Solutions of the braid equation related to a Hopf algebra

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Abstract

Two solutions T and T' of the braid equation acting on $\mathcal{A}\otimes\mathcal{A}$ (where $\mathcal{A}$ is a Hopf algebra) are described. If $\mathcal{A}$ is cocommutative then $T=\sigma$. If $\mathcal{A}$ is commutative then $T'=\sigma$ (σ denotes the flip: $\sigma(a\otimes b)=b\otimes a$ for any $a,b\in\mathcal{A}$).

Let $\mathcal{A}$ be a Hopf algebra [1]. The multiplication, unite, comultiplication, counite and antipode (coinverse) will be denoted by m, I, Δ, e and S respectively. We shall use the Sigma notation of Sweedler [1]:

$$\Delta(b)=\sum_{(b)} b^{(1)}\otimes b^{(2)}$$

$$(\mathrm{id}\otimes\Delta)\Delta(b)=(\Delta\otimes\mathrm{id})\Delta(b)=\sum_{(b)} b^{(1)}\otimes b^{(2)}\otimes b^{(3)}$$

Let T and T' be linear operators acting on $\mathcal{A}\otimes\mathcal{A}$ introduced by the formulae:

$$T(a\otimes b)=\sum_{(b)} b^{(2)}\otimes aS(b^{(1)})b^{(3)},$$

$$T'(a\otimes b)=\sum_{(b)} b^{(1)}\otimes S(b^{(2)})ab^{(3)},$$

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for any $a, b \in \mathcal{A}$. The mappings T and T' are invertible. One can easily check that

$$T^{-1}(a \otimes b) = \sum_{(a)} bS^{-1}(a^{(3)})a^{(1)} \otimes a^{(2)},$$

$$T'^{-1}(a \otimes b) = \sum_{(a)} a^{(3)}bS^{-1}(a^{(2)}) \otimes a^{(1)}$$

for any $a, b \in \mathcal{A}$. Moreover

$$T((a \otimes I)\Delta(b)) = (I \otimes a)\Delta(b)$$

for any $a, b \in \mathcal{A}$.

If $\mathcal{A}$ is cocommutative then $\sum_{(b)} b^{(1)} \otimes b^{(2)} = \sum_{(b)} b^{(2)} \otimes b^{(1)}$ and $T(a \otimes b) = \sum_{(b)} b^{(1)} \otimes aS(b^{(2)})b^{(3)} = b \otimes a$. Similarly if $\mathcal{A}$ is commutative then $T'(a \otimes b) = \sum_{(b)} b^{(1)} \otimes aS(b^{(2)})b^{(3)} = b \otimes a$.

Proposition 1 *Operators T and T' satisfy the braid equation:*

$$(T \otimes id)(id \otimes T)(T \otimes id) = (id \otimes T)(T \otimes id)(id \otimes T) \tag{1}$$
$$(T' \otimes id)(id \otimes T')(T' \otimes id) = (id \otimes T')(T' \otimes id)(id \otimes T') \tag{2}$$

Proof of (1):

For any $b \in \mathcal{A}$ we set:

$$\mathrm{ad}(b) = \sum_{(b)} b^{(2)} \otimes S(b^{(1)})b^{(3)}.$$

One can easily verify that

$$T(a \otimes b) = (I \otimes a)\mathrm{ad}(b) \tag{3}$$
$$(\mathrm{ad} \otimes \mathrm{id})\mathrm{ad}(b) = (\mathrm{id} \otimes \Delta)\mathrm{ad}(b) \tag{4}$$
$$T[(a \otimes b)\Delta(c)] = (I \otimes a)\mathrm{ad}(b)\Delta(c) \tag{5}$$

and (using(4))

$$(T \otimes \mathrm{id})(\mathrm{id} \otimes T)(q \otimes c) = (I \otimes q)(\mathrm{id} \otimes \Delta)\mathrm{ad}(c) \tag{6}$$

for any $a, b, c \in \mathcal{A}$ and $q \in \mathcal{A} \otimes \mathcal{A}$.

Let $a, b, c \in \mathcal{A}$. Using (6) and (5) we get:

$$\begin{aligned}(\mathrm{id} \otimes T)(T \otimes \mathrm{id})(\mathrm{id} \otimes T)(a \otimes b \otimes c) &= (\mathrm{id} \otimes T)[(I \otimes a \otimes b)(\mathrm{id} \otimes \Delta)\mathrm{ad}(c)] \\ &= (I \otimes I \otimes a)(I \otimes \mathrm{ad}(b))(\mathrm{id} \otimes \Delta)\mathrm{ad}(c).\end{aligned}$$

On the other hand using (3) and (6) we obtain

$$\begin{aligned}(T \otimes \mathrm{id})(\mathrm{id} \otimes T)(T \otimes \mathrm{id})(a \otimes b \otimes c) &= (T \otimes \mathrm{id})(\mathrm{id} \otimes T)\{[(I \otimes a)\mathrm{ad}(b)] \otimes c\} \\ &= (I \otimes I \otimes a)(I \otimes \mathrm{ad}(b))(\mathrm{id} \otimes \Delta)\mathrm{ad}(c).\end{aligned}$$

and (1) follows.

Q.E.D.

Proof of (2):

It is not difficult to write down the computational proof of (2). However it is more interesting to notice that (2) is a theorem of the Hopf algebra theory because of the duality principle.

It is well known that the set of axioms of the Hopf algebra theory remains unchanged when we replace $\Delta \mapsto m$, $m \mapsto \Delta$, $e \mapsto I$, $I \mapsto e$ and $S \mapsto S$, reversing in the same moment the order of compositions. Therefore the set of theorems is closed under this procedure. Applying this procedure to (1) we get (2) (one can easily see that $T \mapsto T'$).

Q.E.D.

References

[1] M.E. Sweedler: Hopf algebras, Benjamin, New York 1969