigma(a \otimes x) \;=\; u(x \otimes a)u^{-1}$$

where as before u is a bicharacter $u = \sum_s^{\oplus} u^s$. Then

$$\Delta \;=\; (\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\Delta_c \otimes \Delta_d)$$

Summarizing: the quantum Lorentz group

$$QLG \;=\; (A_c \otimes A_d,\ (\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\Delta_c \otimes \Delta_d)).$$

This fact is of great importance and simplifies the study of the representation theory for *QLG*. Any representation v of *QLG* can be described as the pair (v_c, v_d) of representations $S_qU(2)$ and $S_q\widehat{U}(2)$ respectively acting in the same space and satisfying a compatibility condition.

Let

$$v \in \mathrm{M}(CB(H) \otimes A)$$

be an unitary representation of the quantum Lorentz group *QLG* acting in a Hilbert space H and let

$$\begin{aligned} v_c &= v\,|_{S_qU(2)} := (\mathrm{id} \otimes p_c)v \\ v_d &= v\,|_{S_q\widehat{U}(2)} := (\mathrm{id} \otimes p_d)v \end{aligned}$$

Then $v_c \in \mathrm{M}(CB(H) \otimes A_c)$ and $v_d \in \mathrm{M}(CB(H) \otimes A_d)$ are unitary representations of $S_qU(2)$ and $S_q\widehat{U}(2)$ respectively

$$v = (v_c)_{12}(v_d)_{13}$$

and

$$(v_d)_{12}(v_c)_{13} \;=\; (\mathrm{id} \otimes \sigma)(v_c)_{12}(v_d)_{13}$$

Conversely, if unitary representations v_c, v_d acting in the same Hilbert space satisfy the last condition then $v := (v_c)_{12}(v_d)_{13}$ is an unitary representation of *QLG*. We shall refer to this as the compatibility condition.

Now we can associate with v two sets of operators $\{J_+, J_-, q^{J_3}\}$, $\{\alpha_c, \gamma_c\}$ acting in H via the correspondence:

$$\begin{array}{ccccc} \boxed{\begin{array}{c} v_c = v\,|_{S_qU(2)} \\ \text{representation} \\ \text{of } S_qU(2) \end{array}} & \Longleftrightarrow & \boxed{\psi_d \in \mathrm{Rep}(A_d, H)} & \Longleftrightarrow & \boxed{\begin{array}{c} \text{operators} \\ J_+, J_-, q^{J_3} \\ \text{satisfying (3.3)} \end{array}} \\ \\ \boxed{\begin{array}{c} v_d = v\,|_{S_q\widehat{U}(2)} \\ \text{representation} \\ \text{of } S_q\widehat{U}(2) \end{array}} & \Longleftrightarrow & \boxed{\psi_c \in \mathrm{Rep}(A_c, H)} & \Longleftrightarrow & \boxed{\begin{array}{c} \text{operators} \\ \alpha_c, \gamma_c, \alpha_c^*, \gamma_c^* \\ \text{satisfying (3.1)} \end{array}} \end{array} \qquad (4.3)$$

The compatibility condition in terms of this operators means:

$$\left.\begin{array}{ll} q^{J_3}\alpha_c = \alpha_c q^{J_3}, & q^{J_3}\gamma_c = q^{-1}\gamma_c q^{J_3}, \\ J_+\alpha_c = q\alpha_c J_+ - q^{\frac{3}{2}}\gamma_c^* q^{J_3}, & J_+\gamma_c = \gamma_c J_+ + q^{-\frac{1}{2}}(\alpha_c^* q^{J_3} - \alpha_c q^{-J_3}), \\ J_+\alpha_c^* = q^{-1}\alpha_c^* J_+ + q^{-\frac{1}{2}}\gamma_c^* q^{-J_3}, & J_+\gamma_c^* = \gamma_c^* J_+ \end{array}\right\} \tag{4.4}$$

These relations (as well as the relations (3.3)) have to be supplemented by regularity conditions (like the famous integrability condition of Nelson for Lie algebra relations) stating the existence of sufficiently large invariant domain on which the relations hold. The regularity conditions give the precise meaning to the commutation relations involving unbounded operators. In our case the regularity conditions (as well as the relations themselves) follow from the fact that the considered operators $(J_+, J_-, q^{J_3}, \alpha_c, \gamma_c)$ are related to a unitary representation of the quantum Lorentz group. We have no time to formulate these conditions explicitely. It should be stressed however that they play the essential role in our analysis. This analysis leads to the complete classification of all irreducible representations of *QLG*. In what follows we briefly present the results.

As we know representation v restricted to $S_qU(2)$ is a representation v_c. Any such a representation is a direct sum of irreducible ones. Let $\mathrm{Sp}\ v$ be the spin spectrum of v : a (half-) integer $s \in \mathrm{Sp}\, v$ if and only if u^s is contained in v_c, and p be the minimal element of $\mathrm{Sp}\ v$. Assume that v is irreducible. Then for any $s \in \mathrm{Sp}\, v$, the multiplicity of u^s in v_c is 1:

$$H = \sum_{s\in \mathrm{Sp} v} {}^{\oplus} H^s$$

Any H^s is a v_c - invariant subspace in H. Therefore

$$J_+, J_-, q^{J_3} \ :\ H^s \longrightarrow H^s$$

and the action of this operators is well known. It turns out that operators

$$\begin{array}{c} \alpha_c, \gamma_c \\ \alpha_c^*, \gamma_c^* \end{array} \quad : H^s \longrightarrow H^{s-1} \oplus H^s \oplus H^{s+1}$$

(H^{s-1} does not exist for $p = 0$).
Using commutation relations (3.1) , (3.3) and (4.4) one can show that the Casimir operator:

$$C(v) = q^{-1/2}(1-q^2)\gamma_c J_+ - \alpha_c^* q^{J_3+1} - \alpha_c q^{-J_3-1} \tag{4.5}$$

commutes with all operators related to v. In the irreducible case

$$C(v) = c(v) I$$

where $c(v)$ is a (complex) eigenvalue of $C(v)$.

The eigenvalue of $C(v)$ together with $\mathrm{Sp}\, v$ completely determines (up to a unitary equivalence) an irreducible representation v. The table below presents all the possibilities:

Table 1:

Sp v	The eigenvalue of $C(v)$	Remarks
$\{0\}$	$c = \pm(q+q^{-1})$	1- dimensional representations. Sign "-" corresponds to the trivial one
$\{p, p+1, p+2, \ldots\}$ p - positive (half-)integer	$\mid c-2 \mid + \mid c+2 \mid = 2(q^p + q^{-p})$	Principal series
$\{0, 1, 2, \ldots\}$	$-r < c < r$ where $r = q + q^{-1}$	There are three cases: 1. $c \in [-2, 2]$ Principal series with $p = 0$ 2. $c \in]-(q+q^{-1}), -2[$ 3. $c \in]2, (q+q^{-1})[$ Two complementary series

In Fig.1 we showed the admissible values of $c(v)$ described in Table 1. We see that the values corresponding to the principal series belong to ellipses with common focuses located at points -2 and 2. The size of the ellipses depends on initial spin p. For $p = 0$ the ellipse degenerates to the interval [-2,2]. Besides the principal series we have two complementary

Figure 1:

series corresponding to intervals $]-(q+q^{-1}), -2[$ and $]2, q+q^{-1}[$. For these series $p = 0$. The two 1-dimensional representations τ and $\tilde{\tau}$ corresponds to points $\mp(q+q^{-1})$.

To compare this result with the representation theory of classical Lorentz group we use selfadjoint Casimir operators Δ and Δ' considered in [3, p.167 and statement on p.187]. The eigenvalues of Δ and Δ' together with the spin spectrum completely determine an irreducible unitary representation v of the classical Lorentz group. Table 2 shows all the possibilities.

Table 2:

Sp v	The eigenvalue d of $\Delta + i\Delta'$	Remarks
$\{0\}$	$d = 0$	1- dimensional trivial representation.
$\{p, p+1, p+2, \ldots\}$ p - positive (half-)integer	$\mid d-2 \mid - \mathrm{Re}\, d$ $= 2(2p^2 - 1)$	Principal series
$\{0, 1, 2, \ldots\}$	$0 < d$	There are two cases: 1. $d \in [2, \infty[$ Principal series with $p = 0$ 2. $d \in]0, 2[$ Complementary series

The admissible values of eigenvalues of $\Delta + i\Delta'$ are presented on Fig.2. In this case the values corresponding to principal series belong to parabolas with common focus at the point 2 and directrices depending on the minimal spin p. For $p = 0$ the parabola degenerates to a half-line $[2, \infty[$.

Figure 2:

There is only one complementary series. It corresponds to the interval $]0, 2[$ and as before $p = 0$ is the associated minimal spin. There is also only one 1-dimensional representation - the trivial one τ. It corresponds to the point 0.

To analyse the classical limit $q \to 1$ one should consider the rescaled quantum Casimir operator

$$C' = 2\frac{C + q + q^{-1}}{q - 2 + q^{-1}}$$

The reader easily verify that with this transformation the ellipses of Fig.1 tend (as $q \to 1$) to parabolas of Fig.2. In particular the degenerated ellipse transforms onto the degenerated parabola. The interval $]-(q+q^{-1}),-2[$ corresponding to the first complementary series transforms onto the interval $]0,2[$. The interval corresponding to the second complementary series is moved onto $]2+t_q, 4+t_q[$ where $t_q = 8(q+q^{-1}-2)^{-1}$. For $q \to 1$, the interval is shifted to infinity and the corresponding complementary series disappears. The same holds for the additional 1-dimensional representation.

5 Gelfand spaces and induced action of QLG

The methods used in the previous sections permit to describe all irreducible representations of the quantum Lorentz group *QLG* in an implicit way. Within this approach one can only derive formulae which show how the generators $\alpha_c, \gamma_c, q^{J_3}, J_+, J_-$ of the representation act on some basic vectors of the carrier Hilbert space. We have no explicit expressions descibing the action of *QLG* itself on that space. In the classical case such expressions can be obtained by realization of the carrier space as a space of functions (satisfying some conditions as e.g. a homogeneity conditions) on some G-manifolds. In many cases it is convenient to impose also some regularity conditions on considered functions: working with nuclear spaces of smooth functions one may use very powerful methods of distribution theory. This technique for the classical Lorentz group was proposed in [2]. The deeper analysis shows that in effect it consists in inducing representations of the Lorentz group from 1-dimensional representations of its parabolic subgroup.

In this section we try to mimic this approach in the quantum case. It turns out that in this context one has to consider also non-unitary representations of *QLG* with no hilbertian structure on the underlying vector space. To this aim it is necessary to generalize the framework presented before.

First of all let us notice that the notion of representation as introduced in Definition 1 is formulated in the C^* -algebra language and is not applicable if H is not endowed with a Hilbert space structure. Instead we shall use the concept of action of *QLG* on a vector space which better suits. For the purpose of our paper it is sufficient to consider right invariant subspaces of $\mathcal{A}$ on which *QLG* acts by right shifts. In this sentence $\mathcal{A}$ denotes the space of all "smooth elements" affiliated with A (the precise definition is given later). $\mathcal{A}$ is a *-subalgebra of A^η. It turns out that $\Delta(\mathcal{A}) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$, where $\hat{\otimes}$ denotes the algebraic tensor product followed by a suitable completion. $D \subset \mathcal{A}$ is right invariant if $\Delta(D) \subset D\hat{\otimes}\mathcal{A}$.

Working with a non-unitary representations of the quantum Lorentz group one may also use the results contained in Sections 2 and 3. One should notice however that in non-unitary case the representations ψ_c and ψ_d are not *-homomorphisms (after all the representation space is not endowed with an invariant scalar product and there is no natural *-involution in the space of operators). Therefore in the sequence of operators $\alpha_c, \alpha_c^*, \gamma_c, \gamma_c^*, J_+, J_-, q^{J_3}$ constructed out of considered representation of *QLG* in the way described in (4.3), $\alpha_c^*(\gamma_c^*, J_-, q^{J_3}$ respectively) is no longer related by a hermitian conjugation to $\alpha_c(\gamma_c, J_+, q^{J_3}$ respectively). $\alpha_c, \alpha_c^*, \gamma_c, \gamma_c^*, J_+, J_-, q^{J_3}$ should be treated as independent variables subjected to the relations (3.1), (3.3) (except the last row) and (4.4) suplemented by their formal hermitian conjugation: For example the relation $\alpha_c\gamma_c = q\gamma_c\alpha_c$ should be suplemented by $\gamma_c^*\alpha_c^* = q\alpha_c^*\gamma_c^*$.

The formula (4.5) expressing the Casimir operator remains in power.

We briefly describe the construction of induced representation in quantum case.

To introduce the parabolic subgroup $P = (A_P, \Delta_P)$ one has to complete the set of Podleś relations (4.1) adding the relation "$\gamma = 0$" (we remind that in the classical case P consists of all upper-triangular matrices belonging to $SL(2, \mathbf{C})$). The corresponding C^* - algebra will be denoted by A_P. By definition

$$A_P \ = \ A/I_\gamma$$

where I_γ is the closed two sided ideal of A generated by γ.

Let $\pi \in \mathrm{Mor}(A, A_P)$ be the canonical epimorphism and $\dot\alpha, \dot\beta, \dot\gamma, \dot\delta \ \eta \ A_P$ be the elements corresponding to $\alpha, \beta, \gamma, \delta \ \eta \ A$: $\dot\alpha = \pi(\alpha)$ and so on. Then $\dot\gamma = 0$ and

$$\begin{array}{ll} \dot\alpha, \dot\delta \ \text{- normal} & \\ \dot\alpha\dot\beta \ = q\dot\beta\dot\alpha, & \dot\beta\dot\alpha^* = q^{-1}\dot\alpha^*\dot\beta, \\ \dot\beta\dot\delta \ = q\dot\delta\dot\beta, & \dot\delta\dot\alpha^* = \dot\alpha^*\dot\delta, \\ \dot\alpha\dot\delta \ = \ I \ = \ \dot\delta\dot\alpha, & \dot\delta\dot\beta^* = q\dot\beta^*\dot\delta, \\ \dot\beta\dot\beta^* = \dot\beta^*\dot\beta + (1-q^2)(\dot\delta^*\dot\delta - \dot\alpha^*\dot\alpha). & \end{array}$$

Applying π to the matrix elements of (4.2) we see that $\pi(\gamma_c) = 0$, $\pi(\alpha_c^*)$ is unitary and $\dot\delta = \pi(\delta) = \pi(\alpha_c^*)\pi(q^{-J_3})$. Therefore $\dot\delta^*\dot\delta = \pi(q^{-2J_3})$, $\dot\delta^*\dot\delta$ is an invertible element affiliated with A_P and

$$\mathrm{Sp}\, \dot\delta^*\dot\delta \ \subset q^{\mathbf{Z}} \cup 0 \tag{5.1}$$

The group structure of P is the one induced by that of *QLG*. The comultiplication Δ_P is the unique element of $\mathrm{Mor}(A_P, A_P \otimes A_P)$ such that

$$\Delta_P \circ \pi \ = \ (\pi \otimes \pi)\Delta.$$

In particular

$$\Delta_P(\dot\alpha) = \dot\alpha \otimes \dot\alpha, \qquad \Delta_P(\dot\beta) = \dot\alpha \otimes \dot\beta + \dot\beta \otimes \dot\delta, \qquad \Delta_P(\dot\delta) = \dot\delta \otimes \dot\delta.$$

Now we shall consider characters of P i.e. 1-dimensional representations of P. We do not assume neither unitarity nor even boundedness. More precisely χ is a character if χ is an invertible element affiliated with A_P and $\Delta_P \chi = \chi \otimes \chi$. It turns out that any character of P is of the form

$$\chi \ = \ \dot\delta^{n_1-1}(\dot\delta^*)^{n_2-1} \ = \ (\mathrm{Phase}\,\dot\delta)^{n_1-n_2} \mid \dot\delta \mid^{n_1+n_2-2} \tag{5.2}$$

where $n_1, n_2 \in \mathbf{C}$, $n_1 - n_2 \in \mathbf{Z}$.

Remark: We have inserted -1 in the exponents to have better correspondence with the Gelfand notation [2] .

Due to the spectral condition (5.1) two pairs $(n_1, n_2), (n_1', n_2')$ give rise to the same character if and only if

$$n_1 - n_1' \ = \ n_2 - n_2' \ = \ \frac{2k\pi i}{\log q} \quad \text{for some } k \in \mathbf{Z}.$$

In such a case we write $(n_1, n_2) \equiv (n_1', n_2')$.

The induced representations considered in this paper act on spaces of "smooth functions" on *QLG*. We say that an element a, affiliated with $A = A_c \otimes A_d$ is smooth if for any $s = 0, 1/2, 1, \ldots$:

$$(\mathrm{id} \otimes \pi^s)a \ \in \ \mathcal{A}_c \otimes B(H^s).$$

The set of smooth elements will be denoted by $\mathcal{A}$. It is clear that $\mathcal{A}$ is a *-subalgebra of A^η. One may also consider smooth elements affiliated with $A \otimes A$. By definition $a\,\eta\,A \otimes A$ is smooth if

$$(\mathrm{id} \otimes \pi^s \otimes \mathrm{id} \otimes \pi^{s'})a \ \in \ (\mathcal{A}_c \otimes B(H^s)) \otimes_{\mathrm{alg}} (\mathcal{A}_c \otimes B(H^{s'})$$

for any $s, s' = 0, 1/2, 1, \ldots$. The set of smooth elements affiliated with $A \otimes A$ may be denoted by $\mathcal{A}\hat{\otimes}\mathcal{A}$ where $\hat{\otimes}$ is the algebraic tensor product followed by a suitable completion. It turns out that $\Delta(\mathcal{A}) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$.

Let χ be a character of P. The representation of *QLG* induced by χ acts by right shifts on the space D_χ of smooth elements which transform under the left action of P according to the representation χ :

$$D_\chi \ = \ \{a \in \mathcal{A} : \ (\pi \otimes \mathrm{id})\Delta a = \chi \otimes a \ \}. \tag{5.3}$$

The reader should notice that the transformation low

$$(\pi \otimes \mathrm{id})\Delta a = \chi \otimes a \tag{5.4}$$

coincides in the classical case with $a(pg) = \chi(p)a(g)$ for all $p \in P$ and $g \in G$ (cf.[1, p.473, formula (1)]).

Since the left and the right shifts commute, D_χ is invariant under the right shifts :

$$\Delta(D_\chi) \ \subset \ D_\chi\hat{\otimes}\mathcal{A}.$$

Therefore $v_\chi := \Delta|_{D_\chi}$ is a smooth action of *QLG* on D_χ. In other words D_χ carries a representation of *QLG*. This is the representation induced by χ.

To make our notation close to the one used in [2] we write $D_{n_1 n_2}$ (where $n_1, n_2 \in \mathbf{C}$ and $n_1 - n_2 \in \mathbf{Z}$) instead of D_χ for χ given by (5.2). The relation (5.4) can be solved explicitly:

$$D_{n_1 n_2} \ = \ \{\sigma^{-1}(q^{-(n_1+n_2-2)J_3} \otimes \alpha_{ck}\gamma_c^m\gamma_c^{*n}) : \ m - n - k = n_1 - n_2\}^{\substack{\text{linear} \\ \text{span}}} \tag{5.5}$$

The space $D_\chi = D_{n_1 n_2}$ in the classical setting appeared for the first time in the beautiful monograph [2] by Gelfand and collaborators. To commemorate this fact D_χ will be called the Gelfand spaces. We have

Theorem 3

Let $n_1, n_2 \in \mathbf{C}$, $n_1 - n_2 \in \mathbf{Z}$ and $p = \frac{1}{2} \mid n_1 - n_2 \mid$.

Then the Casimir operator (4.5) *and the spin spectrum of the representation $v_{n_1 n_2}$ of G induced by the character* (5.2) *is given by*

$$\begin{aligned} \mathrm{Sp}\, v_{n_1 n_2} \ &= \ \{p, p+1, p+2, \ldots\} \\ C(v_{n_1 n_2}) \ &= \ -(q^{n_1} + q^{-n_1})I \end{aligned} \tag{5.6}$$

Moreover the spin spectrum is simple: each u^s enters to $v_{n_1 n_2}\mid_{S_qU(2)}$ at most once.

The technique of generalized functions (distributions) developed in [2] works in our case as well. It gives the full description of :

- Invariant bilinear and sesquilinear functionals on $D_\chi \times D_{\chi'}$
- Intertwining operators $D_\chi \longrightarrow D_{\chi'}$
- The set of all χ such that on D_χ there exists a positive invariant sesquilinear form.

Let χ be the character of P related to the pair (n_1, n_2) via the formula (5.2). Then χ^* is related to $(\bar{n}_2, \bar{n}_1)$ and (cf.(5.3))

$$(D_{n_1 n_2})^* \;=\; D_{\bar{n}_2 \bar{n}_1}.$$

The same relation follows from (5.5). Due to this fact the invariant bilinear functionals on $D_{n_1 n_2} \times D_{n'_1 n'_2}$ are in one-to-one correspondence with invariant sesquilinear functionals on $D_{\bar{n}_2 \bar{n}_1} \times D_{n'_1 n'_2}$.

Let

$$\begin{array}{rccc} S: & D_\chi \times D_{\chi'} & \longrightarrow & \mathbf{C} \\ & (x, y) & \longmapsto & (x \mid y)_S \end{array}$$

be a sesquilinear form on $D_\chi \times D_{\chi'}$. Then S gives rise to an $\mathcal{A}$-valued sesquilinear form on $(D_\chi \hat{\otimes} \mathcal{A}) \times (D_{\chi'} \hat{\otimes} \mathcal{A})$:

$$(x \otimes a \mid y \otimes b)_S := (x \mid y)_S \, a^* b.$$

We say that S is invariant if

$$(\Delta\, x \mid \Delta\, y)_S = (x \mid y)_S \, I_{\mathcal{A}}$$

for any $x \in D_\chi$, $y \in D_{\chi'}$.

Theorem 4
Let $n_1, n_2, n'_1, n'_2 \in \mathbf{C}$, $n_1 - n_2, n'_1 - n'_2 \in \mathbf{Z}$. Assume that there exists a non-zero invariant sesquilinear form on $D_{n_1 n_2} \times D_{n'_1 n'_2}$. Then we have the following four possibilities:

1.
$$(n'_1, n'_2) \;\equiv (-\bar{n}_2, -\bar{n}_1) \tag{5.7}$$

2.
$$(n'_1, n'_2) \;\equiv (\bar{n}_2, \bar{n}_1) \tag{5.8}$$

3.
$$(n'_1, n'_2) \;\equiv (-\mathrm{Re}\, n_2, \mathrm{Re}\, n_1) \tag{5.9}$$

where $\mathrm{Re}\, n_1 = 1, 2, \ldots$ *and* $\mathrm{Im}\, n_1 \equiv 0 \mod (2\pi/\log q)$

4.
$$(n'_1, n'_2) \;\equiv (\mathrm{Re}\, n_2, -\mathrm{Re}\, n_1) \tag{5.10}$$

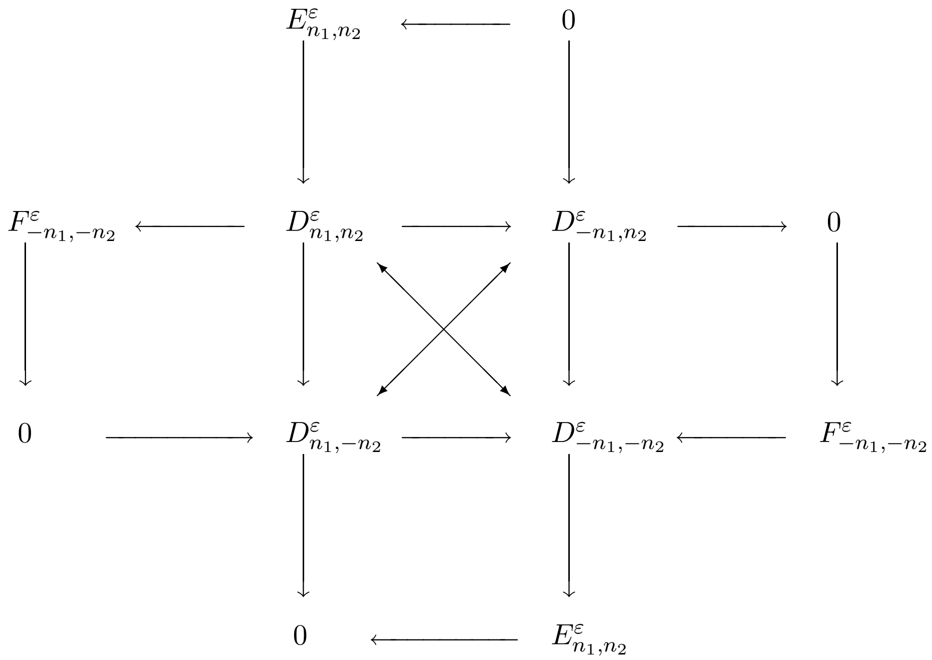
where $\mathrm{Re}\, n_2 = 1, 2, \ldots$ *and* $\mathrm{Im}\, n_2 \equiv 0 \mod (2\pi/\log q)$

In all these cases the invariant sesquilinear form is unique (up to a scalar factor).

The above theorem leads in a standard way to the following description of non-trivial intertwining operators acting between Gelfand spaces. An intertwiner $T : D_\chi \longrightarrow D_{\chi'}$ is trivial if $T = 0$ or $\chi = \chi'$ and $T = \lambda I$.

Theorem 5

1. Let n_1, n_2 *be positive integers,* $\varepsilon = 0$ *or* $i\pi/\log q$ *and* $D^\varepsilon_{n_1 n_2} := D_{n_1+\varepsilon, n_2+\varepsilon}$. *Then* $D^\varepsilon_{n_1 n_2}$ *contains the only one nontrivial invariant subspace* $E^\varepsilon_{n_1 n_2}$ *and* $\dim E^\varepsilon_{n_1 n_2} = n_1 n_2$, $D^\varepsilon_{-n_1,-n_2}$ *contains the only one nontrivial invariant subspace* $F^\varepsilon_{-n_1,-n_2}$ *and* $\operatorname{codim} F^\varepsilon_{n_1 n_2} = n_1 n_2$, $D^\varepsilon_{-n_1,n_2}$, $D^\varepsilon_{n_1,-n_2}$ *have no nontrivial invariant subspace. Moreover we have the following diagram of nontrivial intertwiners (except the ones that starts or ends at 0; these are obviously trivial):*



(5.11)

All intertwiners are unique up to a complex factor and any subsequence containing exactly two spaces $D^\varepsilon_{\pm n_1, \pm n_2}$ *in succession is exact.*

2. Let χ *be a character of* P *such that the space* D_χ *has not appeared in the diagram* (5.11) *and* $\chi' := (\dot\delta\dot\delta^*)^{-2}\chi^{-1}$. *(The reader should notice that* χ' *corresponds to the pair* $(-n_1, -n_2)$, *where* (n_1, n_2) *is related to* χ *via* (5.2)*). Then there exists unique (up to a scalar factor) bijective intertwiner*

$$D_\chi \overset{T}{\longleftrightarrow} D_{\chi'}.$$

Spaces D_χ *and* $D_{\chi'}$ *contain no non-trivial invariant subspace.*

3. The intertwiners listed in the above two points are the only non-trivial intertwiners acting between the Gelfand spaces.

Remark: The finite-dimensional representations acting on $E^{\varepsilon}_{n_1 n_2}$ of point 1 were studied in [4] . They all are non-unitary excepting the the cases of two 1-dimensional representations E^{ε}_{11}. Let us note also that by virtue of the diagram (5.11) the representations acting on $F^{\varepsilon}_{-n_1,-n_2}$, $D^{\varepsilon}_{-n_1,n_2}$ and $D^{\varepsilon}_{n_1,-n_2}$ are equivalent.

6 Gelfand spaces with unitary actions of QLG

Using Theorem 4 one can easily select all Gelfand spaces $D_{n_1 n_2}$ endowed with an invariant sesquilinear form $S : D_{n_1 n_2} \times D_{n_1 n_2} \to \mathbf{C}$. Due to the uniqueness of S it is automatically hermitian (after a suitable choise of the phase of the numerical factor). If S is positive then by the standard procedure $D_{n_1 n_2}$ can be completed to a Hilbert space $H_{n_1 n_2}$ and the action of *QLG* on $D_{n_1 n_2}$ extends in a natural way to a unitary representation (denoted again by $v_{n_1 n_2}$) of *QLG* on $H_{n_1 n_2}$. We shall show that in this way we can obtain all infinite-dimensional irreducible unitary representations listed in Table 1.

Let in Theorem 4, $(n'_1, n'_2) = (n_1, n_2)$. One can easily check that this relation is incompatible with (5.9) and with (5.10). Therefore only the first two possibilities remain.

Solving (5.7) we get

$$(n_1, n_2) \;=\; (p + \frac{i\rho}{2}, -p + \frac{i\rho}{2}) \tag{6.1}$$

where $p \in \mathbf{Z}/2$ and $\rho \in \mathbf{R}$. In this case S is automaticaly positive. Clearly ρ is defined mod $(4\pi/\log q)$ so we may assume that $\rho \in]2\pi/\log q, -2\pi/\log q]$. Moreover we may assume that $p \geq 0$ (and that $\rho \geq 0$ for $p = 0$): according to Theorem 5 simultaneous change of sign of p and ρ leads to an equivalent representations. The same theorem shows that $v_{n_1 n_2}$ is irreducible. Using Theorem 3 we get

$$\begin{aligned} Sp\, v_{n_1 n_2} \;&=\; \{p, p+1, p+2, \ldots\} \\ c(v_{n_1 n_2}) \;&=\; a\cos\varphi + ib\sin\varphi \end{aligned} \tag{6.2}$$

where $a = 2\cosh(p\log q)$, $b = 2\sinh(p\log q)$ and $\varphi = \pi + (\rho\log q)/2 \in [0, 2\pi[$ ($\varphi \in [0,\pi]$ for $p = 0$). For fixed p the values of (6.2) runs over the whole (degenerated for $p = 0$) ellipse $| c - 2 | + | c + 2 | = 2(q^p + q^{-p})$. Therefore (cf.Table 1) the representations $v_{n_1 n_2}$ (where n_1, n_2 are given by (6.1)) exhaust all the representations of the principal series.

Solving (5.8) we get

$$(n_1, n_2) \;=\; (\rho + \varepsilon, \rho + \varepsilon)$$

where $\rho \in \mathbf{R}$ and $\varepsilon = 0,\ i\pi/\log q$. In this case S is strictly positive if and only if $| \rho | < 1$. As before we may assume that $\rho \geq 0$. The case $\rho = 0$ was covered by (6.1). Therefore it is sufficient to consider $\rho \in]0, 1[$. Using Theorem 3 we get

$$\begin{aligned} Sp\, v_{n_1 n_2} \;&=\; \{0, 1, 2, \ldots\} \\ c(v_{n_1 n_2}) \;&=\; \mp 2\cosh(\rho\log q) \end{aligned} \tag{6.3}$$

where the upper (lower) sign corresponds to $\varepsilon = 0$ ($\varepsilon = i\pi/\log q$). The values of (6.3) covers the two intervals $]-(q + q^{-1}), -2[$ and $]2, q + q^{-1}[$. Therefore (cf. Table 1) in this case $v_{n_1 n_2}$ runs over all representations of the two complementary series. For the classical case the solution with $\varepsilon = i\pi/\log q$ do not exist and we have only one complementary series.

References

[1] Barut,A.O., Rączka,R.: Theory of Group Representations and Applications. PWN-Polish Scientific Publishers, Warszawa 1977

[2] Gelfand,I.M.,Graev,M.I.,Vilenkin,N.J.:Generalized Functions vol.5:Integral Geometry and Representation Theory. Academic Press, New York-London 1966

[3] Naimark,M.A.: Linear Representations of the Lorentz Group. Inernational Series on Monographs in Pure and Applied Mathematics vol.63, The Macmillan Company New York 1964

[4] Podleś,P.,Woronowicz,S.L.: Quantum deformation of Lorentz group. Commun.Math.-Phys. **130**, 381-431 (1990)

[5] Pusz,W.: Irreducible unitary representations of quantum Lorentz group. Commun. Math.Phys.**152**,591-626(1993)

[6] Pusz,W.,Woronowicz,S.L.: Representations of quantum Lorentz group on Gelfand spaces. in preparation

[7] Woronowicz,S.L.: Unbounded elements affiliated with C^* - algebras and non-compact quantum groups. Commun.Math. Phys. **136**, 399-432 (1991)