A remark on manageable multiplicative unitaries*

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Abstract

We propose a weaker condition for multiplicative unitary operators related to quantum groups, than the condition of manageability introduced by S.L. Woronowicz. We prove that all the main results of the theory of manageable multiplicative unitaries remain true under this weaker condition. We also show that multiplicative unitaries arising naturally in the construction of some recent examples of non-compact quantum groups satisfy our condition, but fail to be manageable.

1 Introduction

The theory of multiplicative unitary operators initiated by S. Baaj and G. Skandalis in [2] has played a central role in the modern approach to quantum groups. A unitary operator $W \in B(H \otimes H)$ is called multiplicative if it satisfies the pentagon equation (cf. [2]):

$$W_{23}W_{12} = W_{12}W_{13}W_{23}. \tag{1.1}$$

However this condition alone does not guarantee that W is a multiplicative unitary related to a quantum group. S. Baaj and G. Skandalis proposed a condition called regularity which unfortunately did not fit all applications (cf. [1], Proposition 4.2). In [5] the condition of regularity was replaced by another one called manageability. In [3] it is shown that all quantum groups possess a manageable multiplicative unitary which is called the Kac-Takesaki operator.

As one might expect the manageability condition is often difficult to check in particular examples. Moreover the natural choice for the multiplicative unitaries in specific examples like the quantum "$ax+b$" and "$az+b$" groups turns out not to be manageable (cf. [6], [7] and Section 5).

The aim of this paper is to weaken the manageability condition in such a way that it suits the above mentioned examples (cf. Section 5). The condition we propose is the following: let H be a Hilbert space and let $W \in B(H \otimes H)$ be a multiplicative unitary. We will suppose that there exist two positive selfadjoint operators $\widehat{Q}$ and Q on H with $\ker Q = \ker \widehat{Q} = \{0\}$ and a unitary $\widetilde{W} \in B(\overline{H} \otimes H)$ such that

$$W^*(\widehat{Q} \otimes Q)W = \widehat{Q} \otimes Q$$

*Partially supported by Komitet Badań Naukowych, grant No. 2 P0A3 030 14 and the Foundation for Polish Science

and

$$(x \otimes u|W|z \otimes y) = \left(\overline{z} \otimes Qu \middle| \widetilde{W} \middle| \overline{x} \otimes Q^{-1}y\right)$$

for all $x, z \in H$, $y \in D(Q^{-1})$ and $u \in D(Q)$. We hereby take opportunity to change the name "manageable" and we shall call a multiplicative unitary satisfying the above condition a modular multiplicative unitary. The modularity is reflected in the existence of the scaling group and the polar decomposition of the coinverse (cf. Theorem 2.3).

In case $\widehat{Q} = Q$ we retain manageability and in particular any manageable multiplicative unitary is modular. It turns out that all the results obtained in [5] are true with these weaker assumptions. One may try to adapt the proofs from [5] to this new situation, however we encountered some difficulties with this programme. Instead we will construct a new multiplicative unitary (on a different Hilbert space) which is manageable and describes the same quantum group.

In Section 3 we will use an auxiliary separable Hilbert space K and a pair (r, s) of closed operators acting on K such that s is selfadjoint, r is positive selfadjoint and

$$r^{it} s r^{-it} = s - tI$$

for all $t \in \mathbb{R}$. An example of such a pair (r, s) on $K = L^2(\mathbb{R})$ can be obtained by taking

$$\big(sf\big)(x) = xf(x), \qquad f \in L^2(\mathbb{R}), \quad x \in \mathbb{R}$$

and letting r be the analytic generator of the translation group:

$$\big(r^{it} f\big)(x) = f(x - t), \qquad f \in L^2(\mathbb{R}), \quad x, t \in \mathbb{R}$$

i.e. $r = \exp(-D)$ where $D = \frac{1}{i}\partial_x$.

Let us briefly recall the leg numbering notation which we already used in (1.1). Suppose H is a Hilbert space and T is an operator in H. Then by T_k we shall denote the operator

$$\underbrace{I \otimes \cdots \otimes I}_{k-1} \otimes T \otimes \underbrace{I \otimes \cdots \otimes I}_{n-k}$$

acting on $H^{\otimes n}$. A more sophisticated version of this notational convention applies to operators acting on a tensor product of H with itself. Let $U \in B(H \otimes H)$. Then U_{kl} denotes the operator acting as U on the k-th and l-th copies of H sitting inside $H^{\otimes n}$ and as identity on all remaining copies of H in $H^{\otimes n}$. We say that this operator has legs in the k-th and l-th factors of the tensor product $H^{\otimes n}$. We will also be using this notation when dealing with tensor products of different Hilbert spaces.

Let H be a separable Hilbert space and let $\overline{H}$ be the complex conjugate of H. For any $x \in H$ the corresponding element of $\overline{H}$ will be denoted by $\overline{x}$. Then $H \ni x \mapsto \overline{x} \in \overline{H}$ is an antiunitary map. In particular $(\overline{x}|\overline{y}) = (y|x)$ for any $x, y \in H$. For any closed operator m acting on H the symbol $m^\top$ will denote the transpose of m. By definition $D(m^\top) = \overline{D(m^*)}$ and

$$m^\top \overline{x} = \overline{m^* x}$$

for any $x \in D(m^*)$. If $m \in B(H)$ then $m^\top$ is the bounded operator on $\overline{H}$ such that $\big(\overline{x}\big|m^\top\big|\overline{y}\big) = (y|m|x)$ for all $x, y \in H$. Clearly $B(H) \ni m \mapsto m^\top \in B(\overline{H})$ is an antiisomorphism of C^*-algebras. Setting $\overline{\overline{x}} = x$ we identify $\overline{\overline{H}}$ with H. With this identification $m^{\top\top} = m$ for any $m \in B(H)$.

2 The results

Definition 2.1 *Let H be a Hilbert space and let $W \in B(H \otimes H)$ be a multiplicative unitary operator. We say that W is modular if there exist two positive selfadjoint operators Q and $\widehat{Q}$ on H and a unitary operator $\widetilde{W} \in B(\overline{H} \otimes H)$ such that* $\ker Q = \ker \widehat{Q} = \{0\}$,

$$W(\widehat{Q} \otimes Q)W^* = \widehat{Q} \otimes Q \tag{2.1}$$

and

$$(x \otimes u|W|z \otimes y) = \left(\overline{z} \otimes Qu \middle| \widetilde{W} \middle| \overline{x} \otimes Q^{-1}y\right) \tag{2.2}$$

for all $x, z \in H$, $u \in D(Q)$ and $y \in D(Q^{-1})$.

We begin with an analogue of Proposition 1.4 of [5]. It shows that the dual multiplicative unitary (cf. [2]) of a modular multiplicative unitary is modular. The operators Q and $\widehat{Q}$ exchange their positions.

Proposition 2.2 *Let H be a Hilbert space, W a modular multiplicative unitary and $Q, \widehat{Q}$ and $\widetilde{W}$ the operators related to W in the way described in Definition 2.1. Then*

1. *$\widetilde{W}$ and $\widehat{Q}^\top \otimes Q^{-1}$ commute.*
2. *For any $x \in D(\widehat{Q}^{-1})$, $z \in D(\widehat{Q})$ and $y, u \in H$ we have*

$$(x \otimes u|W|z \otimes y) = \left(\overline{\widehat{Q}z} \otimes u \middle| \widetilde{W} \middle| \overline{\widehat{Q}^{-1}x} \otimes y\right). \tag{2.3}$$

3. *The multiplicative unitary $\widehat{W} = \Sigma W^* \Sigma$ is modular.*

Proof: The proof is almost the same as that of Proposition 1.4 of [5]. The necessary modifications are so easy that we present only the proof of Statement 3 as an example. It is obvious that $\widehat{W}$ commutes with $Q \otimes \widehat{Q}$. Moreover introducing the unitary $\widetilde{\widehat{W}} = \left(\Sigma \widetilde{W}^* \Sigma\right)^{\top \otimes \top}$ we have:

$$\left(x \otimes u \middle| \widehat{W} \middle| z \otimes y\right) = \left(\overline{z} \otimes \widehat{Q}u \middle| \widetilde{\widehat{W}} \middle| \overline{x} \otimes \widehat{Q}^{-1}y\right) \tag{2.4}$$

for any $x, z \in H$, $u \in D(\widehat{Q})$ and $y \in D(\widehat{Q}^{-1})$. Indeed: using in the fourth step formula (2.3) we obtain

$$\begin{aligned} \left(\overline{z} \otimes \widehat{Q}u \middle| \left(\Sigma \widetilde{W}^* \Sigma\right)^{\top \otimes \top} \middle| \overline{x} \otimes \widehat{Q}^{-1}y\right) &= \left(x \otimes \overline{\widehat{Q}^{-1}y} \middle| \Sigma \widetilde{W}^* \Sigma \middle| z \otimes \overline{\widehat{Q}u}\right) \\ = \left(\overline{\widehat{Q}^{-1}y} \otimes x \middle| \widetilde{W}^* \middle| \overline{\widehat{Q}u} \otimes z\right) &= \overline{\left(\overline{\widehat{Q}u} \otimes z \middle| \widetilde{W} \middle| \overline{\widehat{Q}^{-1}y} \otimes x\right)} = \overline{(y \otimes z|W|u \otimes x)} \\ &= (u \otimes x|W^*|y \otimes z) = (x \otimes u|\Sigma W^* \Sigma|z \otimes y) \end{aligned}$$

and (2.4) follows. It shows that $\widehat{W}$ is modular. Q.E.D.

Now we will present the main result of the paper.

Theorem 2.3 *Let H be a separable Hilbert space and let $W \in B(H \otimes H)$ be a modular multiplicative unitary. Define*

$$\left.\begin{array}{rcl} A & = & \left\{(\omega \otimes \mathrm{id})W : \omega \in B(H)_*\right\}^{\text{norm closure}}, \\ \widehat{A} & = & \left\{(\mathrm{id} \otimes \omega)W^* : \omega \in B(H)_*\right\}^{\text{norm closure}}. \end{array}\right\} \tag{2.5}$$

Then

1. *A and $\widehat{A}$ are nondegenerate separable C^*-subalgebras in $B(H)$.*
2. *$W \in M\big(\widehat{A} \otimes A\big)$.*
3. *There exists a unique $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that*

$$(\mathrm{id} \otimes \Delta)W = W_{12}W_{13}. \tag{2.6}$$

   Moreover

   (i) Δ is coassociative: $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,

   (ii) $\big\{\Delta(a)(I \otimes b) : a, b \in A\big\}$ and $\big\{(a \otimes I)\Delta(b) : a, b \in A\big\}$ are linearly dense subsets of $A \otimes A$.

4. *There exists a unique closed linear operator κ on the Banach space A such that $\big\{(\omega \otimes \mathrm{id})W : \omega \in B(H)_*\big\}$ is a core for κ and*

$$\kappa\big((\omega \otimes \mathrm{id})W\big) = (\omega \otimes \mathrm{id})W^*$$

   for any $\omega \in B(H)_*$. Moreover

   (i) the domain of κ is a subalgebra of A and κ is antimultiplicative: for any $a, b \in D(\kappa)$ we have $ab \in D(\kappa)$ and $\kappa(ab) = \kappa(b)\kappa(a)$,

   (ii) the image $\kappa(D(\kappa))$ coincides with $D(\kappa)^*$ and $\kappa(\kappa(a)^*)^* = a$ for all $a \in D(\kappa)$,

   (iii) the operator κ admits the following polar decomposition:

$$\kappa = R \circ \tau_{i/2},$$

   where $\tau_{i/2}$ is the analytic generator of a one parameter group $\{\tau_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms of the C^*-algebra A and R is an involutive normal anti-automorphism of A,

   (iv) R commutes with automorphisms τ_t for all $t \in \mathbb{R}$, in particular $D(\kappa) = D(\tau_{i/2})$,

   (v) R and $\{\tau_t\}_{t\in\mathbb{R}}$ are uniquely determined.

5. *We have*

   (i) $\Delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Delta$ for all $t \in \mathbb{R}$,

   (ii) $\Delta \circ R = \sigma(R \otimes R)\Delta$,

   where σ denotes the flip map $\sigma\colon A \otimes A \ni a \otimes b \mapsto b \otimes a \in A \otimes A$.

6. *Let $\widetilde{W}$ and Q be the operators related to W as in Definition 2.1. Then*

   (i) *for any $t \in \mathbb{R}$ and $a \in A$ we have $\tau_t(a) = Q^{2it} a Q^{-2it}$,*

   (ii) *writing a^R instead of $R(a)$ we have $W^{\top \otimes R} = \widetilde{W}^*$.*

Apart from Statement 5 the conclusion of Theorem 2.3 is the same as that of Theorem 1.5 of [5]. The only difference lies in the weaker condition imposed on W.

3 The modified multiplicative unitary

In this section for a given modular multiplicative unitary W acting on $H \otimes H$ we shall construct unitary $W_{\rm M}$ acting on $H_{\rm M} \otimes H_{\rm M}$. The Hilbert space $H_{\rm M} = K \otimes H$, where K is the Hilbert space with a pair (r, s) of operators described in Section 1.

Let $W \in B(H \otimes H)$ be a modular multiplicative unitary. Define a unitary operator $X \in B(H_{\rm M})$ by

$$X = (I \otimes Q)^{i(s \otimes I)} (I \otimes \widehat{Q})^{-i(s \otimes I)} = Q_2^{is_1} \widehat{Q}_2^{-is_1}. \tag{3.1}$$

Let us notice that

$$X^*(r \otimes Q)X = r \otimes \widehat{Q}. \tag{3.2}$$

Indeed: for any $t \in \mathbb{R}$ we have

$$\begin{aligned}
(r \otimes Q)^{it} X &= (r^{it} \otimes Q^{it})X = r_1^{it} Q_2^{it} Q_2^{is_1} \widehat{Q}_2^{-is_1} \\
&= r_1^{it} Q_2^{i(s_1 + tI)} \widehat{Q}_2^{-is_1} = Q_2^{is_1} r_1^{it} \widehat{Q}_2^{-is_1} \\
&= Q_2^{is_1} \widehat{Q}_2^{-i(s_1 - tI)} r_1^{it} = Q_2^{is_1} \widehat{Q}_2^{-is_1} \widehat{Q}_2^{it} r_1^{it} = X(r \otimes \widehat{Q})^{it}
\end{aligned}$$

and (3.2) follows. Using the same method one can easily check that (2.1) implies that

$$Q_2^{it} W Q_2^{-it} = \widehat{Q}_1^{-it} W \widehat{Q}_1^{it}. \tag{3.3}$$

Now we can define a unitary operator $W_{\rm M} \in B(K \otimes H \otimes K \otimes H)$:

$$W_{\rm M} = X_{12} W_{24} X_{12}^*. \tag{3.4}$$

Notice that

$$W_{\rm M} = (\alpha \otimes \beta) W, \tag{3.5}$$

where α and β are injective unital and normal $*$-homomorphisms

$$\left.\begin{aligned}
\alpha\colon B(H) \ni m &\longmapsto X(I \otimes m)X^* \in B(K \otimes H), \\
\beta\colon B(H) \ni m &\longmapsto I \otimes m \in B(K \otimes H).
\end{aligned}\right\} \tag{3.6}$$

Proposition 3.1 *$W_{\rm M}$ is a manageable multiplicative unitary acting on $H_{\rm M} \otimes H_{\rm M}$.*

Proof: First we shall prove that $W_{\rm M}$ is a multiplicative unitary. We have to verify that

$$(W_{\rm M})_{23} (W_{\rm M})_{12} = (W_{\rm M})_{12} (W_{\rm M})_{13} (W_{\rm M})_{23}$$

which reads as

$$X_{34}W_{46}X_{34}^*X_{12}W_{24}X_{12}^* = X_{12}W_{24}X_{12}^*X_{12}W_{26}X_{12}^*X_{34}W_{46}X_{34}^* \tag{3.7}$$

on $K\otimes H\otimes K\otimes H\otimes K\otimes H$. By commuting X_{12} through $X_{34}W_{46}X_{34}^*$ on the left hand side of (3.7) and moving X_{12}^* though $X_{34}W_{46}X_{34}^*$ on the right hand side of (3.7) one reduces (3.7) to

$$X_{34}W_{46}X_{34}^*W_{24} = W_{24}W_{26}X_{34}W_{46}X_{34}^*. \tag{3.8}$$

The pentagon equation (1.1) gives us

$$W_{46}W_{24} = W_{24}W_{26}W_{46} \tag{3.9}$$

Taking into account (3.3) and using the fact that operators with different legs commute we infer that the right hand side of (3.9) is equal to

$$\begin{aligned}\text{RHS} &= W_{24}\widehat{Q}_2^{-is_3}\big(\widehat{Q}_2^{is_3}W_{26}\widehat{Q}_2^{-is_3}\big)\widehat{Q}_2^{is_3}W_{46}\\ &= W_{24}\widehat{Q}_2^{-is_3}\big(Q_6^{-is_3}W_{26}Q_6^{is_3}\big)\widehat{Q}_2^{is_3}W_{46}\\ &= Q_6^{-is_3}W_{24}\widehat{Q}_2^{-is_3}W_{26}Q_6^{is_3}W_{46}\widehat{Q}_2^{is_3}.\end{aligned}$$

Thus

$$W_{46}W_{24} = Q_6^{-is_3}W_{24}\widehat{Q}_2^{-is_3}W_{26}Q_6^{is_3}W_{46}\widehat{Q}_2^{is_3}. \tag{3.10}$$

Applying the map $m\mapsto Q_6^{is_3}\widehat{Q}_2^{is_3}m\widehat{Q}_2^{-is_3}Q_6^{-is_3}$ to both sides of (3.10) and repeatedly using (3.3) we obtain

$$\begin{aligned}Q_6^{is_3}\widehat{Q}_2^{is_3}W_{46}W_{24}\widehat{Q}_2^{-is_3}Q_6^{-is_3} &= \widehat{Q}_2^{is_3}W_{24}\widehat{Q}_2^{-is_3}W_{26}Q_6^{is_3}W_{46}Q_6^{-is_3}\\ \big(Q_6^{is_3}W_{46}Q_6^{-is_3}\big)\big(\widehat{Q}_2^{is_3}W_{24}\widehat{Q}_2^{-is_3}\big) &= \big(\widehat{Q}_2^{is_3}W_{24}\widehat{Q}_2^{-is_3}\big)W_{26}\big(Q_6^{is_3}W_{46}Q_6^{-is_3}\big)\\ \big(\widehat{Q}_4^{-is_3}W_{46}\widehat{Q}_4^{is_3}\big)\big(Q_4^{-is_3}W_{24}Q_4^{is_3}\big) &= \big(Q_4^{-is_3}W_{24}Q_4^{is_3}\big)W_{26}\big(\widehat{Q}_4^{-is_3}W_{46}\widehat{Q}_4^{is_3}\big).\end{aligned}$$

Now we apply the map $m\mapsto Q_4^{is_3}mQ_4^{-is_3}$ to both sides of the last equality and use (3.1) to obtain (3.8) which proves that $W_{\rm M}$ is a multiplicative unitary.

In order to prove manageability of $W_{\rm M}$ we have to construct the operators required by Definition 1.2 of [5]. Let

$$Q_{\rm M} = r\otimes Q \tag{3.11}$$

and

$$\widetilde{W}_{\rm M} = \big(X_{12}^\top\big)^*\,\widetilde{W}_{24}X_{12}^\top,$$

where the symbol $\top$ denotes the transposition $B(H_{\rm M})\ni m\longmapsto m^\top\in B(\overline{H_{\rm M}})$ (cf. Section 1). Take $\xi,\xi'\in K\otimes H$ and $\eta,\eta'\in D(r)\otimes_{\rm alg}D(Q)\subset D(Q_{\rm M})$. Using selfadjointness of r, the equation (2.1) and the fact that operators with different legs commute we obtain:

$$\left.\begin{aligned}&(\xi\otimes\eta|W_{\rm M}|\xi'\otimes Q_{\rm M}\eta') = (\xi\otimes\eta|X_{12}W_{24}X_{12}^*|\xi'\otimes Q_{\rm M}\eta')\\ &= (X^*\xi\otimes\eta|W_{24}|X^*\xi'\otimes Q_{\rm M}\eta') = (X^*\xi\otimes\eta|W_{24}|X^*\xi'\otimes r_1Q_2\eta')\\ &= (X^*\xi\otimes\eta|W_{24}|r_3(X^*\xi'\otimes Q_2\eta')) = (r_3(X^*\xi\otimes\eta)|W_{24}|X^*\xi'\otimes Q_2\eta')\\ &= (X^*\xi\otimes r_1\eta|W_{24}|X^*\xi'\otimes Q_2\eta') = \Big(\overline{X^*\xi'}\otimes Q_2r_1\eta\Big|\widetilde{W}_{24}\Big|\overline{X^*\xi}\otimes\eta'\Big)\\ &= \Big(X^\top\overline{\xi'}\otimes Q_{\rm M}\eta\Big|\widetilde{W}_{24}\Big|X^\top\overline{\xi}\otimes\eta'\Big) = \Big(\overline{\xi'}\otimes Q_{\rm M}\eta\Big|\,\big(X_{12}^\top\big)^*\,\widetilde{W}_{24}X_{12}^\top\Big|\overline{\xi}\otimes\eta'\Big)\\ &= \Big(\overline{\xi'}\otimes Q_{\rm M}\eta\Big|\widetilde{W}_{\rm M}\Big|\overline{\xi}\otimes\eta'\Big).\end{aligned}\right\} \tag{3.12}$$

Since $D(r)\otimes_{\rm alg} D(Q)$ is a core for $Q_{\rm M}$ we have (3.12) for all $\eta,\eta'\in D(Q_{\rm M})$. Now replacing η' by $Q_{\rm M}^{-1}\eta'$ we obtain

$$(\xi\otimes\eta|W_{\rm M}|\xi'\otimes\eta')=\left(\overline{\xi'}\otimes Q_{\rm M}\eta\middle|\widetilde{W}_{\rm M}\middle|\overline{\xi}\otimes Q_{\rm M}^{-1}\eta'\right)$$

for all $\xi,\xi'\in K\otimes H$, $\eta\in D(Q_{\rm M})$ and $\eta'\in D(Q_{\rm M}^{-1})$.
It remains to prove that $W_{\rm M}$ commutes with $Q_{\rm M}\otimes Q_{\rm M}$. Using formula (3.2), (2.1) and again (3.2) we obtain

$$\begin{aligned}(Q_{\rm M}\otimes Q_{\rm M})W_{\rm M} &= (r\otimes Q\otimes r\otimes Q)(X\otimes I\otimes I)W_{24}(X^*\otimes I\otimes I)\\ &= (X\otimes I\otimes I)(r\otimes\widehat{Q}\otimes r\otimes Q)W_{24}(X^*\otimes I\otimes I)\\ &= (X\otimes I\otimes I)W_{24}(r\otimes\widehat{Q}\otimes r\otimes Q)(X^*\otimes I\otimes I)\\ &= (X\otimes I\otimes I)W_{24}(X^*\otimes I\otimes I)(r\otimes Q\otimes r\otimes Q) \;=\; W_{\rm M}(Q_{\rm M}\otimes Q_{\rm M}).\end{aligned}$$

We have thus checked that $W_{\rm M}$ satisfies all the conditions of Definition 1.2 from [5]. Q.E.D.

We are now free to use the theory presented in [5]. All objects constructed for $W_{\rm M}$ with help of Theorem 1.5 of that paper will be denoted by letters with a subscript $_{\rm M}$. For example

$$\left.\begin{aligned}A_{\rm M} &= \left\{(\Phi\otimes{\rm id})W_{\rm M}:\Phi\in B(H_{\rm M})_*\right\}^{\text{norm closure}},\\ \widehat{A}_{\rm M} &= \left\{({\rm id}\otimes\Phi)W_{\rm M}^*:\Phi\in B(H_{\rm M})_*\right\}^{\text{norm closure}}.\end{aligned}\right\}\tag{3.13}$$

We also have $\Delta_{\rm M}$, $\kappa_{\rm M}$, $\{\tau_{{\rm M}t}\}_{t\in\mathbb{R}}$ and $R_{\rm M}$.

4 Proof of Theorem 2.3

Ad 1. We know that $A_{\rm M}$ and $\widehat{A}_{\rm M}$ defined by (3.13) are nondegenerate separable C^*-subalgebras of $B(H_{\rm M})$. Recall that α and β are ultra-weakly continuous injections of $B(H)$ into $B(H_{\rm M})$. Therefore for any normal functional ω on $B(H)$ there exits $\Phi,\Phi'\in B(H_{\rm M})_*$ such that

$$\omega=\Phi\circ\alpha=\Phi'\circ\beta.$$

Keeping this fact in mind, remembering the definitions (2.5) and formula (3.5) we have

$$\begin{aligned}\beta(A) &= \left\{\beta\big((\omega\otimes{\rm id})W\big):\omega\in B(H)_*\right\}^{\text{norm closure}}\\ &= \left\{(\omega\otimes\beta)W:\omega\in B(H)_*\right\}^{\text{norm closure}}\\ &= \left\{(\Phi\circ\alpha\otimes\beta)W:\Phi\in B(H_{\rm M})_*\right\}^{\text{norm closure}}\\ &= \left\{(\Phi\otimes{\rm id})W_{\rm M}:\Phi\in B(H_{\rm M})_*\right\}^{\text{norm closure}}=A_{\rm M}.\end{aligned}$$

Similarly we prove that

$$\alpha\big(\widehat{A}\big)=\widehat{A}_{\rm M}.$$

Now it is easy to see that A and $\widehat{A}$ are nondegenerate separable C^*-subalgebras in $B(H)$.
Ad 2. We know that $W_{\rm M}\in M\big(\widehat{A}_{\rm M}\otimes A_{\rm M}\big)$. In other words (cf. (3.5))

$$(\alpha\otimes\beta)W\in M\big(\alpha\big(\widehat{A}\big)\otimes\beta(A)\big)=M\big((\alpha\otimes\beta)\big(\widehat{A}\otimes A\big)\big)$$

and it follows that $W \in M\big(\widehat{A} \otimes A\big)$.
Ad 3. We have $\Delta_{\rm M} \in \mathrm{Mor}(A_{\rm M}, A_{\rm M} \otimes A_{\rm M})$ and a $*$-isomorphism $\beta\colon A \to A_{\rm M}$, so we can define $\Delta = (\beta \otimes \beta)^{-1}\Delta_{\rm M}\beta$. This provides us with a coassociative $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that $\big\{\Delta(a)(I \otimes b) : a, b \in A\big\}$ and $\big\{(a \otimes I)\Delta(b) : a, b \in A\big\}$ are linearly dense subsets of $A \otimes A$. Furthermore notice that

$$\begin{aligned}
(\mathrm{id} \otimes \Delta)W &= \big(\mathrm{id} \otimes (\beta \otimes \beta)^{-1}\Delta_{\rm M}\beta\big)W \\
&= \big(\alpha^{-1}\alpha \otimes (\beta \otimes \beta)^{-1}\Delta_{\rm M}\beta\big)W \\
&= \big(\alpha^{-1} \otimes (\beta \otimes \beta)^{-1}\Delta_{\rm M}\big)(\alpha \otimes \beta)W \\
&= \big(\alpha^{-1} \otimes (\beta \otimes \beta)^{-1}\Delta_{\rm M}\big)W_{\rm M} \\
&= (\alpha \otimes \beta \otimes \beta)^{-1}(\mathrm{id} \otimes \Delta_{\rm M})W_{\rm M} \\
&= (\alpha \otimes \beta \otimes \beta)^{-1}(W_{\rm M})_{12}(W_{\rm M})_{13} \\
&= (\alpha \otimes \beta \otimes \beta)^{-1}(W_{\rm M})_{12}(\alpha \otimes \beta \otimes \beta)^{-1}(W_{\rm M})_{13} \\
&= (\alpha \otimes \beta \otimes \mathrm{id})^{-1}(W_{\rm M})_{12}(\alpha \otimes \mathrm{id} \otimes \beta)^{-1}(W_{\rm M})_{13} = W_{12}W_{13}.
\end{aligned}$$

Remark. Despite a fairly complicated way of introducing the comultiplication on A we can still recover formula (5.1) of [5] i.e.

$$\Delta(a) = W(a \otimes I)W^* \tag{4.1}$$

for all $a \in A$ (cf. [2], Théorème 3.8). Indeed: take $a = (\omega \otimes \mathrm{id})W$ then using (2.6) and (1.1) we obtain

$$\begin{aligned}
\Delta(a) &= (\omega \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)W \\
&= (\omega \otimes \mathrm{id} \otimes \mathrm{id})W_{12}W_{13} \\
&= (\omega \otimes \mathrm{id} \otimes \mathrm{id})W_{12}W_{13} \\
&= (\omega \otimes \mathrm{id} \otimes \mathrm{id})W_{23}W_{12}W_{23}^* \\
&= W\big((\omega \otimes \mathrm{id})W \otimes I\big)W^*.
\end{aligned}$$

For an arbitrary $a \in A$ we use the continuity argument. This also proves the uniqueness of Δ.
Ad 4. Since β is an isomorphism $A \to A_{\rm M}$ we can define $\kappa = \beta^{-1}\kappa_{\rm M}\beta$. Now it is important to notice (cf. the proof of Statement 1) that

$$\left.\begin{aligned}
\beta^{-1}\big((\Phi \otimes \mathrm{id})W_{\rm M}\big) &= (\Phi \circ \alpha \otimes \mathrm{id})W \\
\beta^{-1}\big((\Phi \otimes \mathrm{id})W_{\rm M}^*\big) &= (\Phi \circ \alpha \otimes \mathrm{id})W^*.
\end{aligned}\right\} \tag{4.2}$$

Then first of all it follows from (4.2) that

$$\beta\left(\big\{(\omega \otimes \mathrm{id})W : \omega \in B(H)_*\big\}\right) = \big\{(\Phi \otimes \mathrm{id})W_{\rm M} : \Phi \in B(H_{\rm M})_*\big\}.$$

Furthermore for $\omega \in B(H)_*$ we have $\omega = \Phi \circ \alpha$ for some $\Phi \in B(H_{\rm M})_*$ and using (4.2) we get

$$\begin{aligned}
\kappa\big((\omega \otimes \mathrm{id})W\big) &= \beta^{-1}\kappa_{\rm M}\beta\big((\omega \otimes \mathrm{id})W\big) = \beta^{-1}\kappa_{\rm M}\big((\Phi \otimes \mathrm{id})W_{\rm M}\big) \\
&= \beta^{-1}\big((\Phi \otimes \mathrm{id})W_{\rm M}^*\big) = (\omega \otimes \mathrm{id})W^*.
\end{aligned}$$

Since $\{(\Phi\otimes\mathrm{id})W_{\mathrm{M}} : \Phi\in B(H_{\mathrm{M}})_*\}$ is a core for κ_{M} we see that $\{(\omega\otimes\mathrm{id})W : \omega\in B(H)_*\}$ is a core for κ. Now setting

$$\left.\begin{array}{rcl}\tau_t &=& \beta^{-1}\tau_{\mathrm{M}t}\beta, \qquad t\in\mathbb{R},\\ R &=& \beta^{-1}R_{\mathrm{M}}\beta\end{array}\right\} \tag{4.3}$$

we see that assertions (i) – (v) follow directly from analogous statements for $A_{\mathrm{M}}, \kappa_{\mathrm{M}}, R_{\mathrm{M}}$ and $\{\tau_{\mathrm{M}t}\}_{t\in\mathbb{R}}$ (cf. [5], Theorem 1.5, Statement 4.) and the fact the β is a normal $*$-isomorphism of A onto A_{M}.

Ad 6. We know (cf. [5], Theorem 1.5, Statement 5) that for any $a_{\mathrm{M}}\in A_{\mathrm{M}}$ and any $t\in\mathbb{R}$

$$\tau_{\mathrm{M}t}(a_{\mathrm{M}}) = Q_{\mathrm{M}}^{2it}a_{\mathrm{M}}Q_{\mathrm{M}}^{-2it}. \tag{4.4}$$

Thus formula (i) follows from the first line of (4.3), (4.4), (3.11) and the definition of β. From the results of [5] (formula (1.14)) we know that

$$W_{\mathrm{M}}^{\top\otimes R_{\mathrm{M}}} = \widetilde{W}_{\mathrm{M}}^*. \tag{4.5}$$

Notice that

$$\widetilde{W}_{\mathrm{M}} = (\alpha^\top\otimes\beta)\widetilde{W}, \tag{4.6}$$

where $\alpha^\top : B(\overline{H})\to B(\overline{H_{\mathrm{M}}})$ is a normal $*$-monomorphism given by

$$\alpha^\top(m) = \left(X^\top\right)^*(I\otimes m)X^\top.$$

It is easy to check that

$$\top\circ\alpha = \alpha^\top\circ\top. \tag{4.7}$$

Finally recall that from the definition of R (4.3) it follows that

$$R_{\mathrm{M}}\circ\beta = \beta\circ R. \tag{4.8}$$

Now taking into account (4.6) and (4.5) and using (3.5), (4.7) and (4.8) we obtain

$$\begin{array}{rcl}(\alpha^\top\otimes\beta)\widetilde{W}^* &=& \widetilde{W}_{\mathrm{M}}^* = W_{\mathrm{M}}^{\top\otimes R_{\mathrm{M}}}\\ &=& \big((\alpha\otimes\beta)W\big)^{\top\otimes R_{\mathrm{M}}}\\ &=& (\alpha^\top\otimes\beta)\left(W^{\top\otimes R}\right),\end{array}$$

which gives formula (ii).

Ad 5. Recall (cf. [5], Theorem 1.5, Statement 5 and formula (5.1)) that for any $a_{\mathrm{M}}\in A_{\mathrm{M}}$ we have

$$\Delta_{\mathrm{M}}(a_{\mathrm{M}}) = W_{\mathrm{M}}(a_{\mathrm{M}}\otimes I)W_{\mathrm{M}}^*.$$

Now formula (i) follows from an easy computation:

$$\begin{array}{rcl}\Delta_{\mathrm{M}}\big(\tau_{\mathrm{M}t}(a_{\mathrm{M}})\big) &=& W_{\mathrm{M}}\big(\tau_{\mathrm{M}t}(a_{\mathrm{M}})\otimes I\big)W_{\mathrm{M}}^*\\ &=& W_{\mathrm{M}}(Q_{\mathrm{M}}^{2it}a_{\mathrm{M}}Q_{\mathrm{M}}^{-2it}\otimes I)W_{\mathrm{M}}^*\\ &=& W_{\mathrm{M}}(Q_{\mathrm{M}}^{2it}\otimes Q_{\mathrm{M}}^{2it})(a_{\mathrm{M}}\otimes I)(Q_{\mathrm{M}}^{-2it}\otimes Q_{\mathrm{M}}^{-2it})W_{\mathrm{M}}^*\\ &=& (Q_{\mathrm{M}}^{2it}\otimes Q_{\mathrm{M}}^{2it})W_{\mathrm{M}}(a_{\mathrm{M}}\otimes I)W_{\mathrm{M}}^*(Q_{\mathrm{M}}^{-2it}\otimes Q_{\mathrm{M}}^{-2it})\\ &=& (\tau_{\mathrm{M}t}\otimes\tau_{\mathrm{M}t})\Delta_{\mathrm{M}}(a_{\mathrm{M}}),\end{array}$$

where in the second last equality we used the fact that W_{M} commutes with $Q_{\mathrm{M}}\otimes Q_{\mathrm{M}}$.
For the proof of (ii) we will need two propositions:

Proposition 4.1 *There exists a closed densely defined linear operator* $\mathcal{T}\colon B(H)\to B(\overline{H})$ *such that* $\ker\mathcal{T}=\{0\}$, $D(\mathcal{T})$ *is a subalgebra of* $B(H)$ *and* $\mathcal{T}$ *is antimultiplicative. Moreover the set*

$$\big\{(\mathrm{id}\otimes\omega)W:\omega\in B(H)_*\big\}$$

is contained in the domain of $\mathcal{T}$ *and*

$$\mathcal{T}\big((\mathrm{id}\otimes\omega)W\big)=(\mathrm{id}\otimes\omega)\widetilde{W}.$$

Proof: Consider a one parameter group $\mathbb{R}\ni t\mapsto\sigma_t\in\mathrm{Aut}\big(B(H)\big)$, $\sigma_t(m)=\widehat{Q}^{it}m\widehat{Q}^{-it}$. Let σ_{-i} be its analytic generator ([8]). Now for $m\in D(\sigma_{-i})$ define

$$\mathcal{T}(m)=\left(\sigma_{-i}(m)\right)^{\top}.$$

It follows (cf. [8]) that $\mathcal{T}$ is a densely defined closed operator whose domain is a subalgebra of $B(H)$ and that $\mathcal{T}$ is antimultiplicative.
Now from formula (2.3) we infer that for any $\omega\in B(H)_*$ we have

$$(x|(\mathrm{id}\otimes\omega)W|z)=\Big(\overline{\widehat{Q}z}\Big|(\mathrm{id}\otimes\omega)\widetilde{W}\Big|\overline{\widehat{Q}^{-1}x}\Big)$$

for any $x\in D(\widehat{Q}^{-1})$ and $z\in D(\widehat{Q})$. This can be rephrased as

$$\Big(\overline{z}\Big|(\mathrm{id}\otimes\omega)\widetilde{W}\Big|\overline{x}\Big)=\Big(\widehat{Q}x\Big|(\mathrm{id}\otimes\omega)W\Big|\widehat{Q}^{-1}z\Big)$$

for any $z\in D(\widehat{Q}^{-1})$ and $x\in D(\widehat{Q})$. Therefore $\widehat{Q}\big((\mathrm{id}\otimes\omega)W\big)\widehat{Q}^{-1}$ extends to a bounded operator on H and

$$\Big(\widehat{Q}\big((\mathrm{id}\otimes\omega)W\big)\widehat{Q}^{-1}\Big)^{\top}=(\mathrm{id}\otimes\omega)\widetilde{W}\in B(\overline{H}).$$

This shows that any element of the form $(\mathrm{id}\otimes\omega)W$ lies in the domain of $\mathcal{T}$ and that $\mathcal{T}\big((\mathrm{id}\otimes\omega)W\big)=(\mathrm{id}\otimes\omega)\widetilde{W}$. Q.E.D.

Formula (4.1) allows us to define $\Delta(m)$ for any $m\in B(H)$ which justifies its use in the statement of the next proposition.

Proposition 4.2 *We have*

$$(\mathrm{id}\otimes\Delta)\widetilde{W}=\widetilde{W}_{13}\widetilde{W}_{12}. \tag{4.9}$$

Proof: Take $\mu,\nu\in B(H)_*$ and denote by $\nu*\mu$ the normal functional $(\mu\otimes\nu)\circ\Delta$. Now using (2.6) and the fact that $\mathcal{T}$ defined in Proposition 4.1 is antimultiplicative we compute:

$$\begin{aligned}
(\mathrm{id}\otimes\mu\otimes\nu)(\mathrm{id}\otimes\Delta)\widetilde{W} &= (\mathrm{id}\otimes\nu*\mu)\widetilde{W}\\
&= \mathcal{T}\big((\mathrm{id}\otimes\nu*\mu)W\big)\\
&= \mathcal{T}\big((\mathrm{id}\otimes\mu\otimes\nu)(\mathrm{id}\otimes\Delta)W\big)\\
&= \mathcal{T}\big((\mathrm{id}\otimes\mu\otimes\nu)W_{12}W_{13}\big)\\
&= \mathcal{T}\big((\mathrm{id}\otimes\mu)W(\mathrm{id}\otimes\nu)W\big)\\
&= \mathcal{T}\big((\mathrm{id}\otimes\nu)W\big)\mathcal{T}\big((\mathrm{id}\otimes\mu)W\big)\\
&= (\mathrm{id}\otimes\nu)\widetilde{W}(\mathrm{id}\otimes\mu)\widetilde{W}\\
&= (\mathrm{id}\otimes\mu\otimes\nu)\widetilde{W}_{13}\widetilde{W}_{12}.
\end{aligned}$$

Now since functionals of the form $\mu \otimes \nu$ separate elements of $B(H \otimes H)$ we obtain (4.9). Q.E.D.

We will use formula (4.9) to prove assertion (ii) of point 5 of our theorem. Applying $*$ to both sides of (4.9) we get

$$(\mathrm{id} \otimes \Delta)\widetilde{W}^* = \widetilde{W}^*_{12}\widetilde{W}^*_{13}. \tag{4.10}$$

Notice that due to (ii) of point 6 of our theorem the left hand side of (4.10) is equal to

$$(\top \otimes \Delta \circ R)W$$

while the right hand side of (4.10) equals

$$\left(W^{\top\otimes R}\right)_{12}\left(W^{\top\otimes R}\right)_{13} = (\top \otimes R \otimes R)W_{13}W_{12} = (\top \otimes R \otimes R)(\mathrm{id} \otimes \sigma)(\mathrm{id} \otimes \Delta)W.$$

In other words

$$(\top \otimes \Delta \circ R)W = (\top \otimes R \otimes R)(\mathrm{id} \otimes \sigma)(\mathrm{id} \otimes \Delta)W = (\top \otimes \sigma \circ (R \otimes R) \circ \Delta)W. \tag{4.11}$$

Applying $(\omega \circ \top \otimes \mathrm{id})$ to both sides of (4.11) and taking into account (2.5) we obtain formula (ii).

5 Applications

In this section we will briefly present two examples of quantum groups whose naturally occurring multiplicative unitaries are modular, but not manageable. These groups are the quantum "$ax + b$" and "$az + b$" groups constructed in [7] and [6] respectively. The algebras A_x and A_z of continuous functions vanishing at infinity on these groups are generated ([4]) by $a_x, a_x^{-1}, b_x, \beta$ and a_z, a_z^{-1}, b_z affiliated with A_x and A_z. These distinguished elements are subject to relations

$$\left(\begin{array}{c} a_x \text{ and } b_x \text{ are selfadjoint} \\ a_x \text{ is strictly positive and} \\ a_x^{it} b_x a_x^{-it} = e^{\hbar t} b_x \\ \text{for any } t \in \mathbb{R}, \\ \beta^2 = \chi(b_x \neq 0),\ \beta a_x = a_x \beta \\ \text{and } \beta b_x = -b_x \beta \end{array}\right) \qquad \left(\begin{array}{c} a_z \text{ and } b_z \text{ are normal operators} \\ \operatorname{Sp} a_z, \operatorname{Sp} b_z \subset \overline{\Gamma},\ \ker a_z = \{0\} \\ (\operatorname{Phase} a_z) b_z (\operatorname{Phase} a_z)^* = e^{\frac{2\pi i}{N}} b_z \\ |a_z|^{-it} b_z |a_z|^{it} = e^{\frac{2\pi}{N} t} b_z \\ \text{for any } t \in \mathbb{R} \end{array}\right)$$

where $\overline{\Gamma} = \{0\} \cup \bigcup_{k=0}^{N-1} e^{\frac{2\pi i}{N}k}\mathbb{R}_+$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ and $\hbar \in]0, \pi[$ and $N \in \mathbb{N}$ are deformation parameters satisfying additional requirements. The natural choices for multiplicative unitary operators for these groups are

$$W_x = F_\hbar \left(e^{i\frac{i}{\hbar}} b_x^{-1} a_x \otimes b_x, ie^{\frac{i\pi^2}{2\hbar}} (\beta \otimes \beta)\chi\left(b_x \otimes b_x < 0\right)\right)^* e^{\frac{i}{\hbar}\log(|b_x|^{-1}) \otimes \log a_x} \tag{5.1}$$

for the "$ax + b$" group and

$$W_z = F_N\left(a_z b_z^{-1} \otimes b_z\right) \chi\left(b_z^{-1} \otimes I, I \otimes a_z\right) \tag{5.2}$$

for the "$az+b$" group. In the above formulae we choose representations in which b_x and b_z are invertible and $F_\hbar, F_N$ and χ are special functions. It can be shown that both W_x and W_z are multiplicative unitary operators, but neither of them is manageable. Nevertheless they are both modular with

$$\begin{aligned} \widehat{Q}_x &= |b_x|^{\frac{1}{2}}, & Q_x &= (a_x)^{\frac{1}{2}}, \\ \widehat{Q}_z &= |b_z|, & Q_z &= |a_z| \end{aligned}$$

and

$$\widetilde{W}_x = F_\hbar\left(-e^{i\frac{\hbar}{2}}\left(b_x^{-1}a_x\right)^\top \otimes e^{i\frac{\hbar}{2}}b_x a_x^{-1}, -(\beta\otimes\beta)\chi\big(e^{i\frac{\hbar}{2}}(b_x^{-1}a_x)^\top \otimes b_x > 0\big)\right) e^{\frac{i}{\hbar}\log(a_x)^\top \otimes \log a_x},$$

$$\widetilde{W}_z = F_N\left(-\left(a_z b_z^{-1}\right)^\top \otimes e^{\frac{2\pi i}{N}} a_z^{-1} b_z\right)^* \chi\left(b_z^{-1\top} \otimes I, I\otimes a_z\right).$$

In both constructions [7] and [6] a clever trick was used to obtain manageability of the unitaries (5.1) and (5.2). This trick was the basis of our construction of the modified multiplicative unitary presented in Section 3. Using Theorem 2.3 one is able to carry out the construction of the two quantum groups without having to resort to some slightly unintuitive means (cf. [7] Theorem 2.1 and [6] Theorem 3.1).

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