A simple construction of twistor forms (conformal Yano-Killing tensors) in anti-de Sitter spacetime

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Conformal Yano–Killing tensors

Let $Q_{\mu\nu}$ be a skew-symmetric tensor field. Contracting the Weyl tensor $W^{\mu\nu\kappa\lambda}$ with $Q_{\mu\nu}$ we obtain a natural object which can be integrated over two-surfaces. The result does not depend on the choice of the surface if $Q_{\mu\nu}$ fulfills the following condition introduced by Penrose

$$Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \eta_{\sigma[\lambda} Q_{\kappa]}{}^{\delta}{}_{;\delta} = 0\,. \tag{1}$$

one can rewrite equation (1) in a generalized form for n-dimensional spacetime with metric $g_{\mu\nu}$:

$$Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \frac{3}{n-1} g_{\sigma[\lambda} Q_{\kappa]}{}^{\delta}{}_{;\delta} = 0 \tag{2}$$

or in the equivalent form:

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1}\left(g_{\sigma\lambda} Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda} Q_{\sigma)}{}^{\mu}{}_{;\mu}\right)\,. \tag{3}$$

Let us define

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) := Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{n-1}\left(g_{\sigma\lambda} Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda} Q_{\sigma)}{}^{\mu}{}_{;\mu}\right) \tag{4}$$

Definition 1. *A skew-symmetric tensor $Q_{\mu\nu}$ is a* conformal Yano–Killing tensor *(or simply CYK tensor) for the metric g iff* $\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) = 0$.

The CYK tensor is a natural generalization of the Yano tensor with respect to the conformal rescalings. More precisely, for any positive scalar function $\Omega > 0$ and for a given metric $g_{\mu\nu}$ we obtain:

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q, g) = \Omega^{-3}\mathcal{Q}_{\lambda\kappa\sigma}(\Omega^3 Q, \Omega^2 g) \,. \tag{5}$$

The formula (5) and the above definition of CYK tensor gives the following

Theorem 1. *If $Q_{\mu\nu}$ is a CYK tensor for the metric $g_{\mu\nu}$ than $\Omega^3 Q_{\mu\nu}$ is a CYK tensor for the conformally rescaled metric $\Omega^2 g_{\mu\nu}$.*

It is interesting to notice, that a tensor $A_{\mu\nu}$ — a "square" of the CYK tensor $Q_{\mu\nu}$ defined as follows:

$$A_{\mu\nu} := Q_\mu{}^\lambda Q_{\lambda\nu}$$

fulfills the following equation:

$$A_{(\mu\nu;\kappa)} = g_{(\mu\nu}A_{\kappa)} \quad \text{with} \quad A_\kappa = \frac{2}{n-1} Q_\kappa{}^\lambda Q_\lambda{}^\delta{}_{;\delta} \tag{6}$$

which simply means that the symmetric tensor $A_{\mu\nu}$ is a conformal Killing tensor. This can be also described by the following

Theorem 2. *If $Q_{\mu\nu}$ is a skew-symmetric conformal Yano–Killing tensor than $A_{\mu\nu} := Q_\mu{}^\lambda Q_{\lambda\nu}$ is a symmetric conformal Killing tensor.*

Remark CYK tensor is a solution of the following conformally invariant equation:

$$\left(\Box + \frac{1}{6}\mathcal{R}\right) Q = \frac{1}{2} W(Q, \cdot) \qquad (\mathbf{n{=}4})$$

$\mathcal{R} := R_{\mu\nu} g^{\mu\nu}$ – scalar curvature, $R_{\mu\nu}$ – symmetric Ricci tensor.

Moreover, if Q is a CYK tensor and the metric is Einstein then

$$K^{\mu} := Q^{\mu\lambda}{}_{;\lambda}$$

is a Killing vector field.

More precisely, one can show

$$K_{(\mu;\nu)} = \frac{n-1}{n-2} R_{\sigma(\mu} Q_{\nu)}{}^{\sigma}$$

which implies the following

Theorem 3. *If $g_{\alpha\beta}$ is an Einstein metric, i.e. $R_{\mu\nu} = \lambda g_{\mu\nu}$, then K^{μ} is a Killing vectorfield.*

Let us restrict ourselves to **four-dimensional** manifold ($n = 4$). The Hodge-dual of $Q_{\mu\nu}$ defined as follows

$$*Q_{\kappa\lambda} = \frac{1}{2}\varepsilon_{\kappa\lambda}{}^{\mu\nu}Q_{\mu\nu} \; .$$

gives also a two-form. Multiplying CYK equation

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1}\left(g_{\sigma\lambda}K_\kappa - g_{\kappa(\lambda}K_{\sigma)}\right)$$

by $\frac{1}{2}\varepsilon^{\alpha\beta\lambda\kappa}$ we get:

$$*Q_{\alpha\beta;\sigma} = \frac{2}{3}g_{\sigma[\alpha}\chi_{\beta]} + \frac{1}{3}\varepsilon_{\alpha\beta\sigma\kappa}K^\kappa \; , \tag{7}$$

where $\chi_\mu := *Q^\nu{}_{\mu;\nu}$ and $K_\mu = Q^\nu{}_{\mu;\nu}$. Multiplying the above equality by $\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}$, we obtain a similar formula:

$$Q_{\mu\nu;\sigma} = \frac{2}{3}g_{\sigma[\mu}K_{\nu]} - \frac{1}{3}\varepsilon_{\mu\nu\sigma\beta}\chi^\beta \; . \tag{8}$$

Finally, symmetrization of indices α and σ in (7) gives:

$$*Q_{\alpha\beta;\sigma} + *Q_{\sigma\beta;\alpha} = \frac{2}{3}\left(g_{\sigma\alpha}\chi_\beta - g_{\beta(\alpha}\chi_{\sigma)}\right) ,$$

which implies the following

Theorem 4. *$Q_{\mu\nu}$ is a CYK tensor iff $*Q_{\mu\nu}$ is a CYK tensor.*

Pullback of CYK tensor to submanifold of codimension one

Let N be a differential manifold of dimension $n+1$ and $\overset{(n+1)}{g}$ its metric tensor (the signature of the metric plays no role). Moreover, we assume that there exists a coordinate system (x^A), where $A = 0, \ldots, n$, in which $\overset{(n+1)}{g}$ takes the following form:

$$\overset{(n+1)}{g} = f(u)h + s\mathrm{d}u^2, \tag{9}$$

where s is equal to 1 or -1, $u \equiv x^n$, f is a certain function, and h is a certain tensor, which does not depend on u. The metric (9) possesses a conformal Killing vector field $\sqrt{f}\partial_u$. Tensor $f(u)h$ is a metric tensor on a submanifold $M := \{u = \text{const.}\}$. We will denote it by $\overset{(n)}{g}$. We will distinguish all objects associated with the metric $\overset{(n)}{g}$ by writing (n) above their symbols. Similar notation will be used for objects associated with the metric $\overset{(n+1)}{g}$.

It turns out that:

Theorem 5. *If Q is a CYK tensor of the metric $\overset{(n+1)}{g}$ in N, then its pullback to the submanifold M is a CYK tensor of the metric $\overset{(n)}{g}$.*

Theorem 5 we apply to (anti-)de Sitter spacetime:

Let N be a five-dimensional differential manifold with a global coordinate system (y^A). We define the metric tensor η on the manifold N by the formula:

$$\begin{aligned} \eta &= \eta_{AB}\mathrm{d}y^A \otimes \mathrm{d}y^B = \\ & s\mathrm{d}y^0 \otimes \mathrm{d}y^0 + \mathrm{d}y^1 \otimes \mathrm{d}y^1 + \mathrm{d}y^2 \otimes \mathrm{d}y^2 + \mathrm{d}y^3 \otimes \mathrm{d}y^3 - \mathrm{d}y^4 \otimes \mathrm{d}y^4 \end{aligned} \tag{10}$$

Let $\tilde{M}$ be a submanifold of N defined by:

$$\eta_{AB}y^A y^B = sl^2. \tag{11}$$

The metric η restricted to $\tilde{M}$ is just the (anti-)de Sitter metric.

For $s = 1$ a parametrization of $\tilde{M}$ takes the following form:

$$y^0 = l\sqrt{1-\bar{r}^2}\cosh\bar{t}, \tag{12}$$

$$y^1 = l\bar{r}\sin\theta\cos\phi, \tag{13}$$

$$y^2 = l\bar{r}\sin\theta\sin\phi, \tag{14}$$

$$y^3 = l\bar{r}\cos\theta, \tag{15}$$

$$y^4 = l\sqrt{1-\bar{r}^2}\sinh\bar{t}. \tag{16}$$

If $s = -1$, the analogous formulae are the following:

$$y^0 = l\sqrt{1+\bar{r}^2}\cos\bar{t}, \tag{17}$$

$$y^1 = l\bar{r}\sin\theta\cos\phi, \tag{18}$$

$$y^2 = l\bar{r}\sin\theta\sin\phi, \tag{19}$$

$$y^3 = l\bar{r}\cos\theta, \tag{20}$$

$$y^4 = l\sqrt{1+\bar{r}^2}\sin\bar{t}. \tag{21}$$

Let us notice that functions l, $\bar{t}$, $\bar{r}$, θ and ϕ can be considered as the local coordinate system on N. Substituting formulae (12)–(16) or (17)–(21) into definition (10) of the metric η we get:

$$\eta = s\mathrm{d}l^2 + l^2\left[(-1+s\bar{r}^2)\mathrm{d}\bar{t}^2 + \frac{1}{1-s\bar{r}^2}\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\Omega_2\right]. \tag{22}$$

Identifying the (anti-)de Sitter spacetime with the submanifold $\tilde{M}$ enables one to find all Killing vector fields of the metric $\tilde{g}$. The vector fields

$$L_{AB} := y_A \frac{\partial}{\partial y^B} - y_B \frac{\partial}{\partial y^A}$$

(where $y_A := \eta_{AB} y^B$) are the Killing fields of the metric η.

For $s = 1$ we get:

$$L_{40} = -\frac{\partial}{\partial \bar{t}}\,, \tag{23}$$

$$L_{i4} = \frac{x^i}{\sqrt{1-\bar{r}^2}} \cosh \bar{t} \frac{\partial}{\partial \bar{t}} + \sqrt{1-\bar{r}^2} \sinh \bar{t} \frac{\partial}{\partial x^i}\,, \tag{24}$$

$$L_{i0} = -\frac{x^i}{\sqrt{1-\bar{r}^2}} \sinh \bar{t} \frac{\partial}{\partial \bar{t}} - \sqrt{1-\bar{r}^2} \cosh \bar{t} \frac{\partial}{\partial x^i}\,, \tag{25}$$

$$L_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}\,, \tag{26}$$

where in the coordinate system on N instead of spherical coordinates $\bar{r}, \theta, \phi$ we use Cartesian $x^k := \frac{y^k}{l} = \bar{r} n^k, k = 1, 2, 3$.

If $s = -1$ in coordinate system $(l, \bar{t}, x^k)$ we have:

$$L_{40} = \frac{\partial}{\partial \bar{t}}, \tag{27}$$

$$L_{i4} = \frac{x^i}{\sqrt{1+\bar{r}^2}} \cos \bar{t} \frac{\partial}{\partial \bar{t}} + \sqrt{1+\bar{r}^2} \sin \bar{t} \frac{\partial}{\partial x^i}, \tag{28}$$

$$L_{i0} = -\frac{x^i}{\sqrt{1+\bar{r}^2}} \sin \bar{t} \frac{\partial}{\partial \bar{t}} + \sqrt{1+\bar{r}^2} \cos \bar{t} \frac{\partial}{\partial x^i}, \tag{29}$$

$$L_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}. \tag{30}$$

It is easy to notice that those fields are tangent to $\tilde{M}$ and therefore their restrictions to the submanifold are Killing fields of the induced metric. The fields defined on N as well as their restrictions to $\tilde{M}$ will be denoted by the same symbol L_{AB}. Restricting the fields L_{AB} to $\tilde{M}$ we get 10 linearly independent Killing fields of the metric $\tilde{g}$. This is the maximum number of the independent Killing fields the four-dimensional metric can have, so L_{AB} span the space of the Killing fields of the metric $\tilde{g}$.

Asymptotic anti-de Sitter spacetime

For asymptotic analysis let us change the radial coordinate in the anti-de Sitter metric as follows

$$z := \frac{l}{r + \sqrt{r^2 + l^2}}\,, \qquad \bar{r} = \frac{r}{l} = \frac{1 - z^2}{2z}\,,$$

which implies that

$$\tilde{g}_{\text{AdS}} = \frac{l^2}{z^2}\left[\mathrm{d}z^2 - \left(\frac{1 + z^2}{2}\right)^2 \mathrm{d}\bar{t}^2 + \left(\frac{1 - z^2}{2}\right)^2 \mathrm{d}\Omega_2\right]. \tag{31}$$

The above particular form of $\tilde{g}_{\text{AdS}}$ is well adopted to the so-called conformal compactification. More precisely, the metric g on the interior $\tilde{M}$ of a compact manifold M with boundary ∂M is said to be conformally compact if $g \equiv \Omega^2 \tilde{g}$ extends continuously (or with some degree of smoothness) as a metric to M, where Ω is a defining function for the scri $\mathscr{I} = \partial M$, i.e. $\Omega > 0$ on $\tilde{M}$ and $\Omega = 0$, $\mathrm{d}\Omega \neq 0$ on ∂M. In the case of AdS metric (31) we have

$$g_{\text{AdS}} = \Omega^2 \tilde{g}_{\text{AdS}}, \quad \text{where} \quad \Omega := \frac{z}{l}\,.$$

Four-dimensional asymptotic AdS spacetime metric $\tilde{g}$ assumes in canonical coordinates the following form:

$$\tilde{g} = \tilde{g}_{\mu\nu} \mathrm{d}z^{\mu} \otimes \mathrm{d}z^{\nu} = \frac{l^2}{z^2} \left(\mathrm{d}z \otimes \mathrm{d}z + h_{ab} \mathrm{d}z^a \otimes \mathrm{d}z^b \right) \tag{32}$$

and the three-metric h obeys the following asymptotic condition:

$$h = h_{ab} \mathrm{d}z^a \otimes \mathrm{d}z^b = \overset{(0)}{h} + z^2 \overset{(2)}{h} + z^3 \chi + O(z^4) \, . \tag{33}$$

Let us observe that the term χ vanishes for the pure AdS given by (31). Moreover, the terms $\overset{(0)}{h}$ and $\overset{(2)}{h}$ have the standard form

$$\overset{(0)}{h} = \frac{1}{4} (\mathrm{d}\Omega_2 - \mathrm{d}\bar{t}^2) \, , \tag{34}$$

$$\overset{(2)}{h} = -\frac{1}{2} (\mathrm{d}\Omega_2 + \mathrm{d}\bar{t}^2) \, . \tag{35}$$

For generalized (asymptotically locally) anti-de Sitter spacetimes tensors $\overset{(0)}{h}$ and $\overset{(2)}{h}$ need not to be conformally "trivial", i.e. in the form (34) and (35) respectively. Such more general situation has been considered e.g. by Anderson, Chruściel, Graham, Skenderis. Let us stress that in the general case only the induced metric $\overset{(0)}{h}$ may be changed freely beyond the conformal class, $\overset{(2)}{h}$ is always given by

$$\overset{(2)}{h}_{ab} = \frac{1}{4}\overset{(0)}{h}_{ab}\mathcal{R}\left(\overset{(0)}{h}\right) - \mathcal{R}_{ab}\left(\overset{(0)}{h}\right) . \tag{36}$$

Moreover, $\overset{(0)}{h}$ and χ form a symplectic structure on conformal boundary.
However, we assume the standard asymptotic AdS: The induced metric h on $\mathscr{I}$ is in the conformal class of the "Einstein static universe", i.e.

$$\overset{(0)}{h} = \exp(\omega)(\mathrm{d}\Omega_2 - \mathrm{d}\bar{t}^2) \tag{37}$$

for some smooth function ω. This implies that our $\mathscr{I}$ is a timelike boundary.

Functions y^A given by equations (17–21) and restricted to $\tilde{M}$ can be expressed in coordinate system $(z^\mu) \equiv (z^0, z^1, z^2, z^3) \equiv (\bar{t}, \theta, \phi, z)$ as follows

$$y^0 = \Omega^{-1} \frac{1+z^2}{2} \cos \bar{t} \,, \tag{38}$$

$$y^k = \Omega^{-1} \frac{1-z^2}{2} n^k \,, \tag{39}$$

$$y^4 = \Omega^{-1} \frac{1+z^2}{2} \sin \bar{t} \,, \tag{40}$$

where $k = 1, 2, 3$, and

$$n := \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$

is a radial unit normal in Euclidean three-space (identified with a point on a unit sphere parameterized by coordinates (θ, ϕ)).

Let us denote a (constant in ambient space) CYK tensor in AdS spacetime by

$$ {}^{[AB]}\tilde{Q} := l \mathrm{d}y^A \wedge \mathrm{d}y^B \, . $$

Coordinates y^A restricted to $\tilde{M}$, given by equations (38–40), lead to the following explicit formulae for two-forms ${}^{[AB]}\tilde{Q}$:

$$ {}^{[04]}\tilde{Q} = \frac{1}{4}\Omega^{-3}(1-z^4)\mathrm{d}\bar{t} \wedge \mathrm{d}z \, , \tag{41} $$

$$ \begin{aligned} {}^{[jk]}\tilde{Q} = {} & \frac{1}{4}\Omega^{-3}\left[(1-z^4)(n^j \mathrm{d}n^k - n^k \mathrm{d}n^j) \wedge \mathrm{d}z \right. \\ & \left. + z(1-z^2)^2 \mathrm{d}n^j \wedge \mathrm{d}n^k\right] , \end{aligned} \tag{42} $$

$$ \begin{aligned} {}^{[0k]}\tilde{Q} = {} & \frac{1}{4}\Omega^{-3}\left[(1-z^2)^2 \cos\bar{t} \mathrm{d}n^k \wedge \mathrm{d}z + n^k(1+z^2)^2 \sin\bar{t} \mathrm{d}\bar{t} \wedge \mathrm{d}z \right. \\ & \left. + z(1-z^4) \sin\bar{t} \mathrm{d}n^k \wedge \mathrm{d}\bar{t}\right] , \end{aligned} \tag{43} $$

$$ \begin{aligned} {}^{[4k]}\tilde{Q} = {} & \frac{1}{4}\Omega^{-3}\left[(1-z^2)^2 \sin\bar{t} \mathrm{d}n^k \wedge \mathrm{d}z - n^k(1+z^2)^2 \cos\bar{t} \mathrm{d}\bar{t} \wedge \mathrm{d}z \right. \\ & \left. - z(1-z^4) \cos\bar{t} \mathrm{d}n^k \wedge \mathrm{d}\bar{t}\right] . \end{aligned} \tag{44} $$

Finally, for the dual two-forms $*\tilde{Q}$ we have

$$*^{[04]}\tilde{Q} = \left(\frac{1-z^2}{2\Omega}\right)^3 \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi\,, \tag{45}$$

$$*^{[jk]}\tilde{Q} = \frac{1+z^2}{2\Omega^3}\mathrm{d}\bar{t} \wedge \left[z n^l \mathrm{d}z - \frac{1-z^4}{4}\mathrm{d}n^l\right]\epsilon_{jkl}\,, \tag{46}$$

$$*^{[0i]}\tilde{Q} = \frac{1-z^2}{2\Omega^3}\left[\left(\frac{1-z^4}{4}\cos\bar{t}\mathrm{d}\bar{t} + z\sin\bar{t}\mathrm{d}z\right)\wedge n^j \mathrm{d}n^k - \frac{1-z^4}{8}\sin\bar{t}\mathrm{d}n^j \wedge \mathrm{d}n^k\right]\epsilon_{ijk}\,, \tag{47}$$

$$*^{[4i]}\tilde{Q} = \frac{1-z^2}{2\Omega^3}\left[z\cos\bar{t}\, n^j \mathrm{d}n^k \wedge \mathrm{d}z - \sin\bar{t}\left(\frac{1-z^4}{4}\right) n^j \mathrm{d}n^k \wedge \mathrm{d}\bar{t} + \cos\bar{t}\left(\frac{1-z^4}{8}\right)\mathrm{d}n^j \wedge \mathrm{d}n^k\right]\epsilon_{ijk}\,, \tag{48}$$

where

$$\epsilon_{ijk} := \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 1,2,3 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 1,2,3 \\ 0 & \text{in any other cases} \end{cases} .$$

According to Theorem 1 for conformally rescaled metric g_{AdS} we get conformally related CYK tensors $Q := \Omega^{-3}\tilde{Q}$. Their boundary values at conformal infinity $\mathscr{I} := \{z = 0\}$ take the following form:

$$
\begin{aligned}
{}^{[04]}Q\big|_{z=0} &= \frac{1}{4}\mathrm{d}\bar{t} \wedge \mathrm{d}z\,, && (49)\\
{}^{[jk]}Q\big|_{z=0} &= \frac{1}{4}(n^j \mathrm{d}n^k - n^k \mathrm{d}n^j) \wedge \mathrm{d}z\,, && (50)\\
{}^{[0k]}Q\big|_{z=0} &= \frac{1}{4}\left(\cos\bar{t}\mathrm{d}n^k \wedge \mathrm{d}z + n^k \sin\bar{t}\mathrm{d}\bar{t} \wedge \mathrm{d}z\right)\,, && (51)\\
{}^{[4k]}Q\big|_{z=0} &= \frac{1}{4}\left(\sin\bar{t}\mathrm{d}n^k \wedge \mathrm{d}z - n^k \cos\bar{t}\mathrm{d}\bar{t} \wedge \mathrm{d}z\right)\,. && (52)
\end{aligned}
$$

When we define charges associated with CYK tensors, it will be clear that (49) corresponds to the total energy and (50) to the angular momentum. From this point of view CYK tensors (51-52) correspond to the linear momentum and static moment.

Similarly, for dual conformally related CYK tensors $*Q := \Omega^{-3} * \tilde{Q}$ we obtain the following boundary values at conformal infinity:

$$*^{[04]}Q\big|_{z=0} = \frac{1}{8}\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi\,, \tag{53}$$

$$*^{[jk]}Q\big|_{z=0} = \frac{1}{8}\epsilon_{jki}\mathrm{d}n^i \wedge \mathrm{d}\bar{t}\,, \tag{54}$$

$$*^{[0i]}Q\big|_{z=0} = \frac{1}{8}\epsilon_{ijk}\left[\cos\bar{t}\mathrm{d}\bar{t} \wedge n^j \mathrm{d}n^k - \frac{1}{2}\sin\bar{t}\mathrm{d}n^j \wedge \mathrm{d}n^k\right], \tag{55}$$

$$*^{[4i]}Q\big|_{z=0} = \frac{1}{8}\epsilon_{ijk}\left[\frac{1}{2}\cos\bar{t}\mathrm{d}n^j \wedge \mathrm{d}n^k - \sin\bar{t}\, n^j \mathrm{d}n^k \wedge \mathrm{d}\bar{t}\right]. \tag{56}$$

We denote by (z^M) the coordinates on a unit sphere $(M = 1, 2, z^1 = \theta, z^2 = \phi)$ and by γ_{MN} the round metric on a unit sphere:

$$\mathrm{d}\Omega_2 = \gamma_{MN}\mathrm{d}z^M\mathrm{d}z^N = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\,.$$

Let us also denote by ε^{MN} a two-dimensional skew-symmetric tensor on S^2 such that $\sin\theta\varepsilon^{\theta\phi} = 1$. Boundary values for Killing vector fields L_{AB} at $\mathscr{I}$ are:

$$L_{40}\big|_{z=0} = \frac{\partial}{\partial\bar{t}}\,, \quad L_{jk}\big|_{z=0} = \epsilon_{jkl}\varepsilon^{NM}n^l{}_{,M}\frac{\partial}{\partial z^N}\,, \quad L_{12}\big|_{z=0} = \frac{\partial}{\partial\phi}\,, \tag{57}$$

$$L_{i0}\big|_{z=0} = \cos\bar{t}\gamma^{-1}(\mathrm{d}n^i) - \sin\bar{t}n^i\frac{\partial}{\partial\bar{t}}\,, \tag{58}$$

$$L_{i4}\big|_{z=0} = \sin\bar{t}\gamma^{-1}(\mathrm{d}n^i) + \cos\bar{t}n^i\frac{\partial}{\partial\bar{t}}\,. \tag{59}$$

Together with (49-52) and (31) they lead to the following universal formula:

$$^{[AB]}Q = \overset{(0)}{h}(L^{AB}) \wedge \mathrm{d}z\,, \tag{60}$$

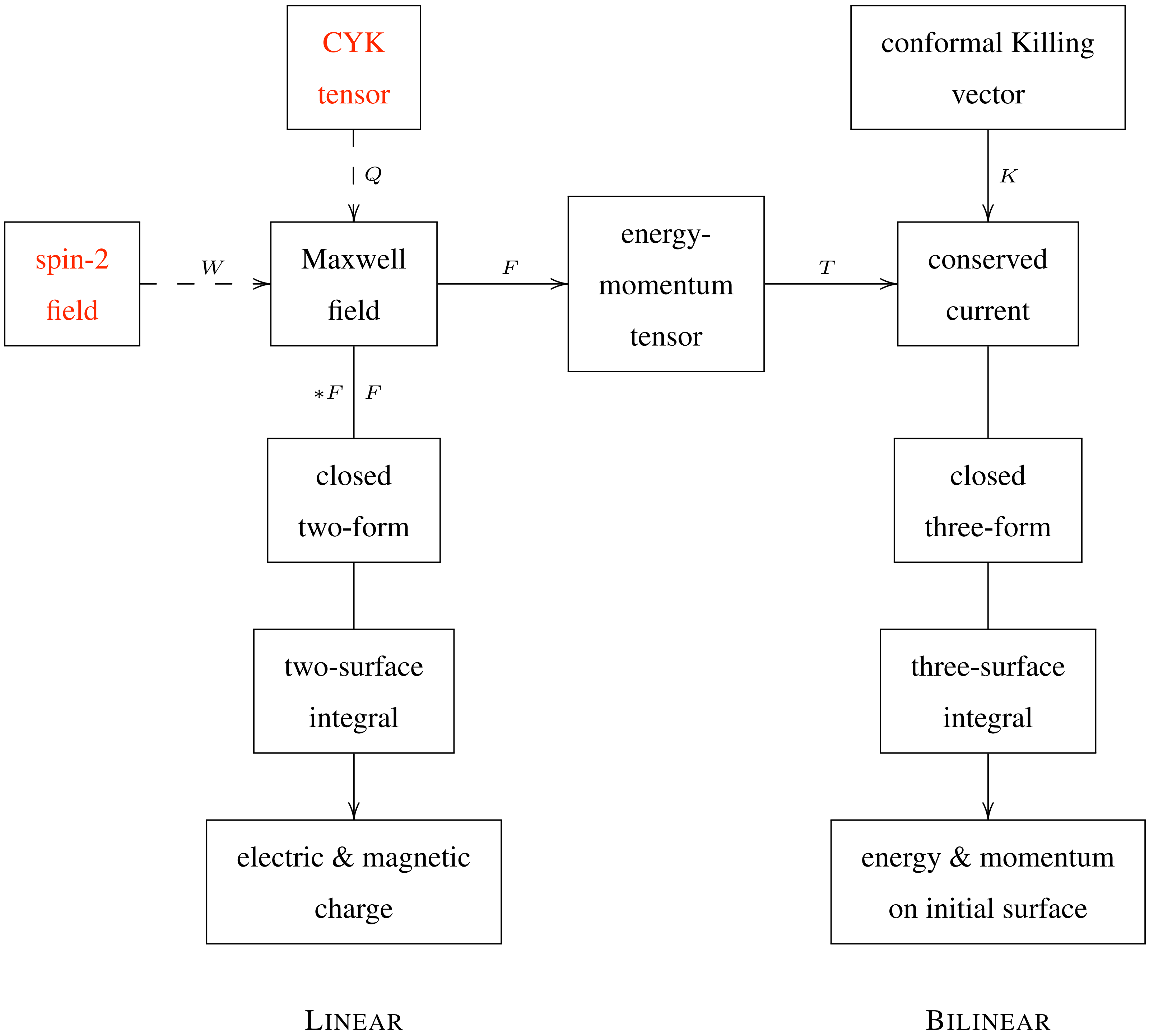
where $L^{AB} := \eta^{AC}\eta^{BD}L_{CD}$. Similarly,

$$*^{[AB]}Q = L^{AB}\rfloor\mathrm{vol}(\overset{(0)}{h})\,, \tag{61}$$

where $\mathrm{vol}(\overset{(0)}{h}) := \sqrt{-\det\overset{(0)}{h}}\,\mathrm{d}\bar{t}\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi$ is a canonical volume three-form on $\mathscr{I}$.

We have constructed all solutions to CYK equation in AdS (and de-Sitter) spacetime via pullback technique from five-dimensional flat ambient space.

The relation between Killing vector fields L and CYK tensors Q has been examined.



CYK
tensor
conformal Killing
vector
Q
K
spin-2
field
W
Maxwell
field
F
energy-
momentum
tensor
T
conserved
current
*F
F
closed
two-form
closed
three-form
two-surface
integral
three-surface
integral
electric & magnetic
charge
energy & momentum
on initial surface
LINEAR
BILINEAR

Let us define the following quantity:

$$H(Q) := \frac{l}{32\pi}\int_C \Omega^{-1}F^{\mu\nu}(Q)\mathrm{d}S_{\mu\nu}\,. \tag{62}$$

For ACYK tensor $\tilde{Q}$ in asymptotic AdS spacetime the corresponding quantity $H(Q)$ is conserved, i.e. does not depend on the choice of spherical cut C. In particular, for the conformal Killing vector field L and $Q(L)$ given by (60) the conserved charge $H(Q(L))$ may be expressed in terms of electric part of Weyl tensor and takes the following form Ashtekar:

$$H(Q(L)) = -\frac{l}{16\pi}\int_C \Omega^{-1}E^a{}_b L^b \mathrm{d}S_a\,. \tag{63}$$

In the Schwarzschild-AdS spacetime

$$ds^2 = -\left(\frac{r^2}{l^2}+1-\frac{2m}{r}\right)\mathrm{d}t^2 + \left(\frac{r^2}{l^2}+1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2 \tag{64}$$

for the Killing vector

$$L = \frac{\partial}{\partial t} = l^{-1}\frac{\partial}{\partial \bar{t}} = l^{-1}\partial_0 \tag{65}$$

definition (62) gives (minus) mass:

$$H(Q(L)) = -\frac{1}{16\pi}\int_C \Omega^{-1}E^a{}_0\mathrm{d}S_a = \frac{3l}{16\pi}\int_C \chi^0{}_0\sqrt{-\det \overset{(0)}{h}}\,\mathrm{d}\theta\mathrm{d}\phi = -m\,. \tag{66}$$

Obviously, the same value $-m$ we obtain for Kerr-AdS metric. Moreover, in the Kerr-AdS spacetime for $L = \frac{\partial}{\partial\phi}$ we obtain the angular momentum:

$$H(Q(L)) = -\frac{l}{16\pi}\int_C \Omega^{-1} E^a{}_\phi \mathrm{d}S_a = \frac{3l^2}{16\pi}\int_C \chi^0{}_\phi \sqrt{-\det \overset{(0)}{h}}\,\mathrm{d}\theta\mathrm{d}\phi = ma\,. \tag{67}$$

Let us observe that our conserved quantity $H(Q(L))$ in terms of the symplectic momenta π^{ab} at $\mathscr{I}$ takes the following form:

$$H(Q(L)) = -\frac{1}{16\pi}\int_C \pi^0{}_b L^b \mathrm{d}\theta\mathrm{d}\phi\,, \tag{68}$$

which is in the same A.D.M. form as the usual linear or angular momentum at spatial infinity in asymptotically flat spacetime.

Remark: In general case, when $\overset{(0)}{h}$ is not conformally flat, it may happen that one obtains asymptotic charge which is no longer conserved — Bondi-like phenomena.

The “topological” charge one can try to define as follows:

$$H(*Q(L)) = \frac{l}{32\pi}\int_{S^2}\Omega^{-1}F^{\mu\nu}(*Q)\mathrm{d}S_{\mu\nu} = -\frac{l}{16\pi}\int_{S^2}\Omega^{-1}B^a{}_bL^b\mathrm{d}S_a\,.$$

We want to stress that, in general, we can meet problems with finding spherical cuts of $\mathscr{I}$. Hence the choice of a domain of integration for the corresponding two-form $\Omega^{-1}F^{\mu\nu}(*Q)\mathrm{d}S_{\mu\nu}$ has to be carefully analyzed. In NUT-AdS spacetime a conformal boundary $\mathscr{I}$ equipped with the metric

$$\overset{(0)}{h} = \frac{1}{4}\left[\mathrm{d}\Omega_2 - \left(\mathrm{d}\bar{t} - 4\bar{l}\sin^2\frac{\theta}{2}\mathrm{d}\phi\right)^2\right] \tag{69}$$

is a non-trivial bundle over S^2 – two-dimensional sphere. However, for L given by (65), when the above formula pretends to define “dual mass” charge, we have

$$-\Omega^{-1}B^a{}_bL^b\mathrm{d}S_a = -\frac{1}{l}\Omega^{-1}B^a{}_0\mathrm{d}S_a \tag{70}$$

$$= \frac{1}{2}\sqrt{-\det\overset{(0)}{h}}\left[\beta^0{}_0\mathrm{d}\theta\wedge\mathrm{d}\phi + \mathrm{d}\bar{t}\wedge\left(\beta^\phi{}_0\mathrm{d}\theta - \beta^\theta{}_0\mathrm{d}\phi\right)\right]$$

$$= 2\bar{l}\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\,. \tag{71}$$

Let us notice that the resulting two-form projects uniquely on the base manifold which is a two-dimensional sphere.

Finally we have

$$2H(*Q(L)) = \frac{l}{16\pi}\int_{S^2} 4\bar{l}\sin\theta \mathrm{d}\theta\mathrm{d}\phi = l\bar{l} = \mathbf{l}$$

which confirms that we can interpret the NUT parameter $\mathbf{l}$ as a dual mass charge.

The construction of global charges in General Relativity has a long history. In the case of asymptotically flat spacetime (asymptotically Minkowskian) the concept of asymptotic CYK tensor led to the strong asymptotic flatness condition at spatial infinity and to the construction of charges in terms of the Weyl tensor which are free from supertranslation ambiguity contrary to the ”superpotentials” based on asymptotic Killing vector fields. There might be similar phenomena for the case of asymptotic AdS. In particular, formulae (60-61) give a hint – the relation between asymptotic KVF and asymptotic CYK tensor which is used to define conserved quantity. Moreover, the universal definition (62) may survive for weaker asymptotics (like in the asymptotically flat case) and for the case when the constructed charge is no longer conserved similarily to Bondi-like phenomena at null infinity. Moreover, CYK tensor enables one to introduce topological charge (magnetic one) which is gravitational analog(ue) of magnetic monopole in electrodynamics.