HAAR WEIGHT ON SOME QUANTUM GROUPS

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Abstract. We present a number of examples of locally compact quantum groups. These are quantum deformations of the group of affine transformations $\mathbb{R}$ ('$ax+b$' group) and $\mathbb{C}$ (Gz group). Starting from a modular multiplicative unitary W we find (under certain technical assumption) a simple formula expressing the (right) Haar weight on the quantum group associated with W. The formula works for quantum '$ax+b$' and '$az+b$' groups.

0. Introduction

It is difficult to overestimate the role of multiplicative unitaries in the present theory of locally compact quantum groups. The concept introduced by Baaj and Skandalis [1] is present in purely theoretical considerations in axiomatic formulation of the theory [2, 3, 4]. It is also very useful, when one considers particular examples of quantum groups (cf. examples presented is Section 3). Usually having the Haar weight h on a quantum group $G = (A, \Delta)$ one uses the GNS-construction to define a Hilbert space H and an embedding $A \subset B(H)$. Then the multiplicative unitary W is defined by taking the linear mapping:

$$A \otimes_{\mathrm{alg}} A \ni a \otimes b \longmapsto \Delta(a)(I \otimes b) \in A \otimes A$$

and pushing it down to the level of $H \otimes H$. This is so called Kac-Takesaki operator [4]. On the other hand in a number of cases, a multiplicative unitary is found independently and the general theory [1, 9, 5] is used to construct the corresponding locally compact quantum group. This was the way used in recent works [11, 12, 6] devoted to quantum deformations of the groups of affine transformations of $\mathbb{R}$ and $\mathbb{C}$. In general the multiplicative unitary that we start with need not coincide with the Kac-Takesaki operator and the problem of existence of the Haar weight must be investigated.

To pass from a multiplicative unitary W to the corresponding quantum group one has to assume that W have certain properties. For example it is sufficient to assume manageability [9] or at least modularity [5].

The Haar weight on the quantum '$ax + b$' and '$az + b$' groups was found by Van Daele some time ago [8]. To this end he had to use very particular properties of the groups. In autumn 2002, I played with the quantum '$az + b$' group acting on the straight line girandole Γ (see formula (3.2) below). The aim was to show that a certain measure on Γ is relatively invariant. The problem reduced to a complicated formula relating the Fourier transform of a special function appearing in the theory with the holomorphic continuation of the function itself. To my surprise that was the same formula as the one used earlier to prove the modularity of the multiplicative unitary staying behind the quantum '$az + b$' group. It was a strong indication that the (relative) invariance of some measure on a homogenous space is closely related to the modularity of the corresponding multiplicative unitary.

Trying to explain this phenomenon when the homogenous space is the group itself (the case of proper homogenous spaces will be considered in a separate paper). I arrived to a simple formula describing a Haar weight on a quantum group coming from a modular multiplicative unitary. It is a special case of a formula found earier by Van Daele [8, 7]. The formula works if a certain technical assumption is satisfied. As a rule this is not the case for manageable multiplicative unitaries. Therefore the modularity introduced in [5] seems to be of real importance.

We shortly describe the content of the paper. In section 1 we recall the basic concepts: multiplicative unitaries, modularity and manageability and the passage from modular multiplicative unitaires to quantum groups. Next we explain what the Haar weight is. Then after a short discussion of the trace and related weights we formulate our main theorem. It contains an explicit formula introducing a faithful lower semicontinuous weight h on a quantum group of the kind mentioned above. If h is locally finite then according to the theorem, h is the Haar weight. Section

2 contains the proof of the formula. It is very straightforward and refers directly to the concept of the right invariance.

The examples are presented in Section 3. These are quantum '$ax+b$' and '$az+b$' groups with different values of the deformation parameter q. Only the result concerning '$az + b$' group with $\Im q \neq 0$ and $|q| \neq 1$ is new (the others were obtained earlier by Van Daele). The main concern was to show that in considered cases our technical assumption (local finiteness) is satisfied. Nevertheless we provided all the necessary information to have a complete description of the considered groups. In particular the modular multiplicative unitaries are presented in full detail including the definitions of the special functions used.

The paper uses the advanced concepts of the theory of C*-algebras and locally compact quantum groups in a very limited way. In the case of any problem concerning the notation we refer to our earlier papers.

1. Basic concepts and the main theorem

Let H be a separable Hilbert space. A unitary operator W acting on $H \otimes H$ is by definition a multiplicative unitary if (using the leg numbering notation) we have:

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

This is the famous pentagon equation of Baaj and Skandalis. An important property of multiplicative unitaries is *the modularity* introduced in [5]. We say a multiplicative unitary W acting on $H \otimes H$ is modular if there exist two strictly positive selfadjoint operators $\widehat{Q}$ and Q on H and a unitary $\widetilde{W} \in B(\overline{H} \otimes H)$ such that

$$W^*(\widehat{Q} \otimes Q)W = \widehat{Q} \otimes Q \qquad \text{and} \tag{1.1}$$

$$(x \otimes y|W|z \otimes u) = \left(\overline{z} \otimes Qy\middle|\widetilde{W}\middle|\overline{x} \otimes Q^{-1}u\right) \tag{1.2}$$

for all $x, z \in H$, $y \in D(Q)$ and $u \in D(Q^{-1})$. In this definition $\overline{H}$ denotes the Hilbert space complex conjugate to H and $H \ni z \mapsto \overline{z} \in \overline{H}$ is the canonical antiunitary identification.

Demanding in addition the equality $\widehat{Q} = Q$ we obtain a stronger condition called *the manageability*. The theory of manageable multiplicative unitaries is developed in [9]. It is shown there that they give rise to objects that in many respects behave like locally compact topological groups. For the purpose of this paper these objects are called (locally compact) quantum groups. In [5] we have shown that all the essential results of the theory of manageable multiplicative unitaries hold for modular ones.

For any $x, y \in H$, ω_{xy} will denote the linear functional on $B(H)$ defined by the formula:

$$\omega_{xy}(a) = (y|a|x)$$

for any $a \in B(H)$. Functionals ω_{xy} are normal. Moreover the set $B(H)_*$ of all normal functionals on $B(H)$ coincides with the norm closed linear span of all functionals of the form ω_{xy}, where $x, y \in H$. If $y = x$ then we write ω_x instead of ω_{xx}. Clearly ω_x is a positive linear functional on $B(H)$.

We have to devote a few lines to the trace and related weights on $B(H)$. The functionals ω_x may by applied to unbounded positive selfadjoint operators: If $T = S^*S$ where S is a closed operator, then by definition

$$\omega_x(T) = \begin{cases} \|Sx\|^2 & \text{if } x \in \mathcal{D}(S) \\ \infty & \text{otherwise} \end{cases}$$

One can easily show that the right hand side is independent of the choice of S in the formula $T = S^*S$.

Let S be a closed operator on H and $c \in B(H)$. We choose two orthonormal basis $e = (e_n)_{n=1,2,\dots}$ and $f = (f_m)_{m=1,2,\dots}$ on H. Assume that $e_n \in \mathcal{D}(S)$ for all $n = 1, 2, \dots$. Then

$$\sum_n \omega_{S^*e_n}(c^*c) = \sum_m \omega_{c^*f_m}(S^*S). \tag{1.3}$$

Indeed both sides equal to $\sum_{nm} |(S^*e_n|c^*f_m)|^2$. The reader should notice that the left hand side of the above formula is independent of the basis f whereas the right hand side does not depend on e. Consequently both sides are independent of the choice of the bases. With a certain abuse of notation, for any positive $a \in B(H)$ we write

$$\sum_n \omega_{S^* e_n}(a) = \mathrm{Tr}(SaS^*).$$

Clearly the mapping $B(H)^+ \ni a \mapsto \mathrm{Tr}(SaS^*) \in \mathbb{R}_+ \cup \{+\infty\}$ is a normal semifinite weight on $B(H)$. In what follows, $S = \widehat{Q}$ will be strictly positive selfadjoint. One can easily show that $\mathrm{Tr}(\widehat{Q}c^*c\widehat{Q}) < \infty$ implies that the range of c^* is contained in the domain of $\widehat{Q}$: $c^*H \subset \mathcal{D}(\widehat{Q})$.

The way from a modular multiplicative unitary W to the corresponding quantum group $G = (A, \Delta)$ is short:

$$A = \{(\omega \otimes \mathrm{id})W : \omega \in B(H)_*\}^{\text{norm closure}} \tag{1.4}$$

and for any $a \in A$

$$\Delta(a) = W(a \otimes I)W^*. \tag{1.5}$$

It is known that A is a C*-algebra and that $\Delta \in \mathrm{Mor}(A, A \otimes A)$. Moreover $\Delta(a)(I \otimes b) \in A \otimes A$ for any $a, b \in A$. Using this fact one can show that for any $a \in A$ and any $\varphi \in A'$ (A' denotes the dual of A), the convolution product

$$\varphi * a = (\mathrm{id} \otimes \varphi)\Delta(a)$$

belongs to A. If a and φ are positive then $\varphi * a$ is positive.

The (right) Haar weight on the quantum group $G = (A, \Delta)$ is by definition (see e.g. [4]), a lower semicontinuous faithful locally finite weight h on A such that

$$h(\varphi * a) = \varphi(I)\, h(a) \tag{1.6}$$

for any positive functional φ on A and any positive $a \in A$ such that $h(a) < \infty$. We recall that h is locally finite if the set $\{c \in A : h(c^*c) < \infty\}$ is dense in A. This concept is the C*-version of the semifiniteness used for normal weights on von Neumann algebras.

The following theorem is a simplified version of a result of A. Van Daele [8, 7]:

Theorem 1.1. *Let $G = (A, \Delta)$ be the quantum group associated with a modular multiplicative unitary W and Q and $\widehat{Q}$ be strictly positive selfadjoint operators entering formulae* (1.2) *and* (1.1)*. For any positive $a \in A$ we put*

$$h(a) = \mathrm{Tr}(\widehat{Q}a\widehat{Q}). \tag{1.7}$$

Clearly h is faithful lower semicontinuous weight h on A. Assume that h is locally finite. Then h is a (*right*) *Haar weight on the quantum group G.*

2. The proof.

This section is devoted to the proof of theorem 1.1. We have to show (1.6). To this end we rewrite (1.2) in a form involving $\widehat{Q}$.

Lemma 2.1. *Let $y, u \in H$ and $x, z \in D(\widehat{Q})$. Then*

$$\left(\widehat{Q}x \otimes y \middle| W \middle| z \otimes u\right) = \left(\overline{\widehat{Q}z} \otimes y \middle| \widetilde{W} \middle| \overline{x} \otimes u\right). \tag{2.1}$$

Proof. Assume for the moment that $y \in \mathcal{D}(\widehat{Q}^{-1})$ and $u \in \mathcal{D}(\widehat{Q})$. Inserting in (1.2), $\widehat{Q}z$, $Q^{-1}y$ and Qu instead of z, y and u respectively we obtain

$$\left(x \otimes Q^{-1}y \middle| W \middle| \widehat{Q}z \otimes Qu\right) = \left(\overline{\widehat{Q}z} \otimes y \middle| \widetilde{W} \middle| \overline{x} \otimes u\right).$$

Now, taking into account (1.1) we get (2.1). Using a simple continuity argument one can extent the validity of (2.1) to all $y, u \in H$. □

The formula (2.1) is appropriate to prove the right-invariance (1.6) with $\varphi \in B(H)_*$. However to show (1.6) in full generality one has to strengthen the above lemma. The C*-algebra of all compact operators acting on H will be denoted by $\mathcal{K}(H)$. According to [5, Statement 2 of Theorem 2.3], the operator W belongs to the multiplier algebra of $\mathcal{K}(H) \otimes A$; $W \in M(\mathcal{K}(H) \otimes A)$. Therefore for any representation π of the C*-algebra A acting on a Hilbert space K, $\pi \in \mathrm{Mor}(A, \mathcal{K}(K))$ and

$(\mathrm{id}\otimes\pi)W \in M(\mathcal{K}(H)\otimes\mathcal{K}(K)) = M(\mathcal{K}(H\otimes K)) = B(H\otimes K)$. In other words $(\mathrm{id}\otimes\pi)W$ is a unitary operator acting on $H\otimes K$. It will be denoted by W_π:

$$W_\pi = (\mathrm{id}\otimes\pi)W.$$

We shall use the following

Lemma 2.2. *There exists a unitary operator $\widetilde{W}_\pi$ acting on $\overline{H}\otimes K$ such that*

$$\left(\widehat{Q}x\otimes y\middle|W_\pi\middle|z\otimes u\right) = \left(\overline{\widehat{Q}z}\otimes y\middle|\widetilde{W}_\pi\middle|\overline{x}\otimes u\right) \tag{2.2}$$

for any $y,u\in K$ and $x,z\in D(\widehat{Q})$.

Proof. Using [5, Statement 6.(ii) of Theorem 2.3] one can easily show that $\widetilde{W}\in M(\mathcal{K}(\overline{H})\otimes A)$. Therefore setting $\widetilde{W}_\pi = (\mathrm{id}\otimes\pi)\widetilde{W}$ we obtain a unitary operator acting on $\overline{H}\otimes K$. Formula (2.1) shows that

$$(\omega_{z,\widehat{Q}x}\otimes\mathrm{id})W = (\omega_{\overline{x},\overline{\widehat{Q}z}}\otimes\mathrm{id})\widetilde{W}.$$

Applying π to both sides we obtain

$$(\omega_{z,\widehat{Q}x}\otimes\mathrm{id})W_\pi = (\omega_{\overline{x},\overline{\widehat{Q}z}}\otimes\mathrm{id})\widetilde{W}_\pi.$$

Therefore

$$(\omega_{z,\widehat{Q}x}\otimes\omega_{uy})W_\pi = (\omega_{\overline{x},\overline{\widehat{Q}z}}\otimes\omega_{uy})\widetilde{W}_\pi.$$

The last formula clearly coincides with (2.2). $\square$

Now the proof of (1.6) is a matter of simple computation. Let φ be a positive linear functional on A. Using the GNS construction we find a representation π of A acting on a Hilbert space K and a vector $x\in K$ such that

$$\varphi(a) = (x|\pi(a)|x) = \omega_x(\pi(a)),$$

for all $a\in A$. Then

$$\begin{aligned}\varphi * a &= (\mathrm{id}\otimes\omega_x)(\mathrm{id}\otimes\pi)\Delta(a)\\ &= (\mathrm{id}\otimes\omega_x)(\mathrm{id}\otimes\pi)(W(a\otimes I)W^*)\\ &= (\mathrm{id}\otimes\omega_x)(W_\pi(a\otimes I)W_\pi^*),\end{aligned}$$

Assume that a is positive. Then $a = c^*c$, where $c\in A$. Let $(e_n)_{n=1,2,\dots}$ be an orthonormal basis in H such that $e_n\in\mathcal{D}(\widehat{Q})$ for all $n=1,2,\dots$. Using (1.7) we obtain

$$\begin{aligned}h(\varphi * a) &= \sum_n(\omega_{\widehat{Q}e_n}\otimes\omega_x)(W_\pi(c^*c\otimes I)W_\pi^*)\\ &= \sum_n\left(\widehat{Q}e_n\otimes x\middle|W_\pi(c^*c\otimes I)W_\pi^*\middle|\widehat{Q}e_n\otimes x\right)\\ &= \sum_n\left\|(c\otimes I)W_\pi^*(\widehat{Q}e_n\otimes x)\right\|^2,\end{aligned}$$

To proceed with our computations we chose an orthonormal basis $(\epsilon_k)_{k=1,2,\dots}$ in K. Then

$$\begin{aligned}h(\varphi * a) &= \sum_{nmk}\left|\left(e_m\otimes\epsilon_k\middle|(c\otimes I)W_\pi^*\middle|\widehat{Q}e_n\otimes x\right)\right|^2\\ &= \sum_{nmk}\left|\left(\widehat{Q}e_n\otimes x\middle|W_\pi\middle|c^*e_m\otimes\epsilon_k\right)\right|^2.\end{aligned}$$

If $h(a)<\infty$ then $c^*e_m\in\mathcal{D}(\widehat{Q})$ for all $m=1,2,\dots$. Inserting in (2.2), e_n, x, c^*e_m and ϵ_k instead of x, y, z and u respectively we obtain

$$\left(\widehat{Q}e_n\otimes x\middle|W_\pi\middle|c^*e_m\otimes\epsilon_k\right) = \left(\overline{\widehat{Q}c^*e_m}\otimes x\middle|\widetilde{W}_\pi\middle|\overline{e_n}\otimes\epsilon_k\right).$$

Therefore

$$
\begin{aligned}
h(\varphi * a) &= \sum_{nmk} \left| \left(\overline{\widehat{Q}c^* e_m} \otimes x \middle| \widetilde{W_\pi} \middle| \overline{e_n} \otimes \epsilon_k \right) \right|^2 \\
&= \sum_{nmk} \left| \left(\overline{e_n} \otimes \epsilon_k \middle| \widetilde{W_\pi}^* \middle| \overline{\widehat{Q}c^* e_m} \otimes x \right) \right|^2 \\
&= \sum_m \left\| \widetilde{W_\pi}^* \left(\overline{\widehat{Q}c^* e_m} \otimes x \right) \right\|^2 = \sum_m \left\| \overline{\widehat{Q}c^* e_m} \otimes x \right\|^2 \\
&= \|x\|^2 \sum_m \left\| \widehat{Q}c^* e_m \right\|^2,
\end{aligned}
$$

where in the forth step we used the unitarity of $\widetilde{W_\pi}$. On the other hand $\|x\|^2 = \varphi(I)$ and by formula (1.3):

$$
\sum_m \left\| \widehat{Q}c^* e_m \right\|^2 = \sum_m \omega_{c^* e_m}(\widehat{Q}^2) = \sum_n \omega_{\widehat{Q}e_n}(a) = h(a).
$$

Inserting these data into our computations we obtain (1.6). The proof of Theorem 1.1 is complete.

3. Examples

In this section we shall show that Theorem 1.1 allows us to reproduce in a simple way the results of Van Daele concerning the Haar measures on quantum '$ax+b$' and '$az+b$' groups. We start with '$az+b$' groups [6, 11]. These groups are quantum deformations of the group of affine transformations of complex plane $\mathbb{C}$.

Let Γ be a selfdual subgroup of the multiplicative group of non-zero complex numbers and α be so called Fresnel function on Γ . The particular form of Γ and α depends on the value of the deformation parameter q. The following value of q were considered so for:

1. q is real and belongs to the open interval $]0,1[$. The quantum '$az+b$' group with this value of q is presented in [11, Appendix A]. In this case Γ is the system of concentric circles with radii forming the geometric progression:

$$
\Gamma = \left\{ q^{i\theta + k} : \theta \in \mathbb{R},\, k \in \mathbb{Z} \right\} \quad \text{and} \quad \alpha(q^{i\theta+k}) = q^{i\theta k}. \tag{3.1}
$$

We say that Γ is a *target plate*.

2. $q = e^{\frac{2\pi i}{N}}$ where N is an even integer strictly larger than 2. The quantum '$az+b$' group with this value of q is presented in [11]. In this case Γ is the system of straight lines dividing $\mathbb{C}$ into N equal sectors:

$$
\Gamma = \left\{ q^k r : k \in \mathbb{Z},\, r > 0 \right\} \quad \text{and} \quad \alpha(q^k r) = e^{\frac{\pi i}{N} k^2} e^{\frac{N}{4\pi i} (\log r)^2}. \tag{3.2}
$$

We say that Γ is a *straight line girandole*.

3. $q = e^{\frac{1}{\rho}}$ where $\rho = c + \frac{iN}{2\pi}$, $c < 0$, N is an even integer and $N \neq 0$. The quantum '$az+b$' group with this value of q is presented in [6]. In this case Γ is the system of N logarithmic spirals dividing $\mathbb{C}$ into equal sectors:

$$
\Gamma = \left\{ e^{\frac{k+it}{\rho}} : k \in \mathbb{Z},\, t \in \mathbb{R} \right\} \quad \text{and} \quad \alpha\left(e^{\frac{n+it}{\rho}} \right) = e^{i\Im \frac{(k+it)^2}{2\rho}}. \tag{3.3}
$$

We say that Γ is a *logarithmic girandole*.

In what follows we shall treat all three cases simultaneously. For any $\gamma, \gamma' \in \Gamma$ we set

$$
\chi(\gamma, \gamma') = \frac{\alpha(\gamma\gamma')}{\alpha(\gamma)\alpha(\gamma')}
$$

It is easy to see that χ is a bicharacter on $\Gamma \times \Gamma$.

Let μ be the Haar measure on Γ and $H = L^2(\Gamma, \mu)$ be the corresponding Hilbert space. We chose the normalization of μ in such a way that the Fourier transform

$$
(\mathcal{F}g)(\gamma) = \int_\Gamma \chi(\gamma, \gamma') g(\gamma') d\mu(\gamma')
$$

is a unitary operator acting on H. We shall use two operators a and b acting on H. Operator b is the multiplication operator:

$$
(bx)(\gamma) = \gamma x(\gamma).
$$

Clearly b is a normal (unbounded) operator acting on H with $\operatorname{Sp} b = \overline{\Gamma}$, where $\overline{\Gamma} = \Gamma \cup \{0\}$ is the closure of Γ. The second operator $a = \mathcal{F} b \mathcal{F}^*$. Then for any $\gamma' \in \Gamma$ we have:

$$(\chi(a, \gamma')x)(\gamma) = x(\gamma\gamma').$$

Operators a and b satisfy the following commutation relations:

$$\chi(a,\gamma)\chi(b,\gamma') = \chi(\gamma,\gamma')\chi(b,\gamma')\chi(a,\gamma) \tag{3.4}$$

for any $\gamma, \gamma' \in \Gamma$. In brief (see [6, 11] for details):

$$ab = q^2 ba, \qquad a^* b = b a^*.$$

The modular multiplicative unitary producing the quantum '$az + b$' group is given by

$$W = \mathbb{F}_q(b^{-1}a \otimes b)\chi(b^{-1} \otimes I, I \otimes a), \tag{3.5}$$

where $\mathbb{F}_q$ is a continuous function on $\overline{\Gamma}$ called *the quantum exponential function.* For $|q| < 1$ (cases 1 and 3) $\mathbb{F}_q$ is given by the formula:

$$\mathbb{F}_q(\gamma) = \prod_{k=0}^{\infty} \frac{1 + \overline{q^{2k}\gamma}}{1 + q^{2k}\gamma} \tag{3.6}$$

For $q = e^{\frac{2\pi i}{N}}$ (case 2), $\mathbb{F}_q(\gamma) = F_N(q^{-2}\gamma)$, where F_N is the function introduced in [11, formula 1.5]. We recall that

$$F_N(q^k r) = \begin{cases} \displaystyle\prod_{s=1}^{\frac{k}{2}} \left(\frac{1 + q^{2s} r}{1 + q^{-2s} r} \right) \frac{f_0(qr)}{1 + r} & \text{for } k \text{ - even} \\[2ex] \displaystyle\prod_{s=0}^{\frac{k-1}{2}} \left(\frac{1 + q^{2s+1} r}{1 + q^{-2s-1} r} \right) f_0(r) & \text{for } k \text{ - odd} \end{cases}$$

where

$$f_0(z) = \exp\left\{ \frac{1}{\pi i} \int_0^\infty \log(1 + a^{-\frac{N}{2}}) \, \frac{da}{a + z^{-1}} \right\}.$$

It is known that the operator W introduced by (3.5) is a modular multiplicative unitary with $Q = |a|$ and $\widehat{Q} = |b|$. The corresponding C*-algebra (1.4) is the crossed product algebra

$$A = \Big\{ g(a)f(b) : g \in C_\infty(\Gamma), f \in C_\infty(\overline{\Gamma}) \Big\}^{\text{norm closed linear envelope}}.$$

We recall that for any locally compact space Λ, $C_\infty(\Lambda)$ denotes the C*-algebra of all continuous functions vanishing at infinity on Λ. Consequently $C_\infty(\Gamma) = \big\{ g \in C_\infty(\overline{\Gamma}) : g(0) = 0 \big\}$.

For any positive $r \in A$ we set:

$$h(r) = \operatorname{Tr}(|b|\, r\, |b|). \tag{3.7}$$

Let $c = g(a)f(b)$. One can verify that the operator $c\,\widehat{Q} = c\,|b|$ is an integral operator:

$$(c\,|b|\,x)(\gamma') = \int_\Gamma K_c(\gamma', \gamma)\, x(\gamma) d\mu(\gamma)$$

with the kernel $K_c(\gamma', \gamma) = (\mathcal{F}g)(\gamma'\gamma^{-1}) f(\gamma)\, |\gamma|$. Therefore

$$\begin{aligned} h(c^*c) &= \int_{\Gamma\times\Gamma} |K_c(\gamma',\gamma)|^2 \, d\mu(\gamma')d\mu(\gamma) \\ &= \int_{\Gamma\times\Gamma} \left|(\mathcal{F}g)(\gamma'\gamma^{-1})\right|^2 |f(\gamma)|^2 \, |\gamma|^2 \, d\mu(\gamma')d\mu(\gamma) \\ &= \int_\Gamma |g(\gamma)|^2 \, d\mu(\gamma) \, \int_{\overline{\Gamma}} |f(\gamma)|^2 \, |\gamma|^2 \, d\mu(\gamma), \end{aligned} \tag{3.8}$$

where in last step we used the right invariance of the Haar measure μ and the unitarity of $\mathcal{F}$. The reader should notice that the measure $d\gamma = |\gamma|^2 \, d\mu(\gamma)$ is locally finite on $\overline{\Gamma}$. Therefore the intersection $L^2(\overline{\Gamma}, d\gamma) \cap C_\infty(\overline{\Gamma})$ is dense in $C_\infty(\overline{\Gamma})$. On the other hand, obviously, $L^2(\Gamma, \mu) \cap C_\infty(\Gamma)$ is dense in $C_\infty(\Gamma)$. Formula (3.8) shows now that $\{c \in A : h(c^*c) < \infty\}$ is dense in A and using Theorem 1.1 we get

Theorem 3.1. *Formula* (3.7) *defines the Haar weight on the quantum '$az + b$' group.*

Operator Q appearing in (1.2) is of great interest. It implements a one parameter group $(\tau_t)_{t\in\mathbb{R}}$ of *scaling automorphisms* of A: $\tau_t(a) = Q^{2it}aQ^{-2it}$. In our case $Q = |a|$ and $\widehat{Q} = |b|$. Using the commutation relations (3.4) one can show that $Q^{2it}\widehat{Q}Q^{-2it} = \left|q^{2it}\right|\widehat{Q}$. Using (1.7) we obtain the relative invariance of the Haar weight:

$$h\circ\tau_t = \left|q^{-4it}\right| h$$

for any $t \in \mathbb{R}$. If q is not real then the Haar weight is not invariant under scaling group. Instead we have the relative invariance (up to a positive factor). The relative invariance of the Haar weight appeared first in the paper of Kustermans and Vaes [2] as a consequence of their axioms. At that time all known quantum groups had the Haar weight invariant under scaling group. Alfons Van Daele was the first person, who noticed that this may not be true for quantum '$ax+b$' and '$az+b$' group. That was the main reason why he computed the Haar weights on these groups finding the first examples of the phenomenon foreseen by Kustermans and Vaes.

Remark 3.2. The infinite product (3.6) is not convergent for $q = e^{\frac{2\pi i}{N}}$ because the factors $\frac{1+\overline{q^{2k}\gamma}}{1+q^{2k}\gamma}$ as well as partial products $\prod_{\ell=0}^{k}\frac{1+\overline{q^{2\ell}\gamma}}{1+q^{2\ell}\gamma}$ are periodic in k. However if in this case for any real positive r, one put

$$\prod_{k=0}^{\infty}\frac{1+q^{-2k-1}r}{1+q^{2k+1}r} = f_0(r)$$

$$\prod_{k=0}^{\infty}\frac{1+q^{-2k}r}{1+q^{2k}r} = \frac{f_0(qr)}{1+r}$$

then the formula (3.6) works also for $q = e^{\frac{2\pi i}{N}}$.

The rest of the section is devoted to the quantum '$ax+b$' group [12]. This is a quantum deformation of the group of affine transformations of $\mathbb{R}$. It is constructed and investigated in [12]. For this group the deformation parameter $q^2 = e^{-i\hbar}$, where $\hbar = \frac{\pi}{2k+3}$ and $k = 0, 1, 2, \ldots$. Let $H = L^2(\mathbb{R})$. We consider three operators acting on H:

$$\begin{aligned}(s\,x)(t) &= i\left(t\tfrac{d}{dt}+\tfrac{1}{2}\right)x(t),\\ (b\,x)(t) &= \quad tx(t),\\ (\beta\,x)(t) &= \quad x(-t).\end{aligned}$$

Operators b, s and β are selfadjoint. Furthermore β is unitary. The operator a is related to s by the formula $a = e^{-\hbar s}$. Clearly a is selfadjoint and strictly positive. We have the formulae: $\beta a = a\beta$, $\beta b = -b\beta$ and $ab = q^2 ba$, where the last equality stands for Zakrzewski relation (cf. [10, Section 2]). To introduce the multiplicative unitary related to the quantum '$ax+b$' group we shall use the appropriate special function $F_\hbar$. This is a function of two variables: the first one runs over $\mathbb{R}$, whereas the second one equals ±1. It is given by the formula

$$F_\hbar(r,\varrho) = \begin{cases} V_\theta(\log r) & \text{for} \quad r>0 \\ \left[1+i\varrho|r|^{\frac{\pi}{\hbar}}\right]V_\theta\left(\log|r|-\pi i\right) & \text{for} \quad r<0. \end{cases}$$

where $\theta = \frac{2\pi}{\hbar}$ and V_θ is the meromorphic function on $\mathbb{C}$ such that

$$V_\theta(x) = \exp\left\{\frac{1}{2\pi i}\int_0^\infty \log(1+t^{-\theta})\,\frac{dt}{t+e^{-x}}\right\}$$

for all $x \in \mathbb{C}$ such that $|\Im x| < \pi$. With this notation

$$W = F_\hbar\left(e^{i\hbar/2}b^{-1}a\otimes b, (-1)^k\beta\otimes\beta\right)^*(|b|\otimes I)^{I\otimes is}.$$

It is known (cf. [5]) that W is a modular multiplicative unitary with $Q = a^{\frac{1}{2}}$ and $\widehat{Q} = |b|^{\frac{1}{2}}$. The corresponding C*-algebra (1.4) is given by

$$A = \left\{g(s)\left(f_1(b)+\beta f_2(b)\right) : g,\ f_1,\ f_2 \in C_\infty(\mathbb{R}),\ f_2(0)=0\right\}^{\text{norm closed linear envelope}}.$$

For any positive $r \in A$ we set:

$$h(r) = \operatorname{Tr}\left(|b|^{\frac{1}{2}}\, r\, |b|^{\frac{1}{2}}\right). \tag{3.9}$$

Let $c = g(s)\,(f_1(b) + \beta f_2(b))$. One can verify that the operator $c\widehat{Q} = c\,|b|^{\frac{1}{2}}$ is an integral operator:

$$(c\,|b|^{\frac{1}{2}}\,x)(t') = \int_{\mathbb{R}} K_c(t',t)\,x(t)dt$$

with the kernel

$$K_c(t',t) = \begin{cases} |t'|^{-\frac{1}{2}}\,\widetilde{g}(\frac{t'}{t})f_1(t) & \text{for } \frac{t'}{t} > 0, \\ |t'|^{-\frac{1}{2}}\,\widetilde{g}(-\frac{t'}{t})f_2(t) & \text{for } \frac{t'}{t} < 0, \end{cases}$$

where

$$\widetilde{g}(\varrho) = \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\varrho^{-i\tau}d\tau$$

for any $\varrho > 0$. Therefore

$$\begin{aligned} h(c^*c) &= \int_{\mathbb{R}\times\mathbb{R}} |K_c(t',t)|^2\; dt'dt \\ &= \int_0^\infty |\widetilde{g}(\varrho)|^2\,\frac{d\varrho}{\varrho}\int_{\mathbb{R}}\left(|f_1(t)|^2 + |f_2(t)|^2\right)dt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}|g(\tau)|^2\,d\tau\int_{\mathbb{R}}\left(|f_1(t)|^2 + |f_2(t)|^2\right)dt. \end{aligned}$$

This formula shows that $\{c \in A : h(c^*c) < \infty\}$ is dense in A and using Theorem 1.1 we get

Theorem 3.3. *Formula* (3.9) *defines the Haar weight on the quantum '$ax+b$' group.*

Also in this case the Haar weight is not invariant with respect to the scaling group. Remembering that $Q = a^{\frac{1}{2}}$ and $\widehat{Q} = |b|^{\frac{1}{2}}$ and using the Zakrzewski relation we obtain $Q^{2it}\widehat{Q}Q^{-2it} = e^{\frac{\hbar t}{2}}\widehat{Q}$ and

$$h\circ\tau_t = e^{-\hbar t}\,h$$

for any $t \in \mathbb{R}$.

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