POSITIVE MAPS OF LOW DIMENSIONAL MATRIX ALGEBRAS

by

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ABSTRACT : It is shown that any positive map from M_2 into M_3 is a sum of completely positive and completely copositive maps. This result does not hold for maps into M_4. A generalization of the Kadison inequality is suggested and proved for positive maps defined on M_2 or M_3.

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INTRODUCTION

A linear mapping from a C^*-algebra $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ is called positive if it sends positive elements of $\mathcal{A}$ into positive elements of $\mathcal{B}$. Such a map is said to be normalized if the image of the unity of $\mathcal{A}$ coincide with the unity of $\mathcal{B}$.

In recent years, positive maps have become a subject of common interest to mathematicians and physicists. There are several reasons for this :

1° The notion of positive map generalizes that of state, representation, Jordan representation, conditional expectation and (semi-) spectral measure.

2° By transposition, a normalized positive map defines an affine mapping between sets of states of C^*-algebras. If these algebras coincide with the C^*-algebra of observables assigned to a physical system, then this affine mapping corresponds to some "operation" which can be performed on the system. Invertible operations such as those representing the time evolution of a closed system or the symmetries of a system have a very simple mathematical structure. According to [7], they are related to a very special class of positive maps known as Jordan automorphisms.

On the other hand, the time evolution of an open system, (i.e. interacting with its surroundings), as well as perturbations of a system caused by a measurement process are described by noninvertible operations. The theory of positive maps is then very useful (cf.[5,9]).

3° In our opinion, there is an important, purely mathematical reason for investigating continuous affine mappings of sets of states of C^*-algebras. To explain it, let us make a few remarks on the Krein-Milman-Choquet theory of convex compact sets. In this theory, a distinguished role is played by sets of probability measures on compact topological spaces. We will call these sets standard. The structure of any compact convex set is investigated by means of morphisms of standard sets onto the given one.

One may think about the "second order" theory of convex sets in which the role of standard sets will be played by sets of states of C^*-algebras. To make the first step towards this new theory, one has to investigate in detail morphisms between new standard objects, i.e. the affine continuous mappings mentioned above.

Despite the important role of positive maps, their structure has been well understood only in some special cases.

The set of all operators from a Hilbert space K into Hilbert space H will be denoted by $B(K,H)$. $B(\mathbb{C}^n, \mathbb{C}^m)$ will be identified with the set $M_{m\times n}$ of all $m\times n$ complex matrices. We also write $B(H)$ instead of $B(H,H)$ and M_n instead of $M_{n\times n}$: $M_n = B(\mathbb{C}^n)$.

Let us recall [11] that a linear map

$$\Phi : \mathcal{A} \longrightarrow \mathcal{B} \tag{0.1}$$

is called n-positive (where n is a natural number) if the tensor product map

$$\Phi \otimes id : \mathcal{A} \otimes M_n \longrightarrow \mathcal{B} \otimes M_n$$

(where id is the identity map in M_n) is positive. The map Φ is called completely positive if it is n-positive for any natural number n.

The structure of completely positive maps is very well understood (see [11]). For example the set of all completely positive maps from M_n into M_m coincides with the convex cone generated by maps of the form

$$M_n \ni a \longmapsto s^* a s \in M_m \tag{0.2}$$

where $s \in M_{n\times m}$.

Lindblad [8] indicates that operations which can be performed on a physical system are related to completely positive maps. His argument is based on two assumptions : that any operation can be applied to a part of a composed system, having no influence to the rest (which seems not to be the case for geometric operations like time inversion) and that C^*-algebra of a composed system coincides with the tensor product of C^*-algebras associated with the components of the system. The latter does not hold, if the components contain particles of the same kind. However, for a large class of operations his argument is essentially correct and these operations would be represented by completely positive maps.

One may introduce a notion, which is in some sense dual to the notion of n-positivity.

Definition :

Map (0.1) is called n-copositive if the tensor product map

$$\Phi \otimes T : \mathcal{A} \otimes M_n \longrightarrow \mathcal{B} \otimes M_n$$

(where T denotes the transposition of $n\times n$ matrices) is positive. Φ is called completely copositive if it is n-copositive for any natural number n.

Let us note that the notions of positivity, 1-positivity and 1-copositivity coincide.

All results derived for n-positive and completely positive maps can be repeated with suitable modifications for copositive maps. In particular the set of all completely copositive maps from M_n into M_m coincides with the convex cone generated by maps of the form

$$M_n \ni a \longmapsto r^* a^T r \tag{0.3}$$

where $r \in M_{n\times m}$.

It turns out that for some pairs $(\mathcal{A}, \mathcal{B})$, any positive map from $\mathcal{A}$ into $\mathcal{B}$ splits into sum of completely positive and completely copositive maps.

This is the case for (M_2, M_2), (M_2, M_3) and (M_3, M_2). The result for (M_2, M_2) is in fact contained in [10] (see also [6]). In the present paper we investigate the case (M_2, M_3) (see Thm 1.2). The result for (M_3, M_2) follows easily from that for (M_2, M_3) by passing to the transposed maps.

For other pairs of matrix algebras (M_n, M_m) (where $n, m \geqslant 2$) the statement is not valid. For (M_3, M_3) a suitable counterexample is given in [3]. The case (M_2, M_4) is analysed in the present paper (thm 1.3). A counterexample for (M_4, M_2) can be obtained from that for (M_2, M_4) by transposition.

The main idea of the paper is based on the duality principle. One finds cones V and V^τ dual to the cones of completely positive and completely copositive maps. Then the cone generated by the completely positive and the completely copositive maps is dual to the intersection $V \cap V^\tau$. We investigate this intersection in detail and discover that in some cases the dual of $V \cap V^\tau$ coincides with the cone of all positive maps.

On the way we get a remarkable generalisation of the Kadison inequality for positive maps from M_2 and M_3 (see section 5).

1. MAIN RESULTS

Let H be a finite dimensional Hilbert space. In the following we shall mainly work with operators acting on $H^2 = H \oplus H$. Any operator $Q \in B(H^2)$ can be represented by a block matrix

$$Q = \begin{pmatrix} A, & B \\ C, & D \end{pmatrix} \tag{1.1}$$

where $A, B, C, D \in B(H)$

For $u \in H^2$, u^1 and u^2 will denote its first and second components respectively :

$$u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \tag{1.2}$$

Vector u will be called simple if u^1 and u^2 are proportional : $u^i = \lambda^i x$, where $\lambda^i \in \mathbb{C}$, $x \in H$ $(i = 1, 2)$.

The action of the operator (1.1) on the vector (1.2) is defined according to the usual rules of matrix calculus :

$$\begin{pmatrix} A, & B \\ C, & D \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} Au^1 + Bu^2 \\ Cu^1 + Du^2 \end{pmatrix}$$

In what follows, an important role will be played by the block transposition operation :

$$B(H^2) \ni Q \longmapsto Q^\tau \in B(H^2)$$

which is defined by

$$\begin{pmatrix} A, & B \\ C, & D \end{pmatrix}^\tau = \begin{pmatrix} A, & C \\ B, & D \end{pmatrix}$$

We shall use Dirac notation : for any x, y belonging to a Hilbert space K, $|x)(y|$ will denote an operator acting in according to the formula :

$$|x)(y|\, u = (y|u)\, x \qquad u \in K$$

Let $u \in H^2$. By simple computation one get :

$$|u)(u| = \begin{pmatrix} |u^1)(u^1| , & |u^1)(u^2| \\ |u^2)(u^1| , & |u^2)(u^2| \end{pmatrix}$$

and

$$|u)(u|^{\tau} = \begin{pmatrix} |u^1)(u^1| , & |u^2)(u^1| \\ |u^1)(u^2| , & |u^2)(u^2| \end{pmatrix}$$

Let us notice that $|u)(u|^{\tau}$ is not in general positive. One can easily check that $|u)(u|^{\tau} \geqslant 0$ if and only if u is a simple vector :

$$u = \begin{pmatrix} \lambda^1 x \\ \lambda^2 x \end{pmatrix} \qquad (1.3)$$

where $\lambda^1, \lambda^2 \in \mathbb{C}$, $x \in H$. In that case

$$|u)(u|^{\tau} = |u^{\tau})(u^{\tau}|$$

where

$$u^{\tau} = \begin{pmatrix} \overline{\lambda^1} x \\ \overline{\lambda^2} x \end{pmatrix} \qquad (1.4)$$

<u>Remark</u> : One should notice that u^{τ} is defined up to a phase factor; (1.4) depends on the representation of u in form (1.3).

After this preparation, we may state our main technical result, which will be proved in section 2.

<u>Proposition 1.1</u>

Let H be a finite dimensional Hilbert space. Then the following two

statements are equivalent :

1. Any positive map

$$\phi : M_2 \dashrightarrow B(H)$$

is a convex combination of maps of the form :

$$M_2 \ni a \longmapsto S^* a S \in B(H) \qquad (1.5)$$

$$M_2 \ni a \longmapsto R^* a^{\tau} R \in B(H) \qquad (1.6)$$

where $R, S : H \longrightarrow \mathbb{C}^2$ are linear maps (cf. formulae (0.2) and (0.3)).

II. For any operator $Q \in B(H^2)$ such that $Q \neq 0$, $Q \geqslant 0$ and $Q^{\tau} \geqslant 0$, there exists a simple vector $u \in H^2$ such that $u \neq 0$, $u \in Q(H^2)$ and $u^{\tau} \in Q^{\tau}(H^2)$.

We shall use this proposition in both directions. It is known (see Introduction) that the statement I holds for dim H = 2. Therefore we have :

<u>Theorem 1.1</u>

Let H be a two dimentional Hilbert space, $Q \in B(H^2)$, $Q \neq 0$, $Q \geqslant 0$ and $Q^{\tau} \geqslant 0$. Then there exists a simple vector $u \in H^2$ such that $u \neq 0$, $u \in Q(H^2)$ and $u^{\tau} \in Q^{\tau}(H^2)$.

In section 3 we prove the statement II for dim H = 3. Consequently:

<u>Theorem 1.2</u>

Let H be a three dimensional Hilbert space and $\phi : M_2 \longrightarrow B(H)$ be a positive map. Then

$$\phi(a) = \sum_{i=1}^{N} S_i^* a S_i + \sum_{j=1}^{M} R_j^* a^{\tau} R_j \qquad (1.7)$$

where $S_i , R_j : H \longrightarrow \mathbb{C}^2$ are linear maps.

In section 4 we disprove the statement II for $\dim H = 4$. Thus :

Theorem 1.3

Assume that $\dim H = 4$. Then there exists a positive map $\Phi : M_2 \longrightarrow B(H)$, which can not be written in form (1.7).

In the last section we conjecture some generalisation of the Kadison inequality and discuss its connection with the statements I and II .

2. POSITIVE MAPS AS LINEAR FUNCTIONALS

This section is devoted to the proof of Prop. 1.1. Its title is motivated by the following

Proposition 2.1

1° The formula

$$\omega(Q) = \mathrm{Tr}\, Q \begin{pmatrix} \Phi\begin{pmatrix}1,0\\0,0\end{pmatrix}, & \Phi\begin{pmatrix}0,1\\0,0\end{pmatrix} \\ \Phi\begin{pmatrix}0,0\\1,0\end{pmatrix}, & \Phi\begin{pmatrix}0,0\\0,1\end{pmatrix} \end{pmatrix} \tag{2.1}$$

where Q runs over $B(H^2)$, defines 1-1 correspondence between the set of all linear maps $\Phi : M_2 \longrightarrow B(H)$ and the set of all linear functionals $\omega : B(H^2) \longrightarrow \mathbb{C}$.

2° Φ is positive if and only if

$$\omega(|u)(u|) \geqslant 0 \tag{2.2}$$

for any simple vector $u \in H^2$.

3° Φ is a convex combination of maps of the form (1.5) if and only if

$$\omega(Q) \geqslant 0 \tag{2.3}$$

for any $Q \geqslant 0$.

4° Φ is a convex combination of maps of the form (1.6) if and only if

$$\omega(Q) \geqslant 0$$

for any $Q \in B(H^2)$ such that $Q^T \geqslant 0$.

Proof :

Ad 1° : Clearly Φ determines ω . On the other hand any linear functional ω can be represented by an operator $F \in B(H^2)$ in the following sense :

$$\omega(Q) = \operatorname{Tr} QF \qquad Q \in B(H^2)$$

Writing F in block matrix form

$$F = \begin{pmatrix} F_{11}, & F_{12} \\ F_{21}, & F_{22} \end{pmatrix}$$

and setting

$$\Phi\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} = \alpha F_{11} + \beta F_{12} + \gamma F_{21} + \delta F_{22}$$

we find the linear map $\Phi : M_2 \longrightarrow B(H)$ corresponding to ω .

Ad 2° : Let $u = \begin{pmatrix} \lambda^1 x \\ \lambda^2 x \end{pmatrix}$. Making use of (2.1) one may check that

$$\omega(|u)(u|) = \left(x \,\middle|\, \Phi\begin{pmatrix} \lambda^1\bar\lambda^1, & \lambda^2\bar\lambda^1 \\ \lambda^1\bar\lambda^2, & \lambda^2\bar\lambda^2 \end{pmatrix} x\right)$$

Let us note that

$$\begin{pmatrix} \lambda^1\bar\lambda^1, & \lambda^2\bar\lambda^1 \\ \lambda^1\bar\lambda^2, & \lambda^2\bar\lambda^2 \end{pmatrix} \geq 0 \qquad (2.4)$$

Therefore (2.2) holds for positive Φ . On the other hand, matrices of the form (2.4) generate whole cone of positive matrices. Therefore (2.2) implies that $(x|\Phi(a)x) \geq 0$ for any positive matrix a.

Ad 3° : At first we note that any linear map

$$S : H \longrightarrow \mathbb{C}^2$$

is determined by a vector $u \in H^2$ in the following sense :

$$Sx = \begin{pmatrix} (u^1|x) \\ (u^2|x) \end{pmatrix} \qquad (2.5)$$

Assume (cf.(1.5)) that

$$\Phi(a) = S^* a S$$

where S is given by (2.5). After a simple computation using (2.1) and (2.5) one get in that case

$$\omega(Q) = (u|Qu)$$

This proves "only if" part of 3° .

Conversely assume (2.3). Then

$$\begin{pmatrix} \Phi\begin{pmatrix} 1,0 \\ 0,0 \end{pmatrix}, & \Phi\begin{pmatrix} 0,1 \\ 0,0 \end{pmatrix} \\ \Phi\begin{pmatrix} 0,0 \\ 1,0 \end{pmatrix}, & \Phi\begin{pmatrix} 0,0 \\ 0,1 \end{pmatrix} \end{pmatrix} \geq 0$$

According to the spectral theorem one may find vectors $u_1, u_2, \dots, u_N \in H^2$ such that

$$\begin{pmatrix} \Phi\begin{pmatrix} 1,0 \\ 0,0 \end{pmatrix}, & \Phi\begin{pmatrix} 0,1 \\ 0,0 \end{pmatrix} \\ \Phi\begin{pmatrix} 0,0 \\ 1,0 \end{pmatrix}, & \Phi\begin{pmatrix} 0,0 \\ 0,1 \end{pmatrix} \end{pmatrix} = \sum_{n=1}^{N} |u_n)(u_n|$$

Then

$$\Phi(a) = \sum_{n=1}^{N} S_n^* a S_n$$

where

$$S_n x = \begin{pmatrix} (u_n^1|x) \\ (u_n^2|x) \end{pmatrix} \qquad x \in H$$

Ad 4° : The proof is almost identical as in Ad 3° and will be omitted.

In brief, Prop. 2.1 provides us a duality between $B(H^2)$ and the space of all linear mappings from M_2 into $B(H)$. It turns out that the cone of positive maps is dual to the cone W generated by $\{|u)(u| : u \in H^2,\ u\text{-simple}\}$. Moreover the cone generated by mappings of the form (1.5) is dual to the cone

$$V = \{Q \in B(H^2) : Q \geqslant 0\}$$

and the cone generated by mappings of the form (1.6) is dual to

$$V^\tau = \{Q \in B(H^2) : Q^\tau \geqslant 0\}$$

Therefore the cone generated by mappings of the form (1.5) and (1.6) is dual to $V \cap V^\tau$.

Statement I (see section 1) can be now expressed in an equivalent dual version :

$$V \cap V^\tau \subset W \tag{2.6}$$

We shall show that (2.6) is equivalent to statement II :

(2.6) ⇒ II : Let $Q \in B(H^2)$, $Q \geqslant 0$ and $Q^\tau \geqslant 0$.
It means that $Q \in V \cap V^\tau$. In virtue of (2.6) we have

$$Q = \sum_{n=1}^{N} |u_n)(u_n|$$

where u_n are simple vectors. Then

$$Q^\tau = \sum_{n=1}^{N} |u_n^\tau)(u_n^\tau|$$

and for each n we have

$$u_n \in Q(H^2) \qquad \text{and} \qquad u_n^\tau \in Q^\tau(H^2)$$

II ⇒ (2.6) : Assume that Q is an extreme element of the cone $V \cap V^\tau$. Then $Q \geqslant 0$, $Q^\tau \geqslant 0$ and according to II one can find a simple vector u such that

$$u \in Q(H^2) \qquad \text{and} \qquad u^\tau \in Q^\tau(H^2)$$

Taking into account these relations, one can easily show, that for sufficiently small positive ε we have

$$Q \geqslant \varepsilon\, |u)(u| \qquad \text{and} \qquad Q^\tau \geqslant \varepsilon\, |u)(u|$$

It means that $Q - \varepsilon|u)(u| \in V \cap V^\tau$. This fact contradicts to the extremality of Q unless Q is proportional to $|u)(u|$. Therefore extreme elements of $V \cap V^\tau$ belong to W and (2.6) follows.

This completes the proof of Prop. 1.1.

3. STATEMENT II FOR dim H = 3

Throughout this section H is a three dimensional Hilbert space. A relatively simple structure of positive maps from M_2 into B(H) discovered in Thm. 1.2 is based on the following :

Proposition 3.1

Let $Q \in B(H^2)$, $Q \neq 0$, $Q \geqslant 0$ and $Q^\tau \geqslant 0$; Then there exists a simple vector $u \in H^2$ such that $u \neq 0$, $u \in Q(H^2)$ and $u^\tau \in Q^\tau(H^2)$.

Present section is devoted to the proof of this proposition We start with the following special case.

Lemma 3.1

Let

$$Q_1 = \begin{pmatrix} I, & B \\ B^*, & C \end{pmatrix} \tag{3.1}$$

where $B, C \in B(H)$; I is the unity of B(H). Assume that

$$Q_1 \geqslant 0 \tag{3.2}$$

and

$$Q_1^\tau \geqslant 0 \tag{3.3}$$

Then there exist a complex number $t \in \mathbb{C}$ and a vector $z \in H$ such that $z \neq 0$ and

$$\begin{pmatrix} z \\ \bar{t} z \end{pmatrix} \in Q_1(H^2) \tag{3.4}$$

$$\begin{pmatrix} z \\ t z \end{pmatrix} \in Q_1^\tau(H^2) \tag{3.5}$$

Proof : One can easily check that for any $x, y \in H$ we have

$$\left(\begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} I, & B \\ B^*, & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right) = \| x + By \|^2 + (y | C - B^* B | y)$$

It shows that (3.2) is equivalent to

$$C - B^* B \geqslant 0 \tag{3.6}$$

In the same way (3.3) is equivalent to

$$C - B B^* \geqslant 0 \tag{3.7}$$

Now, we shall show that (3.4) and (3.5) are implied by the following orthogonality relation :

$$z \perp H_t \tag{3.8}$$

where H_t is a subspace of H spanned by $(B - tI)\, Ker\, (C - B^* B)$ and $(B - tI)^*\, Ker\, (C - B B^*)$

Indeed (3.8) means that

$$z \perp (B - tI)\, Ker\, (C - B^* B) \tag{3.9}$$

and

$$z \perp (B - tI)^*\, Ker\, (C - B B^*) \tag{3.10}$$

(3.9) is equivalent to $(B - tI)^* z \perp Ker\, (C - B^* B)$. For self-adjoint operators, the orthogonal complement of the kernel coincides with the image. Therefore there exists $y \in H$ such that

$$(B - tI)^* z = (C - B^* B) y$$

Now using this relation one easily verifies that

$$\begin{pmatrix} z \\ \bar{t}z \end{pmatrix} = \begin{pmatrix} I, B \\ B^*, C \end{pmatrix} \begin{pmatrix} z + By \\ -y \end{pmatrix}$$

and (3.4) follows. In the same way, starting with (3.10) one proves (3.5).
To prove our lemma it is sufficient to show that there exists a complex number t such that

$$\dim H_t < 3 \tag{3.11}$$

Let us denote dimensions of $Ker(C - B^*B)$ and $Ker(C - BB^*)$ by n_+ and n_- respectively.
Assume for the moment that $n_+ = 3$. Then $C - B^*B = 0$, $Tr(C - BB^*) = Tr(C - B^*B) = 0$ and in virtue of (3.7) we have $C - BB^* = 0$ and $n_- = 3$. In the same way one show that $n_- = 3$ implies $n_+ = 3$.
Therefore

$$(n_+ = 3) \Longleftrightarrow (n_- = 3) \tag{3.12}$$

We shall consider the following cases.

I. $n_+ + n_- \leq 2$
It is obvious, that in this case $\dim H_t \leq 2$ for any $t \in \mathbb{C}$.

II. $n_+ + n_- = 3$
In virtue of (3.12) $n_+ \neq 3 \neq n_-$. To be concrete assume that $n_+ = 2$ and $n_- = 1$. Let (e, f) be a basis of $Ker(C - B^*B)$ and (g) be a basis of $Ker(C - BB^*)$. Then H_t is generated by vectors

$$\begin{aligned} e(t) &= (B - tI)e \\ f(t) &= (B - tI)f \\ g(t) &= (B - tI)^* g \end{aligned} \tag{3.13}$$

Let $w(t) = \det(e(t), f(t), g(t))$. Clearly $w(t)$ is a third order polynomial with respect to t and $\bar{t}$ with leading term of the form $-t^2\bar{t}$. By using the standard topological technics (the index [4] p.226) of point 0 with respect to the path $[0, 2\pi] \ni \varphi \to w(re^{i\varphi})$ is 0 for $r = 0$ and 1 for large r; therefore for some r, the path has to pass through 0) one may show that there exists t such that $w(t) = 0$. For such t vectors (3.13) are linearly dependent and (3.11) follows.

III. $n_+ + n_- = 4$, $B^*B \neq BB^*$
In this case (see (3.12)) $n_+ = n_- = 2$. The operators $C - B^*B$ and $C - BB^*$ are one dimensional :

$$C - B^*B = |x)(x| \tag{3.14}$$

$$C - BB^* = |y)(y| \tag{3.15}$$

where $x, y \in H$. Therefore

$$B^*B - BB^* = |y)(y| - |x)(x| \tag{3.16}$$

Computing traces of both sides of (3.16) we get $(x|x) = (y|y)$.
We may assume that x is not proportional to y. Indeed, otherwise $|x)(x| = |y)(y|$ and B would be a normal operator. That case will be considered latter.
Let us consider vectors x, y, Bx, B^*y. Since $\dim H = 3$, these vectors are linearly dependent :

$$\alpha B^*y + \beta y = \gamma Bx + \delta y \tag{3.17}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. One may assume that α, γ are real and non negative (possible phase factor can be always absorbed by x and y) and that $\alpha + \gamma > 0$ (otherwise $\alpha = \gamma = 0$ and x would be proportional to y).

In virtue of (3.16) (see Appendix) there exists an antiunitary involution J such that $JBJ = B^*$ and $Jx = y$. Applying J to the both sides of (3.17) we get

$$\alpha Bx + \bar{\beta} x = \gamma B^* y + \bar{\delta} y \tag{3.18}$$

Combining (3.17) and (3.18) we obtain

$$(B - \tau I)x = (B - \tau I)^* y \tag{3.19}$$

where $\tau = -\dfrac{\bar{\beta} + \delta}{\alpha + \gamma}$.

Let $z \in H$; $t, s \in \mathbb{C}$. Taking into account (3.16), (3.19) and relation $(x|x) = (y|y)$, one can check by simple computation that

$$\|(B - tI)z + sy\|^2 + |(x|z) + (\tau - t)s|^2$$
$$= \|(B - tI)^* z + sx\|^2 + |(y|z) + (\bar{\tau} - \bar{t})s|^2 \tag{3.20}$$

Now let us consider a family of operators

$$D_t = (B - tI) + \frac{|y)(x|}{t - \tau}$$

where $t \in \mathbb{C}$ and $t \neq \tau$. Determinant $\det D_t$ is a rational function of t and tends to infinity as $t \to \infty$. Since any rational function defined on the compactified complex plane takes any complex value, one can find t such that $\det D_t = 0$. Then there exists a non-zero vector z such that

$$D_t z = 0$$

More explicitely

$$(B - tI)z + \frac{(x|z)}{t - \tau} y = 0 \tag{3.21}$$

It shows that $(B - tI)z$ is proportional to y. Therefore (cf.(3.15)) $(B - tI)z$ is orthogonal to $\mathrm{Ker}\,(C - BB^*)$ and consequently we get (3.10).
Setting in (3.20) $s = \dfrac{(x|z)}{t - \tau}$ and using (3.21) we see that $(B - tI)^* z$ is proportional to x. Therefore (cf.(3.14) $(B - tI)^* z$ is orthogonal to $\mathrm{Ker}\,(C - B^*B)$ and (3.9) follows. This completes the proof in case III.

Let us notice that $n_+ + n_- \geq 5$ iff $n_+ = n_- = 3$ (cf.(3.12). In that case $B^*B = C = BB^*$. Therefore, to end the proof of the lemma it is sufficient to consider the following case :

IV. B is a normal operator.

This is the simplest case. Let t be an eigenvalue of B and z be the corresponding eigenvector : $Bz = tz$ and $B^*z = \bar{t}z$
Then

$$\begin{pmatrix} z \\ \bar{t}z \end{pmatrix} = \begin{pmatrix} I, & B \\ B^*, & C \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z \\ tz \end{pmatrix} = \begin{pmatrix} I, & B^* \\ B, & C \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix}$$

Since we have considered all possible cases, the lemma is proved.

Now we are ready to prove Prop. 3.1. Assume that operator

$$Q = \begin{pmatrix} A, & B \\ B^*, & C \end{pmatrix}$$

fulfills the assumptions of Prop. 3.1. Clearly A is a positive operator. We shall consider two cases.

I. A is invertible.

This case can be easily reduced to that in Lemma 3.1. Let

$$Q_1 = \begin{pmatrix} A^{-1/2}, & 0 \\ 0, & A^{-1/2} \end{pmatrix} Q \begin{pmatrix} A^{-1/2}, & 0 \\ 0, & A^{-1/2} \end{pmatrix}$$

Then Q_1 is of the form (3.1) and satisfies all assumptions of Lemma 3.1. Therefore there exists a simple vector $u_1 \in H^2$ such that $u_1 \neq 0$, $u_1 \in Q_1(H^2)$ and $u_1^\tau \in Q_1^\tau(H^2)$. Now, one can easily check that vector

$$u = \begin{pmatrix} A^{1/2}, 0 \\ 0, A^{1/2} \end{pmatrix} u_1$$

fulfills our requirements : u is simple, $u \neq 0$, $u \in Q(H^2)$ and $u^\tau \in Q^\tau(H^2)$.

II. A is not invertible

Let $H_0 = A(H)$. The projection onto $H_0^\perp$ will be denoted by P_1. We have

$$\begin{pmatrix} 0, P_1 B \\ B^* P_1, C \end{pmatrix} = \begin{pmatrix} P_1, 0 \\ 0, I \end{pmatrix} Q \begin{pmatrix} P_1, 0 \\ 0, I \end{pmatrix} \geqslant 0$$

$$\begin{pmatrix} 0, P_1 B^* \\ B P_1, C \end{pmatrix} = \begin{pmatrix} P_1, 0 \\ 0, I \end{pmatrix} Q^\tau \begin{pmatrix} P_1, 0 \\ 0, I \end{pmatrix} \geqslant 0$$

It follows immediately that

$$P_1 B = B^* P_1 = P_1 B^* = B P_1 = 0 \tag{3.22}$$

Assume that $C P_1 \neq 0$. Then there exists a vector $x \perp H_0$ such that $Cx \neq 0$. Taking into account (3.22) we have

$$\begin{pmatrix} 0 \\ Cx \end{pmatrix} = Q \begin{pmatrix} 0 \\ x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ Cx \end{pmatrix} = Q^\tau \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

and vector $u = u^\tau = \begin{pmatrix} 0 \\ Cx \end{pmatrix}$ satisfies our requirements. Therefore we may assume that

$$C P_1 = P_1 C = 0 \tag{3.23}$$

In breif, relations (3.22) and (3.23) mean that all our operators act in fact in H_0. In that case Prop. 3.1 follows directly from Thm. 1.1 (note that $\dim H_0 < 3$).

<u>Remark</u> :

Methods presented in this section can be used to a direct proof of Thm. 1.1.

4. COUNTEREXAMPLE

In this section, we present a counterexample showing that, in the case $\dim H = 4$, statement II of section 1 is not valid.

Throughout this section $\dim H = 4$. Elements of $B(H)$ will be represented by 4x4 complex matrices.

Let

$$Q = \begin{pmatrix} I, & B \\ B^*, & C \end{pmatrix} \qquad \text{and} \qquad Q^\tau = \begin{pmatrix} I, & B^* \\ B, & C \end{pmatrix}.$$

where

$$B = \begin{pmatrix} 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \\ 0, & 0, & 0, & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} \frac{25}{9}, & 0, & 0, & \frac{20}{9} \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ \frac{20}{9}, & 0, & 0, & \frac{25}{9} \end{pmatrix}$$

With the help of (3.6) and (3.7) one easily checks that Q and Q^τ are positive. We claim that 0 is the only simple vector $u \in H^2$ such that $u \in Q(H^2)$ and $u^\tau \in Q^\tau(H^2)$. This fact is very simple, the proof however needs a lot of computations of arithmetic nature (one works with 8x8 matrices!). In order not to bore the reader we indicate only main stages of these computations.

1° Any vector in $Q(H^2)$ is of the form

$$u = (\alpha, \beta, \gamma, \delta, 5\varepsilon, \alpha, \beta, \gamma + 4\varepsilon) \tag{4.1}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{C}$ (to save the paper we adopt "horizontal notation" for vectors in $H^2 = \mathbb{C}^8$).

2° Vector (4.1) is simple iff either

$$(5\varepsilon, \alpha, \beta, \gamma + 4\varepsilon) = t(\alpha, \beta, \gamma, \delta)$$

where $t \in \mathbb{C}$ or

$$(\alpha, \beta, \gamma, \delta) = 0$$

In the first case we have

$$u = \gamma\left(t^2, t, 1, t^{-1} + \tfrac{4}{5}t^2, t^3, t^2, t, 1 + \tfrac{4}{5}t^3\right) \tag{4.2}$$

for $t \neq 0$ and

$$u = (0, 0, 0, \delta, 0, 0, 0, 0) \tag{4.3}$$

for $t = 0$; in the second case

$$u = (0, 0, 0, 0, 5\varepsilon, 0, 0, 4\varepsilon) \tag{4.4}$$

3° The "transposes" of vectors (4.2), (4.3) and (4.4) are equal to

$$u^\tau = \gamma\left(t^2, t, 1, t^{-1} + \tfrac{4}{5}t^2, t^2\bar{t}, t\bar{t}, \bar{t}, t^{-1}\bar{t} + \tfrac{4}{5}t^2\bar{t}\right) \tag{4.5}$$

$$u^\tau = (0, 0, 0, \delta, 0, 0, 0, 0) \tag{4.6}$$

$$u^\tau = (0, 0, 0, 0, 5\varepsilon, 0, 0, 4\varepsilon) \tag{4.7}$$

respectively.

4° Any vector in $Q^\tau(H^2)$ is of the form

$$v = (\alpha', \beta', \gamma', \delta', \beta' + 4\varepsilon', \gamma', \delta', 5\varepsilon') \tag{4.8}$$

where $\alpha', \beta', \gamma', \delta', \varepsilon' \in \mathbb{C}$

5° Assume that one of vectors (4.5), (4.6), (4.7) is of the form (4.8). Then we get easily $u^\tau = 0$ and $u = 0$.

5. REMARKS ON THE KADISON INEQUALITY

Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras and let

$$\Phi : \mathcal{A} \longrightarrow \mathcal{B}$$

be a normalized (i.e. $\Phi(1) = 1$) positive map. The famous Kadison inequality says that for any self-adjoint element $a \in \mathcal{A}$

$$\Phi(a^2) \geqslant \Phi(a)^2 \tag{5.1}$$

For 2-positive normalized maps we have a stronger version of the Kadison inequality [1]

$$\Phi(a^* a) \geqslant \Phi(a)^* \Phi(a) \tag{5.2}$$

for any $a \in \mathcal{A}$.
For normalized, 2-copositive maps instead of (5.2) we have

$$\Phi(a^* a) \geqslant \Phi(a) \Phi(a)^* \tag{5.3}$$

One can easily invent an inequality which is stronger than (5.1) and weaker than (5.2) and (5.3).

Definition :
We say that Φ satisfies the strong Kadison inequality if for any $b, c \in \mathcal{A}$ such that $c \geqslant b^* b$ and $c \geqslant b b^*$ we have :

$$\Phi(c) \geqslant \Phi(b)^* \Phi(b) \tag{5.4}$$

and

$$\Phi(c) \geqslant \Phi(b) \Phi(b)^* \tag{5.5}$$

In our opinion, the following conjecture is very reasonable.

Conjecture :
Any normalised positive map satisfies the strong Kadison inequality.

We shall prove this conjecture in some special cases.

Theorem 5.1
Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a normalised positive map. Assume that $\Phi = \Phi_1 + \Phi_2$, where Φ_1 is 2-positive and Φ_2 is 2-copositive. Then Φ satisfies the strong Kadison inequality.

Proof : Let us remind that Φ_1 is 2-positive iff

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \geqslant 0 \quad \Longrightarrow \quad \begin{pmatrix} \Phi_1(a), & \Phi_1(b) \\ \Phi_1(c), & \Phi_1(d) \end{pmatrix} \geqslant 0$$

for all $a, b, c, d \in \mathcal{A}$. Φ_2 is 2-copositive iff

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \geqslant 0 \quad \Longrightarrow \quad \begin{pmatrix} \Phi_2(a), & \Phi_2(c) \\ \Phi_2(b), & \Phi_2(d) \end{pmatrix} \geqslant 0$$

Let b, c be elements of $\mathcal{A}$ such that $c \geqslant b^* b$ and $c \geqslant b b^*$. Then (cf.(3.6) and (3.7))

$$\begin{pmatrix} 1, & b \\ b^*, & c \end{pmatrix} \geqslant 0 \quad \text{and} \quad \begin{pmatrix} 1, & b^* \\ b, & c \end{pmatrix} \geqslant 0 \tag{5.6}$$

Since Φ_1 is 2-positive, we have

$$\begin{pmatrix} \Phi_1(1), & \Phi_1(b) \\ \Phi_1(b)^*, & \Phi_1(c) \end{pmatrix} \geqslant 0 \quad \text{and} \quad \begin{pmatrix} \Phi_1(1), & \Phi_1(b^*) \\ \Phi_1(b), & \Phi_1(c) \end{pmatrix} \geqslant 0$$

Since Φ_2 is 2-copositive, we have

$$\begin{pmatrix} \Phi_2(1), & \Phi_2(b)^* \\ \Phi_2(b), & \Phi_2(c) \end{pmatrix} \geqslant 0 \quad \text{and} \quad \begin{pmatrix} \Phi_2(1), & \Phi_2(b) \\ \Phi_2(b)^*, & \Phi_2(c) \end{pmatrix} \geqslant 0$$

Combining these relations in a proper way and remembering that Φ is normalized, we get

$$\begin{pmatrix} 1, & \Phi(b) \\ \Phi(b)^*, & \Phi(c) \end{pmatrix} \geqslant 0 \quad \text{and} \quad \begin{pmatrix} 1, & \Phi(b)^* \\ \Phi(b), & \Phi(c) \end{pmatrix} \geqslant 0$$

and (5.4) and (5.5) follow immediately.

Theorem 5.2

Let $\mathcal{A} = B(H)$, where $\dim H \leq 3$. Then any normalized positive map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies the strong Kadison inequality.

Proof : Let $b, c \in \mathcal{A}$. Assume that $c \geqslant b^* b$ and $c \geqslant b b^*$. We introduce an operator Q acting in H^2 :

$$Q = \begin{pmatrix} 1, & b \\ b^*, & c \end{pmatrix}$$

Then (cf.(5.6))

$$Q \geqslant 0 \quad \text{and} \quad Q^{\tau} \geqslant 0$$

Now, we use relation (2.6) proven for $\dim H \leq 3$ (cf. Thm. 1.1 and Prop. 3.1). It means that

$$Q = \sum_{n=1}^{N} |u_n)(u_n| \tag{5.7}$$

where u_n are simple vectors

$$u_n = \begin{pmatrix} \lambda_n^1 x_n \\ \lambda_n^2 x_n \end{pmatrix}$$

In obvious notation, (5.7) is equivalent to

$$\begin{pmatrix} 1, & b \\ b^*, & c \end{pmatrix} = \sum_{n=1}^{N} \begin{pmatrix} \lambda_n^1 \overline{\lambda_n^1} |x_n)(x_n| , & \lambda_n^1 \lambda_n^2 |x_n)(x_n| \\ \lambda_n^2 \overline{\lambda_n^1} |x_n)(x_n| , & \lambda_n^2 \overline{\lambda_n^2} |x_n)(x_n| \end{pmatrix}$$

$$= \sum_{n=1}^{N} |x_n)(x_n| \otimes \begin{pmatrix} \lambda_n^1 \overline{\lambda_n^1} , & \lambda_n^1 \overline{\lambda_n^2} \\ \lambda_n^2 \overline{\lambda_n^1} , & \lambda_n^2 \overline{\lambda_n^2} \end{pmatrix}$$

Applying to the both sides the mapping $\Phi \otimes id$ (where id is the identity map in M_2) we get

$$\begin{pmatrix} 1, & \Phi(b) \\ \Phi(b)^*, & \Phi(c) \end{pmatrix} = \sum_{n=1}^{N} \Phi(|x_n)(x_n|) \otimes \begin{pmatrix} \lambda_n^1 \overline{\lambda_n^1} , & \lambda_n^1 \overline{\lambda_n^2} \\ \lambda_n^2 \overline{\lambda_n^1} , & \lambda_n^2 \overline{\lambda_n^2} \end{pmatrix}$$

Since tensor product of positive elements is positive, all summands on R.H.S. are positive and we obtain

$$\begin{pmatrix} 1, & \Phi(b) \\ \Phi(b)^*, & \Phi(c) \end{pmatrix} \geqslant 0$$

(5.4) follows immediately. Replacing b by b^* we get (5.5).

Remark I

Thm. 5.2 does not follow directly from Thm. 5.1. One can show (see[12]) that there exists a positive map acting from M_2 which can not be written as a sum of 2-positive and 2-copositive maps.

Remark II

Inequality (5.1) can be proved in the similar way. One has to use the following

Theorem 5.3

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{A}_+$ denote the cone of positive elements of $\mathcal{A}$. Then the convex cone in $\mathcal{A} \otimes M_2$ generated by all $a \otimes \alpha$, where $a \in \mathcal{A}_+$ and α runs over all real positive matrices, coincides with

$$\left\{ \begin{pmatrix} A, B \\ B, C \end{pmatrix} \in \mathcal{A} \otimes M_2 \ : \ \begin{pmatrix} A, B \\ B, C \end{pmatrix} \geqslant 0 \right\}$$

This theorem in turn follows from the known results concerning positive maps acting from real 2x2 matrices ([2]).

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APPENDIX : ALMOST NORMAL OPERATORS

By using the spectral theorem one can easily show that any normal operator B is anti isomorphic to its adjoint B^* i.e. that

$$B^* = JBJ$$

where J is an antiunitary involution. It turns out that the same result holds for "almost normal" operators B such that $[B^*, B]$ is two dimensional.

Theorem A

Let B be an operator acting in a finite dimensional Hilbert space H. Assume that

$$B^*B - BB^* = |y)(y| - |x)(x| \tag{A.1}$$

Then there exists an antiunitary involution J acting in H such that

$$Jx = y \tag{A.2}$$

$$JBJ = B^* \tag{A.3}$$

Proof : We shall constantly meet expressions containing products of many operators B and B^* taken in different order acting on vectors x and y . In order to deal with these expressions we have to introduce a convenient notation.

In what follows, variables denoted by capital letter B furnished with different subscripts run over the two element set $\{B, B^*\}$:

$$B_1, B_2, \ldots, B_1', B_2', \ldots \ \in \{B, B^*\}$$

Similarly variables denoted by small letter z (also with different subscripts) run over the two element set $\{x, y\}$. Moreover z^* denotes x (resp. y) whenever $z = y$ (resp. $z = x$)

For any $t \in \mathbb{R}$ we put

$$A(t) = B + tB^*$$

By direct computation one checks that

$$\frac{1}{t}\left(A(t)^* A(t) - A(t)A(t)^*\right) = |y)(y| - |x)(x|$$

We shall prove that

$$(x|A(t)^n x) = (y|A(t)^n y) \tag{A.4}$$

for any non-negative integer n. Indeed

$$(y|A(t)^n y) - (x|A(t)^n x) = \mathrm{Tr}\ A(t)^n \{|y)(y| - |x)(x|\}$$

$$= \frac{1}{t}\ \mathrm{Tr}\ A(t)^n \{A(t)^* A(t) - A(t)A(t)^*\} = 0$$

The last equality follows from the well known property of the trace : $\mathrm{Tr}\ C\ D = \mathrm{Tr}\ D\ C$.

Equation (A.4) can be rewritten in a more sophisticated way :

$$(z|A(t)^n z_1) = (z_1^*|A(t)^n z^*) \tag{A.5}$$

In fact (A.5) reduces to (A.4) if $z = z_1$. If $z \neq z_1$, then $z = z_1^*$, $z_1 = z^*$ and (A.5) is evidently satisfied.

Both sides of (A.5) are polynomials of order n with respect to t. Comparing corresponding coefficients we get :

$$\sum_{\sigma \in \Delta_{nk}} (z|B_{\sigma,1} B_{\sigma,2} \cdots B_{\sigma,n} z_1) = \sum_{\sigma \in \Delta_{nk}} (z_1^*|B_{\sigma,n} \cdots B_{\sigma,2} B_{\sigma,1} z^*) \tag{A.6}$$

where σ runs over the set Δ_{nk} of all k-element subsets of $\{1, 2, \ldots, n\}$ and

$$B_{\sigma,r} = \begin{cases} B & \text{if} \quad r \in \sigma \\ B^* & \text{if} \quad r \notin \sigma \end{cases}$$

Equation (A.6) will be used in the following form :

$$\sum_{\sigma \in \Delta_{nk}} \left\{(z|B_{\sigma,1} \cdots B_{\sigma,n} z_1) - (z_1^*|B_{\sigma,n} \cdots B_{\sigma,1} z^*)\right\} = 0 \tag{A.7}$$

As we shall see latter, the existence of the antiunitary involution J satisfying (A.2) and (A.3) is equivalent to the following equality

$$(z|B_1 B_2 \cdots B_m z_1) = (z_1^*|B_m \cdots B_2 B_1 z^*) \tag{A.8}$$

For $m = 0$ this relation has been verified (put $n = 0$ in (A.5)). Assume now, that (A.8) holds for all $m \leq n-1$. Then using (A.1) we have

$$(z|B_1 \cdots B_k B^* B B_{k+3} \cdots B_n z_1) - (z|B_1 \cdots B_k B B^* B_{k+3} \cdots B_n z_1) =$$

$$= (z|B_1 \cdots B_k y)(y|B_{k+3} \cdots B_n z_1) - (z|B_1 \cdots B_k x)(x|B_{k+3} \cdots B_n z_1) =$$

$$= (x|B_k \cdots B_1 z^*)(z_1^*|B_n \cdots B_{k+3} x) - (y|B_k \cdots B_1 z^*)(z_1^*|B_n \cdots B_{k+3} y) =$$

$$= (z_1^*|B_n \cdots B_{k+3} x)(x|B_k \cdots B_1 z^*) - (z_1^*|B_n \cdots B_{k+3} y)(y|B_k \cdots B_1 z^*) =$$

$$= (z_1^*|B_n \cdots B_{k+3} B B^* B_k \cdots B_1 z^*) - (z_1^*|B_n \cdots B_{k+3} B^* B B_k \cdots B_1 z^*).$$

Therefore

$$\left(z\middle|B_1 \dots B_k B^* B B_{k+1} \dots B_n z_1\right) - \left(z_1^*\middle|B_n \dots B_{k+1} B B^* B_k \dots B_1 z^*\right) =$$

$$= \left(z\middle|B_2 \dots B_k B B^* B_{k+1} \dots B_n z_1\right) - \left(z_1^*\middle|B_n \dots B_{k+1} B^* B B_k \dots B_1 z^*\right).$$

This result shows that the difference

$$\left(z\middle|B_1 \dots B_n z_1\right) - \left(z_1^*\middle|B_n \dots B_1 z^*\right) = \alpha(k)$$

is independent of the order of operators in the sequence $B_1, \dots, B_n$. It depends only on z, z', n and total number k of entries of B in $B_1, B_2, \dots, B_n$. In particular all summands in (A.7) are equal to $\alpha(k)$. Therefore $\alpha(k) = 0$ and (A.8) is verified for $m = n$.

By the induction principle, (A.8) holds for all nonnegative integers m.

Let

$$u = B_1 B_2 \dots B_n z \qquad v = {'B_1} {'B_2} \dots {'B_m} {'z}$$

$$u^* = B_1^* B_2^* \dots B_n^* z^* \qquad v^* = {'B_1^*} {'B_2^*} \dots {'B_m^*} {'z^*}$$

Then, it follows immediately from (A.8) that

$$(u|v) = (v^*|u^*) \tag{A.9}$$

Let H_0 be a subspace of H generated by all vectors of the form $B_1 \dots B_n z$. Relation (A.9) shows that there exists an anti-unitary involution J_0 acting in H_0 such that

$$J_0 B_1 B_2 \dots B_n z = B_1^* B_2^* \dots B_n^* z^* \tag{A.10}$$

If $H_0 = H$, then this involution solves our problem : relations (A.2) and (A.3) follow directly from (A.10). In the general case

$$H = H_0 \oplus H_1$$

$$B = C_0 \oplus C_1$$

where H_1 is the orthogonal complement of H_0, C_0 is the restriction of B to H_0 and C_1 is the similar restriction to H_1. Since $x, y \in H_0$, the operator C_1 is normal. Making use of the spectral theorem one can find an antiunitary involution J_1 acting in H_1 such that $J_1 C_1 J_1 = C_1^*$. Then the involution

$$J = J_0 \oplus J_1$$

satisfies (A.2) and (A.3).

<u>Remark</u>

The theorem remains valid in the case $\dim H = \infty$, if one assume that B is a Hilbert Schmidt operator. The same proof applies.

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