DUALITY IN THE C^*-ALGEBRA THEORY

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JUNE 1983

CPT-83/P.1520

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In Mathematics, the word "representation" has many meanings. One speaks about representations of groups, algebras, commutation relations, Lie algebras, etc. Usually the things that are being represented are objects of a certain category and representations are simply category morphisms into special objects whose structure is considered to be well understood.

For example in the case of topological groups one considers the group Aut H of all unitary operators acting on a given Hilbert space H. Aut H is endowed with the strong operator topology. A representation (or more precisely a unitary representation) of a topological group G is then a continuous group homomorphism $\pi : G \to \mathrm{Aut}\,H$. We say that H is the carrier Hilbert space of π or in other words that π is acting on H.

In many cases (especially in the field of interest of quantum physics) like in the example above, the special objects that serve as targets for representations are related to Hilbert spaces. For these representations one can speak about the carrier Hilbert spaces. Also the other important notions known in the group representation theory such as unitary equivalence, subrepresentation, direct sum and direct integral of representations and intertwining operators are meaningful in this general case.

Let us notice that the notions of unitary equivalence, subrepresentations can be defined in terms of intertwining operators. Indeed representations π and π' are unitarily equivalent iff there exists a unitary operator intertwining π and π'; a representation π' acting on $H' \subset H$ is a subrepresentation of a representation π acting on H iff the orthogonal projection $E : H \longrightarrow H'$ intertwines π and π'; $\pi = \pi_1 \oplus \pi_2$ (where π_1, π_2, π are acting on H_1, H_2, $H_1 \oplus H_2$ respectively) iff the canonical projection $P_i : H_1 \oplus H_2 \longrightarrow H_i$ intertwines π and π_i for $i = 1,2$.

On the other hand it is easy to list the characteristic properties of intertwining operators (cf. Def. 1 in the sequel). This way we arrive to the concept of W^*-category [2]. Our definition of a W^*-category differs from the one used in [2] in four important points

At first the sets of morphisms considered in [2] are abstract dual Banach spaces whereas in our approach they are weakly closed sub spaces of $B(H)$ and the composition of morphisms coincides with the product of operators. One may say that W^*-categories considered in this paper have concrete Hilbert space realization.

Secondly in order to avoid the well known set theoretical problems (the use of classes not being sets) we restrict ourselves to a fixed Hilbert space.

At third we use additional axioms saying that W^*-categories are closed with respect to the direct sum and taking sub-object operations.

At last we assume that the set of objects is endowed with a topological structure. We stress this point by saying that we deal with topological W^*-categories. Topology is necessary to introduce the direct integral operation (we shall not discuss this point detailly).

For any C^*-algebra A the set $\mathrm{Rep}_H A$ of all nondegenerate representations of A acting on the Hilbert space H in a natural way becomes a topological W^*-category. It turns out that $\mathrm{Rep}_H A$ can be considered as the object dual to A. Indeed it is possible to reconstruct A if the topological W^*-category $\mathrm{Rep}_H A$ is given. At this point a notion of operator function defined on a W^*-category is very important. The operator functions were introduced in [9] and [4], see also [3] where they are called decomposable functions and [2]. W^*-categories serve as natural domains for operator functions. We show that the C^*-algebra A is canonically isomorphic with the algebra of all "vanishing at infinity" continuous operator functions defined on $\mathrm{Rep}_H A$.

At the end of the paper we introduce an important property of topological W^*-categories called "locally compactness" which is characteristic for topological W^*-categories of the form $\mathrm{Rep}_H A$, where A is a C^*-algebra. The algebra A has unity if and only if the considered topological W^*-category is "compact".

In principle this property can be used to determine in which cases the structure of a set of representations of an object (e.g. a group, an algebra, a commutation relation) is isomorphic to $\mathrm{Rep}_H A$. If this is the case, the C^*-algebra A is uniquely determined.

DEFINITION 1

Let H be a Hilbert space and R be a set. Assume that for any pair (r,r') of elements of R a weakly closed linear subspace $\mathrm{Mor}(r,r')$ of $B(H)$ is given. We say that R is a W^*-category if the following axioms are satisfied.

I. $I \in \mathrm{Mor}(r,r)$ for any $r \in R$

II. For any $r, r', r'' \in R$ and $a, b \in B(H)$

$$\begin{pmatrix} a \in \mathrm{Mor}(r,r') \\ b \in \mathrm{Mor}(r',r'') \end{pmatrix} \Rightarrow \left(ba \in \mathrm{Mor}(r,r'') \right)$$

III. For any $r, r' \in R$ and $a \in B(H)$

$$\left(a \in \mathrm{Mor}(r,r') \right) \Rightarrow \left(a^* \in \mathrm{Mor}(r',r) \right)$$

IV. For any $r, r' \in R$

$$\left(I \in \mathrm{Mor}(r,r') \right) \Rightarrow \left(r = r' \right)$$

V. For any $r \in R$ and any $u \in B(H)$ such that $u^*u = I$, $uu^* \in \mathrm{Mor}(r,r)$ there exists $r' \in R$ such that $u \in \mathrm{Mor}(r',r)$

VI. For any family $(r_\alpha)_{\alpha \in A}$ of elements of R and any family of isometries $(u_\alpha)_{\alpha \in A}$ acting on H such that $\sum_\alpha u_\alpha u_\alpha^* = I$ there exists $r' \in R$ such that $u_\alpha \in \mathrm{Mor}(r_\alpha, r)$ $\alpha \in A$

To stress the connection between R and H we shall say that R is a W^*-category acting on H. One can easily see that the properties described in axioms IV, V, VI can be expressed in the following single statement.

THEOREM 1

Let R be a W^*-category acting on H, $(r_\alpha)_{\alpha\in A}$ be a family of elements of R and $(u_\alpha)_{\alpha\in A}$ be a family of operators acting on H indexed by the same nonempty set A. Assume that H is generated by the union of the images of u_α : $H = \bigvee_{\alpha\in A} u_\alpha H$ and that

$$u_\alpha^* u_\beta \in \mathrm{Mor}(r_\beta, r_\alpha)$$

for any $\alpha,\beta \in A$. Then there exists one and only one element $r \in R$ such that

$$u_\alpha \in \mathrm{Mor}(r_\alpha, r)$$

for any $r \in R$.

The proof of this theorem is standard and will be omitted.

DEFINITION 2

Let R be a W^*-category acting on a Hilbert space H. A mapping

$$F : R \longrightarrow B(H)$$

will be called an operator function defined on R if for any $r,r' \in R$ and any $a \in \mathrm{Mor}(r,r')$ we have

$$a F(r) = F(r') a$$

The set of all operator functions defined on a W^*-category R will be denoted by $\mathcal{F}(R)$. For any $F,G \in \mathcal{F}(R)$, $\lambda \in \mathbb{C}$ and $r \in R$ we set

$$(F+G)(r) \overset{df}{=} F(r) + G(r)$$

$$(\lambda\cdot F)(r) \overset{df}{=} \lambda F(r)$$

$$(F\cdot G)(r) \overset{df}{=} F(r)\, G(r)$$

$$(F^*)(r) \overset{df}{=} F(r)^*.$$

One can easily check that $F+G$, $\lambda\cdot F$, $F\cdot G$ and F^* are operator functions and that $\mathcal{F}(R)$ endowed with these algebraic operations becomes a $*$-algebra.

Assume now for the moment that for some $F \in \mathcal{F}(R)$ one can find a sequence (r_n) in R such that $\lim \|F(r_n)\| = \infty$. Let u_n be a sequence of isometries acting on H such that $\sum_n u_n u_n^* = I$. According to Def. 1. VI there exists $r \in R$ such that $u_n \in \mathrm{Mor}(r_n, r)$. Then we have $u_n F(r_n) = F(r) u_n$, $F(r_n) = u_n^* F(r$ and all the norms $\|F(r_n)\|$ are not larger than $\|F(r)\|$. The contradiction that we obtained this way shows that for any $F \in \mathcal{F}(R)$

$$\|F\| \overset{df}{=} \sup_{r\in R} \|F(r)\|$$

is finite. Clearly the above formula defines a norm on $\mathcal{F}(R)$ and $\mathcal{F}(R)$ becomes a normed $*$-algebra. The following theorem gives more informations about $\mathcal{F}(R)$.

THEOREM 2

Let R be a W^*-category acting on H. Then $\mathcal{F}(R)$ is a W^*-algebra. Moreover $\mathcal{F}(R)$ is rich in the following sense: for any r, $r' \in R$

$$\mathrm{Mor}(r,r') = \left\{ a \in B(H) : \begin{array}{l} aF(r) = F(r')a \\ \text{for all } F \in \mathcal{F}(R) \end{array} \right\}$$

<u>Proof</u> : This theorem follows directly from the Murray von Neumann double commutant theorem. For details see [2].

As it was mentioned in the Introduction the really interesting objects can be obtained by combining the W^*-category structure and the topological structure. These objects are called topological W^*-categories. More precisely R is a topological W^*-category if R is a topological space and R is a W^*-category. For the moment we do not formulate any compatibility condition relating these two structures.

Let R be a topological W^*-category and $F \in \mathcal{F}(R)$. We say that F is a continuous operator function if F is a continuous map from the topological space R into $B(H)$ endowed with the $*$-strong topology. The set of all continuous operator functions defined on R will be denoted by $C(R)$:

$$C(R) \stackrel{df}{=} \left\{ F \in \mathcal{F}(R) : \begin{array}{l} \text{For any } x \in H \text{ the mappings} \\ R \ni r \longrightarrow F(r)x \in H \\ R \ni r \longrightarrow F(r)^* x \in H \\ \text{are continuous.} \end{array} \right\}$$

It is well known that algebraic operations restricted to a bounded subset of $B(H)$ are continuous with respect to the $*$-strong topology. Remembering that the limit of a uniformly converging sequence of continuous mappings is a continuous mapping we get the following result :

<u>THEOREM 3</u>

Let R be a topological W -category. Then $C(R)$ is a C^*-algebra with unity.

The unity is the operator function $\mathbb{1} \in \mathcal{F}(R)$ such that $\mathbb{1}(r) = I$ for any $r \in R$. Clearly $\mathbb{1} \in C(R)$.

Let R be a topological W^*-category and $r \in R$. It is easy to see that the map :

$$(*) \qquad C(R) \ni F \longrightarrow F(r) \in B(H)$$

is a representation of $C(R)$. A representation π of $C(R)$ will be called singular if π is disjoint with all representations of the form (*).

Now we can introduce the class of all continuous operator functions vanishing at infinity.

$$C_\infty(R) \stackrel{df}{=} \left\{ F \in C(R) : \begin{array}{l} \pi(F) = 0 \text{ for any singular} \\ \text{representation } \pi \text{ of } C(R) \end{array} \right\}$$

For the completeness reasons we state the following obvious result :

<u>THEOREM 4</u>

Let R be a topological W^*-category. Then $C_\infty(R)$ is a C^*-algebra (without unity in general). $C_\infty(R)$ is a closed ideal in $C(R)$.

In the general case, when the topology of R is not well compatible with its W^*-category structure, the algebras $C(R)$ and $C_\infty(R)$ may be very small (e.g. : $C(R) = \{\lambda \mathbb{1} : \lambda \in \mathbb{C}\}$, $C_\infty(R) = \{0\}$). Therefore we need some axioms expressing the compatibility of the W^*-category structure with the topology. These axioms should be formulated in terms of open sets and elements of Mor(.,.) . Unfortunately at the present moment we are not able to formulate the compatibility condition in the way satisfying the above requirement. Instead we shall use the following two axioms.

VII. The algebra $C(R)$ separates the points of R.

VIII. The topology of R is the weakest one such that for all $F \in C(R)$ and $x \in H$ the mappings

$$R \ni r \longrightarrow F(r)x \in H$$

are continuous.

More precisely VII says that for any two distinct elements $r, r' \in R$ one can find $F \in C(R)$ such that $F(r) \neq F(r')$. Axiom VIII means that for any neighbourhood $\mathcal{O}$ of a point $r \in R$ one can find $F_1, F_2, \dots, F_N \in C(R)$; $x_1, x_2, \dots, x_N \in H$ and $\varepsilon > 0$ such that

$$\left\{ r' \in R : \begin{array}{c} \|(F_i(r') - F_i(r))x_i\| < \varepsilon \\ i = 1, 2, \dots, N \end{array} \right\} \subset \mathcal{O}.$$

Clearly VII implies that the topological space R is Hausdorff.

Now we shall disucss the topological W^*-categories related to C^*-algebras. For the simplicity we shall assume that the algebras are separable (otherwise one has to consider nonseparable Hilbert spaces).

Let A be a C^*-algebra. We denote by Rep A (or more precisely by $Rep_H A$) the set of all nondegenerate representations of A acting on the Hilbert space H. We recall that a representation π of A is said to be nondegenerated (essential) if 0 in the only vector $x \in H$ such that $\pi(a)x = 0$ for all $a \in A$.

For any $\pi, \pi' \in Rep\,A$ we denote by $Mor(\pi, \pi')$ the set of all intertwining operators :

$$Mor(\pi, \pi') = \left\{ b \in B(H) : \begin{array}{c} b\pi(a) = \pi'(a)b \\ \text{for any } a \in A \end{array} \right\}$$

One can easily check that $Mor(\pi, \pi')$ is a weakly closed linear subspace of B(H) and that the axioms I-VI are satisfied. It means that Rep A is a W^*-category.

It is interesting that in this case the algebra of all operator functions admits very simple description. Indeed, using the Murray-von Neumann double commutant theorem (for details see [2]) one easily gets the following result :

THEOREM 5

Let A be a C^*-algebra and H be a Hilbert space. Assume that A and H are separable (in fact it is sufficient to assume that the dimension of H is larger than the cardinality of a dense subset of A). Then the algebra of all operator functions $\mathcal{F}(Rep_H A)$ is canonically isomorphic to the W^*-envelopping algebra A^{**} of A. An element $a \in A^{**}$ corresponds to an operator function $F \in \mathcal{F}(Rep\,A)$ if and only if for any $\pi \in Rep\,A$

$$F(\pi) = \tilde{\pi}(a)$$

where $\tilde{\pi} : A^{**} \to B(H)$ denotes the weakly continuous extension of π.

The most interesting are operator functions corresponding to elements of A. We provide Rep A with the weakest topology such that all these operator functions are continuous. This way Rep A becomes a topological W^*-category. A subset $\mathcal{O} \subset Rep\,A$ is a neighbourhood of a representation $\pi \in Rep\,A$ if and only if $\mathcal{O}$ contains a set of the form

$$\left\{ \pi' \in Rep\,A : \begin{array}{c} \|(\pi'(a_i) - \pi(a_i))x_i\| < \varepsilon \\ i = 1, 2, \dots, N \end{array} \right\}$$

where $a_1, a_2, \dots, a_N \in A$; $x_1, x_2, \dots, x_N \in H$ and $\varepsilon > 0$. This topology on Rep A was considered by many authors (see e.g. [7]).

For the sake of completeness we state the following obvious result :

THEOREM 6

Let A be a C^*-algebra and H be a Hilbert space. Then $\mathrm{Rep}_H A$ is a topological W^*-category satisfying the compatibility axioms VII and VIII.

It is interesting to see which subsets of A^{**} correspond to classes of operator functions with different continuity properties. The answer to this question is given in the following theorem, where we identify elements of A^{**} with the corresponding operator functions.

THEOREM 7

Let A be a separable C^*-algebra and H be a separable Hilbert space. Then

1. $$\left\{ F \in \mathcal{F}(\mathrm{Rep}\,A) \;:\; \begin{array}{l} \text{For any } x,y \in H \text{ the function} \\ \mathrm{Rep}\,A \ni \pi \longrightarrow (x|F(\pi)y) \in \mathbb{C} \\ \text{is continuous} \end{array} \right\} = QM(A)$$

2. $$\left\{ F \in \mathcal{F}(\mathrm{Rep}\,A) \;:\; \begin{array}{l} \text{For any } x \in H \text{ the mapping} \\ \mathrm{Rep}\,A \ni \pi \longrightarrow F(\pi)x \in H \\ \text{is continuous} \end{array} \right\} = LM(A)$$

3. $$C(\mathrm{Rep}\,A) = M(A)$$

4. $$C_\infty(\mathrm{Rep}\,A) = A$$

where QM(A), LM(A), M(A) denote the multiplier algebras of the algebra A (see [5] for the precise definitions).

The proof of this theorem is given in [10]. Here we would like to point out only it is very easy to show that members of QM(A), LM(A), M(A) and A have the postulated continuity properties : for A it follows directly from the definition of the topology on Rep A ; for QM(A), LM(A) and M(A) one has to use the approximative unity in A and the classical result saying that the limit of the uniformly converging sequence of continuous pping. On the other hand the proof that the continuity properties imply the belonging to the multiplier algebras of A is quite difficult. In the case 1, one has to use the Voiculescu result [8], in the case 2, one passes to the algebra with unity and apply 1. Case 3 follows directly from case 2 and case 4 follows easily from case 3.

Let us notice that for the algebras with unity the essentially equivalent result is contained in [7].

Now we go back to the general topological W^*-categories. Let R be a topological W^*-category. We are interested, under what conditions one can find a C^*-algebra A such that R is isomorphic to Rep A. According to Theorem 7 there is only one candidate for A : $A = C_\infty(R)$. Unfortunately in the general case $C_\infty(R)$ may be very small (e.g. $C_\infty(R) = \{0\}$) ; axiom VII says only that C(R) is large enough. The same phenomena we have in the usual theory of topological spaces. Let X be a normal topological space. Then the algebra C(X) of bounded continuous complex valued functions on X is rich enough : it separates the points of X . On the other hand

$$C_\infty(X) \overset{\mathrm{df}}{=} \left\{ F \in C(X) \;:\; \begin{array}{l} \text{For any } \varepsilon > 0 \\ \{z \in X : |F(z)| \geq \varepsilon\} \\ \text{is compact} \end{array} \right\}$$

is in general very small. Indeed only if X is locally compact, the algebra $C_\infty(X)$ contains functions nonvanishing at any given point of X .

Therefore in our case we also need a condition saying that our topological W^*-category is in some sense "locally compact". To express this condition we have to introduce the following notation.

Let Isom(H) denote the set of all isometric operators acting on H equipped with the strong operator topology and let $\mathcal{V}$ denotes the filter of all neighbourhoods of I in Isom(H). For any $\mathcal{O} \subset R$ and any $V \in \mathcal{V}$ we set

$$\mathcal{O}^V \overset{df}{=} \left\{ r \in R \; : \; \begin{array}{ll} \text{there exist} & r' \in \mathcal{O},\ b \in V \\ \text{such that} & b \in Mor(r, r') \end{array} \right\}$$

DEFINITION 3

Let R be a topological W^*-category acting on H, $\mathcal{O} \subset R$, $V \in \mathcal{V}$. We say that $\mathcal{O}$ is V-precompact if for any open covering

$$R = \dot{\bigcup_{\lambda \in \Lambda}} \mathcal{O}_\lambda$$

one can find a finite subset $\Lambda_0 \subset \Lambda$ such that

$$\mathcal{O} \subset \bigcup_{\lambda \in \Lambda_0} \mathcal{O}_\lambda^V$$

DEFINITION 4

A topological W^*-category R is called "compact" if R is V-precompact for any $V \in \mathcal{V}$.

DEFINITION 5

A topological W^*-category R is called "locally compact" if for any $r \in R$ and any $V \in \mathcal{V}$ there exists a V-precompact neighbourhood of r.

It turns out that only for "locally compact" topological W^*-categories R one may find a C^*-algebra A such that R is isomorphic to Rep A. Indeed we have the following result :

THEOREM 8

Let A be a C^*-algebra and H is a Hilbert space. Assume that A and H are separable (in fact like in theorem 5 it is sufficient to assume that dim H is large enough). Then the topological W^*-category Rep A is "locally compact". If A has unity then Rep A is "compact".

The proof of this theorem is given in [10]. It uses the Dixmier Lemma [1], the compactness of the set of all normalized completely positive maps from A into B(H) endowed with the weak topology and the Stinespring theorem [6].

The condition that R is "locally compact" is also sufficient in order to obtain the positive answer to our main question. Namely we have :

THEOREM 9

Let R be a topological W^*-category acting on a Hilbert space H. Assume that R satisfies axioms VII and VIII and that R is "locally compact". Then

1° The algebra $C_\infty(R)$ is rich in the following sense : for any $r \in R$ the representation

$$\pi_r \; : \; C_\infty(R) \ni F \longrightarrow F(r) \in B(H)$$

is nondegenerate.

2°) Any nondegenerate representation of $C_\infty(R)$ acting on H is of the form π_r with $r \in R$ uniquely determined.

3°) For any $r, r' \in R$

$$\mathrm{Mor}(\pi_r, \pi_{r'}) = \mathrm{Mor}(r, r')$$

4°) The mapping

$$R \ni r \longrightarrow \pi_r \in \mathrm{Rep}_H\ C_\infty(R)$$

is a homeomorphism.

Moreover if R is "compact" then there is no singular representations of C(R) and $C_\infty(R) = C(R)$ is a C^*-algebra with unity.

The proof of this theorem is given in [10]. The main step in this proof is to show that "locally compactness" implies that the set of states of C(R) related to the singular representations of C(R) is a closed face in the set of all states of C(R). Then the statement of the theorem follows easily.

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