Boost-invariant flow from string theory – near and far from equilibrium physics and AdS/CFT

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Idea

- heavy ion collisions @ RHIC - strongly coupled quark-gluon plasma (QGP)
- fully dynamical process - need for a new tool
- idea: exchange QCD in favor of $\mathcal{N} = 4$ SYM and use the gravity dual
- there are differences
  - SUSY
  - conformal symmetry at the quantum level
  - no confinement...
- ... but not very important at high temperature
• RHIC suggests that QGP behaves as an almost perfect fluid
• there has been an enormous progress in understanding QGP hydrodynamics with the AdS/CFT
• can the AdS/CFT be used to shed light on far from equilibrium part of the QGP dynamics?
• maybe, but only at $\lambda \gg 1$!
• let’s focus on the boost-invariant flow and use the AdS/CFT to learn about some near and far from equilibrium physics!
• one-dimensional expansion along the collision axis $x^1$

• natural coordinates
  - proper time $\tau$ and rapidity $y$
  - $x^0 = \tau \cosh y$, $x^1 = \tau \sinh y$

• **boost invariance** (no rapidity dependence)
Gauge-gravity duality is an equivalence between

\[ \mathcal{N} = 4 \text{ Supersymmetric Yang-Mills in } \mathbb{R}^{1,3} \]

- strong coupling
- non-perturbative results
- gauge theory operators

Superstrings in curved AdS\(_5 \times S^5\) 10D spacetime

- (super)gravity regime
- classical behavior
- supergravity fields

AdS/CFT dictionary relates

energy-momentum tensor of \( \mathcal{N} = 4 \) SYM to 5D AdS metric
Gravity dual to the boost-invariant flow

- The energy-momentum tensor is specified by $\epsilon(\tau)$

$$T^{\mu\nu} = \text{diag} \left\{ \epsilon(\tau), -\frac{1}{\tau^2} \epsilon(\tau) - \frac{1}{\tau} \epsilon'(\tau), \epsilon(\tau) + \frac{1}{2} \tau \epsilon'(\tau) \right\}$$

- This suggests the metric Ansatz for the gravity dual

$$ds^2 = -e^{a(\tau,z)}d\tau^2 + \tau^2 e^{b(\tau,z)}dy^2 + e^{c(\tau,z)}dx_\perp^2 + dz^2$$

- Einstein equations

$$G_{AB} = R_{AB} - \frac{1}{2} R \cdot g_{AB} - 6 g_{AB} = 0$$

cannot be solved exactly (→ numerics)

- However, there are two regimes

$$\tau \gg 1 \text{ or } \tau \approx 0$$

where analytic calculations can be done
\[ \tau \gg 1 \] regime – hydrodynamics
Holographic reconstruction of space-time from $\epsilon(\tau) \sim \frac{1}{\tau^s}$

- Einstein eqns $\mathcal{G}_{AB}$ can be solved order by order in $z^2$, e.g.

$$a(\tau, z) = 0 + 0 \cdot z^2 + a_4(\tau) \cdot z^4 + a_6(\tau) \cdot z^6 + \ldots$$

where $a_4(\tau) = -\epsilon(\tau)$, $a_6(\tau) = -\frac{1}{4\tau} \epsilon'(\tau) - \frac{1}{12} \epsilon''(\tau)$, \ldots

- assuming $\epsilon(\tau) \sim \frac{1}{\tau^s}$ and choosing in each $a_{2k}(\tau)$ the leading contribution one ends up with $a(\tau, z) = a_{\text{scaling}} (z \cdot \tau^{-s/4})$, etc

- this reduces $\mathcal{G}_{AB}$ to solvable set of ODEs and then requiring regularity of $\mathcal{R}_{ABCD} \mathcal{R}^{ABCD}$ evaluated on $a, b, c_{\text{scaling}}$ fixes $s$ to be $\frac{4}{3}$ leading to $\epsilon(\tau) \sim \frac{1}{\tau^{4/3}}$

- $\epsilon(\tau) \sim \frac{1}{\tau^{4/3}}$ turns out to be the solution of hydrodynamics
Hydrodynamics from ground up

Basics

- long-wavelength effective theory
- vast reduction of \# degrees of freedom
  - velocity \( u^\mu(x) \) constrained by \( u^\mu u_\mu = -1 \)
  - temperature \( T(x) \)
- slow changes \( \rightarrow \) gradient expansion

gradient expansion

- definition of the energy-momentum tensor
  \[
  T^{\mu\nu} = \epsilon \cdot u^\mu u^\nu + p \cdot \Delta^{\mu\nu} - \eta \cdot \left( \Delta^{\mu\lambda} \nabla_\lambda u^\nu + \Delta^{\nu\lambda} \nabla_\lambda u^\mu - \frac{2}{3} \Delta^{\mu\nu} \nabla^\lambda u_\lambda \right) + \ldots
  \]
- EOMs \( \nabla_\mu T^{\mu\nu} = 0 + \) equation of state (e.g. \( \epsilon = 3p \))
Hydrodynamics and $\epsilon(\tau)$

Perfect hydrodynamics

- in conformal boost invariant hydrodynamics
  
  \[ \epsilon(\tau) \sim T(\tau)^4, \quad u^\mu = 1 \cdot [\partial_\tau]^\mu, \quad \eta_{\mu\nu} = \text{diag}\{-1, \tau^2, 1, 1\} \]

- perfect hydro ($\nabla_\mu T^{\mu\nu} = 0$ for $T^{\mu\nu} = \epsilon \cdot u^\mu u^\nu + p \cdot \Delta^{\mu\nu}$) gives
  
  \[ \partial_\tau \epsilon(\tau) = -\frac{\epsilon(\tau) + p(\tau)}{\tau} \]

- which together with $\epsilon = 3p$ leads to \( \epsilon \sim \frac{1}{\tau^{4/3}} \)

Gradient expansion

- remainder: in hydro the expansion parameter is $\frac{1}{L \cdot T}$

- in this setting $T \sim \tau^{-1/3}$, $L^{-1} \sim \nabla u = \tau^{-1}$, so $\frac{1}{L \cdot T} \sim \frac{1}{\tau^{2/3}}$

- one should expect the general structure of $\epsilon(\tau)$ of the form

  \[ \epsilon(\tau) \sim \frac{1}{\tau^{4/3}} \left\{ \#0 + \frac{1}{\tau^{2/3}} \#1 + \frac{1}{\tau^{4/3}} \#2 + \ldots \right\} \]
Boost-invariant flow and gradient expansion

Reminder:

\[
d s^2 = -e^{a(\tau, z)} d\tau^2 + \tau^2 e^{b(\tau, z)} dy^2 + e^{c(\tau, z)} dx_\perp^2 + dz^2
\]

Gravitational gradient expansion:

\[
a(\tau, z) = a_0 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{2/3}} a_1 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{4/3}} a_2 \left( \frac{z}{\tau^{1/3}} \right) + \ldots
\]

\[
b(\tau, z) = b_0 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{2/3}} b_1 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{4/3}} b_2 \left( \frac{z}{\tau^{1/3}} \right) + \ldots
\]

\[
c(\tau, z) = c_0 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{2/3}} c_1 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{4/3}} c_2 \left( \frac{z}{\tau^{1/3}} \right) + \ldots
\]

\[
R^2(\tau, z) = \mathcal{R}_0^2 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{2/3}} \mathcal{R}_1^2 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{4/3}} \mathcal{R}_2^2 \left( \frac{z}{\tau^{1/3}} \right) + \ldots
\]

This is AdS counterpart of hydrodynamics

\[
\epsilon(\tau) = \left( \frac{N_c^2}{2\pi^2} \right) \frac{1}{\tau^{4/3}} \left\{ 1 - 2\eta_0 \frac{1}{\tau^{2/3}} + \left[ \frac{3}{2} \eta_0^2 - \frac{2}{3} (\eta_0 \tau_1^0 - \lambda_1^0) \right] \frac{1}{\tau^{4/3}} + \ldots \right\}
\]
Further developments and why AdS/CFT is useful

Further developments

• corrections to the transport coefficients from finite $\lambda$ and $N_c$

\[
S = \frac{1}{2l_p^3} \int \det g \left\{ \mathcal{R} + \frac{12}{L^2} + \gamma \cdot L^2 \text{Weyl}^2 + \delta \cdot L^6 \text{Weyl}^4 \right\}
\]

• apparent, event horizons and slow evolution

Why useful?

• transport properties at strong coupling

• inspired the correct formulation of second order hydro

• implications in GR as well (fluid/gravity correspondence)
\( \tau \approx 0 \) regime – dynamics far from equilibrium
Initial conditions and early times expansion of $\epsilon(\tau)$

- warp factors can be solved near the boundary given $\epsilon(\tau)$

$$a(\tau, z) = -\epsilon(\tau) \cdot z^4 + \left\{ -\frac{\epsilon'(\tau)}{4\tau} - \frac{\epsilon''(\tau)}{12} \right\} \cdot z^6 + \ldots$$

- for $\epsilon(\tau) = \epsilon_0 + \epsilon_1 \tau + \epsilon_2 \tau^2 + \epsilon_3 \tau^3 + \epsilon_4 \tau^4 + \epsilon_5 \tau^5 + \ldots$

  all $\epsilon_{2k+1}$ must vanish, otherwise $a(0, z) \rightarrow \infty$

- setting $\tau$ to zero in $a(\tau, z)$ for

$$\epsilon(\tau) = \epsilon_0 + \epsilon_2 \tau^2 + \epsilon_4 \tau^4 + \ldots$$

gives

$$a(0, z) = a_0(z) = \epsilon_0 \cdot z^4 + \frac{2}{3} \epsilon_2 \cdot z^6 + \left( \frac{\epsilon_4}{2} - \frac{\epsilon_0^2}{6} \right) \cdot z^8 + \ldots$$

- it defines map between initial profiles in the bulk and $\epsilon(\tau)$
Resummation of the energy density

• energy density power series @ \( \tau = 0 \)

\[
\epsilon(\tau) = \epsilon_0 + \epsilon_2 \tau^2 + \ldots + \epsilon_{2N_{\text{cut}}} \tau^{2N_{\text{cut}}} + \ldots
\]

has a finite radius of convergence and thus

a resummation is needed

• presumably the simplest can be given by Pade approximation

\[
\epsilon_{\text{approx}}(\tau)^3 = \frac{\epsilon^{(0)}_U + \epsilon^{(2)}_U \tau^2 + \ldots + \epsilon^{(N_{\text{cut}}-2)}_U \tau^{N_{\text{cut}}-2}}{\epsilon^{(0)}_D + \epsilon^{(2)}_D \tau^2 + \ldots + \epsilon^{(N_{\text{cut}}-2)}_D \tau^{N_{\text{cut}}+2}}
\]

which uses the uniqueness of the asymptotic behavior

\[
\epsilon \sim \frac{1}{\tau^{4/3}}
\]
• A nice example of initial data in the bulk is given by

\[ a(0, z) = b(0, z) = 2 \log \left\{ \cos \frac{\pi}{2} z^2 \right\} \quad \text{and} \quad c(\tau, z) = 2 \log \left\{ \cosh \frac{\pi}{2} z^2 \right\} \]

leading to the following \( \epsilon(\tau) \) and \( \Delta p(\tau) = 1 - \frac{p_{\parallel}(\tau)}{p_{\perp}(\tau)} \).
Results:

- **AdS/CFT is indispensable** not only near equilibrium
- Gauge/gravity duality may serve as a definition of *strongly coupled non-equilibrium gauge theory*
- transport properties of various plasmas at strong coupling
- estimates of thermalization time

Open questions:

- **Towards colliding shock-waves**
- applications of non-conformal gauge/gravity dualities