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A theorem on kernel in the theory of operator-valued distributions

by

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1. Introduction. Let $\mathcal{S}(\boldsymbol{R}^n)$ denote the topological vector space of test functions for tempered distributions introduced by L. Schwartz [3]. For any two functions $\varphi\in\mathcal{S}(\boldsymbol{R}^n)$, $\psi\in\mathcal{S}(\boldsymbol{R}^m)$ we put

$$(\varphi\otimes\psi)(x,y)\overset{\mathrm{df}}{=}\varphi(x)\psi(y),$$

where $x\in\boldsymbol{R}^n$, $y\in\boldsymbol{R}^m$ and $(x,y)\in\boldsymbol{R}^{n+m}$. It is known that this formula defines a continuous bilinear mapping

$$\otimes:\ \mathcal{S}(\boldsymbol{R}^n)\times\mathcal{S}(\boldsymbol{R}^m)\to\mathcal{S}(\boldsymbol{R}^{n+m}).$$

Let L be a topological vector space. Any continuous linear mapping

$$\boldsymbol{A}:\ \mathcal{S}(\boldsymbol{R}^n)\to L$$

is called a L-valued distribution defined on $\boldsymbol{R}^n$. For the special case $L=\boldsymbol{C}^1$ this definition coincides with the definition of tempered distributions given by L. Schwartz. The second special case $L=L(D)$, where D is a dense linear subset of a Hilbert space H and $L(D)$ denotes the $*$-algebra of operators acting in D (the strict definition of $L(D)$ is given below), is of great importance in the quantum field theory [4]. $L(D)$-valued distributions are often called operator-valued distributions.

We say that the topological vector space L satisfies the theorem on kernel if for any separately continuous bilinear mapping

$$\boldsymbol{B}:\ \mathcal{S}(\boldsymbol{R}^n)\times\mathcal{S}(\boldsymbol{R}^m)\to L$$

there exists a continuous linear mapping

$$\overset{\otimes}{\boldsymbol{B}}:\ \mathcal{S}(\boldsymbol{R}^{n+m})\to L$$

such that $\boldsymbol{B}(\varphi,\psi)=\overset{\otimes}{\boldsymbol{B}}(\varphi\otimes\psi)$ for any $\varphi\in\mathcal{S}(\boldsymbol{R}^n)$ and $\psi\in\mathcal{S}(\boldsymbol{R}^m)$.

It is known that C^1 satisfies the theorem on kernel. This fact discovered by L. Schwartz is of great importance for the theory of tempered distributions. Similarly many interesting problems concerning L-valued distributions can be solved if it is known that L satisfies the theorem on kernel. For example assuming that $L(D)$ satisfies the theorem on kernel one can easily show that for any two $L(D)$-value distributions $\boldsymbol{A}_1, \boldsymbol{A}_2$ defines on $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$ respectively there exists a $L(D)$-value distribution $\boldsymbol{B}$ defined on $\boldsymbol{R}^{n+m}$ such that $\boldsymbol{B}(\varphi\otimes\psi) = \boldsymbol{A}_1(\varphi)\,\boldsymbol{A}_2(\psi)$ for any $\varphi\in S(\boldsymbol{R}^n)$ and $\psi\in S(\boldsymbol{R}^m)$.

One can check that in general case $L(D)$ does not satisfy the theorem on kernel. However, it appears that there exist sufficiently many dense linear subsets $\tilde{D}\subset H$ such that $L(\tilde{D})$ satisfies the theorem on kernel. More exactly, for any dense linear subset $D\subset H$ we shall construct a linear subset $\tilde{D}\supset D$ such that:

1° $L(\tilde{D})$ satisfies the theorem on kernel.

2° For any $L(D)$-valued distribution $\boldsymbol{A}$ there exists one and only one $L(\tilde{D})$-valued distribution $\tilde{\boldsymbol{A}}$ such that $\boldsymbol{A}(\varphi)\subset\tilde{\boldsymbol{A}}(\varphi)$ for all test functions φ. (Relation $A\subset B$, where A, B are operators acting in the Hilbert space H means that $D_A\subset D_B$ and $Au = Bu$ for any $u\in D_A$.)

2. Sequentially complete spaces. Assume that L is a locally convex topological vector space i. e. that the topology of L is given by system of seminorms $\{q_\alpha\colon \alpha\in\Lambda\}$. Let us remind that a sequence $(A_k)_{k=1,2,\ldots}$ of elements of L is called a Cauchy sequence if for any $\alpha\in\Lambda$ and any positive number ε one can find an integer N such that $q_\alpha(A_k - A_{k'})\leqslant\varepsilon$ for any $k, k'\geqslant N$. The space L is sequentially complete if any Cauchy sequence is convergent.

Let $(A_k)_{k=1,2,\ldots}$ be a sequence of elements of the space L. Assume that L is sequentially complete and that

$$\sum_{k=1}^{\infty} q_\alpha(A_k) < \infty$$

for any $\alpha\in\Lambda$. Then the series $\sum_{k=1}^{\infty} A_k$ is convergent and

$$q_\alpha\Big(\sum_{k=1}^{\infty} A_k\Big)\leqslant\sum_{k=1}^{\infty} q_\alpha(A_k) \tag{1}$$

for any $\alpha\in\Lambda$.

THEOREM 1. *Any locally convex and sequentially complete topological vector space satisfies the theorem on kernel.*

Proof. Let

$$S(\boldsymbol{R}^n)\times S(\boldsymbol{R}^m)\ni(\varphi,\psi)\to\boldsymbol{B}(\varphi,\psi)\in L \tag{2}$$

be a partially continuous linear mapping. We have to find a continuous linear mapping

$$S(\boldsymbol{R}^{n+m})\ni\chi\to\overset{\otimes}{\boldsymbol{B}}(\chi)\in L$$

such that $\overset{\otimes}{\boldsymbol{B}}(\varphi\otimes\psi) = \boldsymbol{B}(\varphi,\psi)$ for any $\varphi\in S(\boldsymbol{R}^n)$ and $\psi\in S(\boldsymbol{R}^m)$.

Assume for simplicity that $n = m = 1$ (the proof in the general case is slightly more complicated). Let

$$h_k(x)\overset{\text{df}}{=}\frac{1}{\sqrt{\pi^{1/2}k!\,2^k}}e^{x^2/2}\Big(\frac{d}{dx}\Big)^k e^{-x^2}$$

be the k-th normalized Hermite function. It is known that $h_k\in S(\boldsymbol{R}^1)$. Moreover any element $\varphi\in S(\boldsymbol{R}^1)$ is a linear combination

$$\varphi = \sum_{k=0}^{\infty} C_k(\varphi)h_k,$$

where complex coefficients $C_k(\varphi)$ decrease rapidly when $k\to\infty$. It means that

$$P_N(\varphi)\overset{\text{df}}{=}\sup_k|C_k(\varphi)|(1+k)^N<\infty \tag{3}$$

for any natural N. One can check that (3) introduces the system of norms $\{p_N\colon N\text{-integer}\}$ on the space $S(\boldsymbol{R}^1)$ and that the topology of $S(\boldsymbol{R}^1)$ given by this system coincides with the well known topology introduced by L. Schwartz.

Similarly any element $\chi\in S(\boldsymbol{R}^2)$ can be written as a linear combination

$$\chi = \sum_{k,k'=0}^{\infty} C_{kk'}(\chi)h_k\otimes h_{k'},$$

where $C_{kk'}(\chi)$ decrease rapidly when $k, k'\to\infty$, i. e.:

$$p_{NN'}(\chi)\overset{\text{df}}{=}\sup_{k,k'}|C_{kk'}(\chi)|(1+k)^N(1+k')^{N'}<\infty \tag{4}$$

for any natural N and N'. The topology of $S(\boldsymbol{R}^2)$ is defined by the system of norms $\{p_{NN'}\colon N, N'\text{-integer}\}$ introduced by (4).

$S(\boldsymbol{R}^1)$ is a Fréchet space and therefore according to the theorem of Mazur, Orlicz and Bourbaki (see [2]) we can conclude that (2) is a continuous mapping. It means that for any $\alpha\in\Lambda$ there exist integers $N(\alpha)$, $N'(\alpha)$ and a positive number K_α such that

$$q_\alpha\big(\boldsymbol{B}(\varphi,\psi)\big)\leqslant K_\alpha p_{N(\alpha)}(\varphi)\,p_{N'(\alpha)}(\psi)$$

for any $\varphi, \psi\in S(\boldsymbol{R}^1)$. Setting h_k and $h_{k'}$ instead of φ and ψ we obtain:

$$q_\alpha\big(\boldsymbol{B}(h_k, h_{k'})\big)\leqslant K_\alpha(1+k)^{N(\alpha)}(1+k')^{N'(\alpha)}.$$

On the other hand the formula (4) says that

$$|C_{kk'}(\chi)| \leqslant \frac{p_{N(a)+2,N'(a)+2}(\chi)}{(1+k)^{N(a)+2}(1+k')^{N'(a)+2}}.$$

Combining these two formulas one can check that

$$\sum_{kk'=0}^{\infty} |C_{kk'}(\chi)|\, q_a\big(\boldsymbol{B}(h_k, h_{k'})\big) \leqslant \left(\frac{\pi^2}{6}\right)^2 K_a p_{N(a)+2,N'(a)+2}(\chi), \tag{5}$$

where $\left(\frac{\pi^2}{6}\right)^2 = \sum_{kk'=0}^{\infty} \frac{1}{(1+k)^2(1+k')^2}$. This inequality holds for any $a \in A$. Therefore (L is assumed to be sequentially complete) the series

$$\overset{\otimes}{\boldsymbol{B}}(\chi) \overset{\text{df}}{=} \sum_{kk'=0}^{\infty} C_{kk'}(\chi)\boldsymbol{B}(h_k, h_{k'}) \tag{6}$$

is convergent. It is seen that this formula defines a linear mapping

$$\overset{\otimes}{\boldsymbol{B}}\colon S(\boldsymbol{R}^2) \to L.$$

This mapping is continuous. Indeed by virtue of (5) and (1) we have

$$q_a\big(\overset{\otimes}{\boldsymbol{B}}(\chi)\big) \leqslant \left(\frac{\pi^2}{6}\right)^2 K_a p_{N(a)+2,N'(a)+2}(\chi).$$

Let $\varphi, \psi \in S(\boldsymbol{R}^1)$. Then $C_{kk'}(\varphi\otimes\psi) = C_k(\varphi)C_{k'}(\psi)$ and equation (6) shows that $\overset{\otimes}{\boldsymbol{B}}(\varphi\otimes\psi) = \boldsymbol{B}(\varphi, \psi)$. This completes the proof. ■

3. $L(D)$-valued distributions. Let D be a dense linear subset of a Hilbert space H. Symbol $L(D)$ will denote the set of all (in general unbounded) operators acting in H and such that:

$$D_A = D, \qquad A\, D \subset D,$$
$$D_{A^*} \supset D \quad \text{and} \quad A^* D \subset D,$$

where D_A is the domain of an operator A. For any $A \in L(D)$ the domain D_{A^*} is dense in H and therefore the operator A is preclosed.

It is seen that $L(D)$ is a $*$-algebra. It means that $A+B$, λA, AB and $A^+ \overset{\text{df}}{=} A^*|_D$ belong to $L(D)$ for any $A, B \in L(D)$, $\lambda \in \boldsymbol{C}^1$.

Let $u \in D$. Then the mapping

$$L(D) \ni A \to (u|Au) \in \boldsymbol{C}^1 \tag{7}$$

is a linear functional on $L(D)$. The topology of $L(D)$ induced by the family of all functionals of the form (7) is called weak topology. It can be proved that $L(D)$ provided with the weak topology is a locally convex topological vector space and that the mappings

$$L(D) \ni A \to A^+ \in L(D),$$
$$L(D) \ni A \to AB \in L(D),$$
$$L(D) \ni A \to BA \in L(D),$$

(where B is a fixed element of $L(D)$) are continuous.

A sequence (u_n) of elements of H is called (D)-fundamental if $u_n \in D_{\bar{A}}$ (where $\bar{A}$ is the closure of an operator A) and the sequence $(\bar{A}u_n)$ is convergent for any $A \in L(D)$. Any (D)-fundamental sequence (u_n) is convergent (since $I|_D \in L(D)$). Moreover $\lim u_n \in D_{\bar{A}}$ and

$$\lim \bar{A}u_n = \bar{A}\lim u_n \tag{8}$$

for any $A \in L(D)$.

Let $\tilde{D}$ be the smallest linear subset of the Hilbert space H such that:

1° $D \subset \tilde{D} \subset D_{\bar{A}}$ for any $A \in L(D)$,

2° $\lim u_n \in \tilde{D}$ for any (D)-fundamental sequence (u_n) of elements of $\tilde{D}$.

For any $B \in L(D)$ we put

$$\tilde{B} \overset{\text{df}}{=} \bar{B}|_{\tilde{D}}. \tag{9}$$

It is seen that $B \subset \tilde{B} \subset \bar{B}$. We are going to prove

THEOREM 2.

1° *$\tilde{B} \in L(\tilde{D})$ for any $B \in L(D)$*

2° *The mapping*

$$L(D) \ni B \to \tilde{B} \in L(\tilde{D})$$

is a homomorphism of the $*$-algebras.

3° $\tilde{\tilde{D}} = \tilde{D}$.

Proof. Let D_1 be the set consisting of all elements $u \in \tilde{D}$ such that

$$\bar{B}u \in \tilde{D}, \tag{10}$$
$$\overline{AB}u = \bar{A}\bar{B}u, \tag{11}$$
$$\overline{A+B}u = \bar{A}u + \bar{B}u, \tag{12}$$
$$u \in D_{B^*}, \tag{13}$$
$$B^* u = \overline{B^+}u \tag{14}$$

for any $A, B \in L(D)$. It is seen that $D \subset D_1 \subset \tilde{D}$. Let (u_n) be a (D)-fundamental sequence of elements of D_1. Then

a) $u_\infty = \lim u_n \in \tilde{D}$ (since (u_n) is a (D)-fundamental sequence of elements of $\tilde{D}$).

b) The sequence $(\overline{AB}u_n)$ is convergent for any $A, B \in L(D)$ and equation (11) shows that $(\bar{B}u_n)$ is a (D)-fundamental sequence. By using (8) one can see that $\bar{B}u_\infty = \lim \bar{B}u_n \in \tilde{D}$ (as a limit of a (D)-fundamental sequence of elements $\tilde{D}$).

c) Taking $n \to \infty$ in the both sides of equations

$$\overline{AB}\, u_n = \bar{A}\bar{B}\, u_n,$$

$$\overline{A+B}\, u_n = \bar{A}\, u_n + \bar{B}\, u_n$$

we have (see (8)):

$$\overline{AB}\, u_\infty = \bar{A}\bar{B}\, u_\infty,$$

$$\overline{A+B}\, u_\infty = \bar{A}\, u_\infty + \bar{B}\, u_\infty.$$

d) Taking $n \to \infty$ in the both sides of equation $B^* u_n = \overline{B^+}u$ and remembering that B^* is a closed operator we obtain $u_\infty \in D_{B^*}$ and $B^* u_\infty = \overline{B^+} u_\infty$.

The obtained results imply that $u_\infty \in D_1$. We have proved that any (D)-fundamental sequence of elements of D_1 is convergent to an element of D_1. Now the definition of $\tilde{D}$ says that $D_1 = \tilde{D}$. Therefore the equations (10)–(14) hold for any $u \in \tilde{D}$. Let us notice that $B^* = \tilde{B}^*$. Now the statement 1° of the theorem follows immediately from (10), (13) and (14). The statement 2° is a simple conclusion of (11), (12) and (14).

We are going to prove the statement 3°. Let (u_n) be a $(\tilde{D})$-fundamental sequence of elements of $\tilde{D}$. It is sufficient to prove that $\lim u_n \in \tilde{D}$. The sequence $(\tilde{A}u_n)$ is convergent for any $A \in L(D)$ (since $\tilde{A} \in L(\tilde{D})$). But $\tilde{A}u_n = \bar{A}u_n$. It means that (u_n) is a (D)-fundamental sequence and therefore $\lim u_n \in \tilde{D}$. ∎

Briefly speaking the Theorem 2 says that all operators from $L(D)$ can be extended to the larger domain $\tilde{D}$ and that this extention preserves all algebraic relations. However, one can easily prove that (excluding the trivial case $\tilde{D} = D$) mapping $B \to \tilde{B}$ is not continuous. Therefore the following theorem needs a proof.

THEOREM 3. *Let*

$$\mathbf{A}: \mathcal{S}(\mathbf{R}^n) \to L(D) \tag{15}$$

be a $L(D)$-valued distribution. Then the mapping

$$\tilde{\mathbf{A}}: \mathcal{S}(\mathbf{R}^n) \to L(\tilde{D}), \tag{16}$$

where $\tilde{\mathbf{A}}(\varphi) = \mathbf{A}(\varphi)^\sim$ for any $\varphi \in \mathcal{S}(\mathbf{R}^n)$, is continuous, i. e. *$\tilde{\mathbf{A}}$ is a $L(\tilde{D})$-valued distribution.*

Proof. For any $u \in \tilde{D}$ we put

$$f_u(\varphi) = \big(u \,|\, \mathbf{A}(\varphi)^\sim u\big) \qquad \varphi \in \mathcal{S}(\mathbf{R}^n)$$

f_u is a linear functional on $\mathcal{S}(\mathbf{R}^n)$. Let D_2 be the subset of H consisting of all vectors $u \in \tilde{D}$ such that the functional f_u is continuous. The continuity of (15) means that $D \subset D_2$.

Let (u_n) be (D)-fundamental sequence of elements of D_1. By virtue of (8) one can see that for any $A \in L(D)$:

$$\lim (u_n | \bar{A}u_n) = (u_\infty | \bar{A}u_\infty),$$

where $u_\infty = \lim u_n$. Setting $A = \mathbf{A}(\varphi)$ we get:

$$\lim f_{u_n}(\varphi) = f_{u_\infty}(\varphi)$$

for any $\varphi \in \mathcal{S}(\mathbf{R}^n)$. The functionals f_{u_n} are continuous since $u_n \in D_2$. Therefore we have the sequence of continuous functionals on $\mathcal{S}(\mathbf{R}^n)$, which is convergent at any point $\varphi \in \mathcal{S}(\mathbf{R}^n)$. In this situation the limit functional f_{u_∞} has to be continuous (cf. [1] Chapter II § 1, Theorem 17). It means that $u_\infty \in D_2$. Taking into account the definition of $\tilde{D}$ one can see that $D_2 = \tilde{D}$ i. e. for any $u \in \tilde{D}$ the functional

$$\mathcal{S}(\mathbf{R}^n) \ni \varphi \to \big(u \,|\, \tilde{A}(\varphi)u\big) \in C^1 \tag{17}$$

is continuous. In order to complete the proof let us remind that the mapping (16) is continuous if and only if the functionals (17) are continuous for all $u \in \tilde{D}$. ∎

4. Strong topology in $L(D)$. Let D be a dense linear subset of a Hilbert space H. The topology of $L(D)$ introduced by the family of mappings of the form

$$L(D) \ni A \to BAu \in H,$$

$$L(D) \ni A \to BA^+u \in H,$$

where $u \in D$ and $B \in L(D)$ is called a strong topology. This topology is stronger than the weak topology introduced before. It means that any strongly continuous mapping into $L(D)$ is the more weakly continuous.

The following theorem shows that these two topologies are equivalent from the point view of the operator-valued distribution theory.

THEOREM 4. *Let*

$$\boldsymbol{A}\colon \mathcal{S}(\boldsymbol{R}^n)\to L(D)$$

be a $L(D)$-valued distribution. Then $\boldsymbol{A}$ is a strongly continuous mapping.

Proof. It is sufficient to show that for any $u\in D$ and $B\in L(D)$ the mappings

$$\mathcal{S}(\boldsymbol{R}^n)\ni\varphi\to B\boldsymbol{A}(\varphi)u\in H, \tag{18}$$

$$\mathcal{S}(\boldsymbol{R}^n)\ni\varphi\to B\boldsymbol{A}(\varphi)^+u\in H \tag{19}$$

are continuous.

Assume that $\lim\varphi_n=\varphi$ and $\lim B\boldsymbol{A}(\varphi_n)u=v$ for a sequence (φ_n) of test functions. Then for any $u_1\in D$:

$$\lim\big(u_1|BA(\varphi_n)u\big)=(u_1|v).$$

On the other hand the mapping $\boldsymbol{A}$ is weakly continuous and

$$\lim\big(u_1|B\boldsymbol{A}(\varphi_n)u\big)=\lim\big(B^+u_1|\boldsymbol{A}(\varphi_n)u\big)$$
$$=\big(B^+u_1|\boldsymbol{A}(\varphi)u\big)=\big(u_1|B\boldsymbol{A}(\varphi)u\big).$$

It implies that $B\boldsymbol{A}(\varphi)\,u=v$. This way we have proved that the mapping (18) has a closed graph. By using the closed graph theorem (see for example [1] Chapter II § 3 theorem 4) we conclude that (18) is a continuous mapping. Similarly one can prove the continuity of (19). ■

It can be easily seen that (excluding the trivial case $D=H$) the space $L(D)$ (provided with a weak topology) is not sequentially complete. The situation becomes better if one considers the strong topology.

THEOREM 5. *Let $D=\tilde{D}$. Then the space $L(D)$ provided with the strong topology is sequentially complete.*

Proof. Assume that $(A_n)_{n=1,2,\dots}$ (where $A_n\in L(D)$) is a Cauchy sequence with respect to the strong topology. Then $(BA_nu)_{n=1,2,\dots}$ and $(BA_n^+u)_{n=1,2,\dots}$ are convergent in H for any $B\in L(D)$ and $u\in D$ (the Hilbert space H is complete and any Cauchy sequence of elements of H is convergent). It means that $(A_nu)_{n=1,2,\dots}$ and $(A_n^+u)_{n=1,2,\dots}$ are (D)-fundamental sequences. For any $u\in D$ we put:

$$Au\overset{\text{df}}{=}\lim A_nu. \tag{20}$$

Then

$$Au\in D \tag{21}$$

as a limit of a (D)-fundamental sequence of elements of $D=\tilde{D}$. Let $v\in D$. Then

$$(v|\lim A_n^+u)=\lim(v|A_n^+u)=\lim(A_nv|u)=(Av|u).$$

Therefore $u\in D_{A^*}$ and

$$A^*u=\lim A_n^+u\in D \tag{22}$$

because $(A_n^+u)_{n=1,2,\dots}$ is also (D)-fundamental sequence of elements of $D=\tilde{D}$. Relations (21) and (22) show now that $A\in L(D)$.

By using (8) we get

$$\lim BA_nu=BAu\quad\text{ and }\quad\lim BA_n^+u=BA^+u$$

for any $B\in L(D)$ and $u\in D$. It means that the sequence (A_n) is strongly convergent to the element $A\in L(D)$. ■

5. Final results. By virtue of Theorems 1, 4 and 5 we immediately get:

THEOREM 6. *Let D be a dense linear subset of a Hilbert space H such that $\tilde{D}=D$. Then $L(D)$ satisfies the theorem on kernel.*

For the general case (without assumption that $\tilde{D}=D$) we have:

THEOREM 7. *Let D be a dense linear subset of a Hilbert space H and let*

$$\boldsymbol{B}\colon \mathcal{S}(\boldsymbol{R}^n)\times\mathcal{S}(\boldsymbol{R}^m)\to L(D)$$

be a separately continuous bilinear mapping. Then there exists a $L(\tilde{D})$-valued distribution

$$\overset{\otimes}{\tilde{\boldsymbol{B}}}\colon \mathcal{S}(\boldsymbol{R}^{n+m})\to L(\tilde{D})$$

such that

$$\boldsymbol{B}(\varphi,\psi)\subset\overset{\otimes}{\tilde{\boldsymbol{B}}}(\varphi\otimes\psi)$$

for any $\varphi\in\mathcal{S}(\boldsymbol{R}^n)$ and $\psi\in\mathcal{S}(\boldsymbol{R}^m)$.

This theorem follows immediately from the Theorems 3 and 6.

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