Effective equations of motion on the brane in higher order dilaton gravity

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Outline

introduction

higher order dilaton gravity

effective brane equations

cosmological example

higher order & conclusions
what if ‘some’ extra dimensions existed . . .

- currently observed 4 space-time dimensions
  - Einstein equations of motion
  - Einstein-Hilbert action
    → linear in Riemann tensor

- higher-dimensional space-times
  ~ additional higher curvature terms can be considered

- introducing higher powers of Riemann tensor into the gravity action
  ~ Einstein theory of gravity generalized

- for a given order in the Riemann tensor
  → contribution to the action unique (overall normalization)
    - quadratic contribution: Gauss-Bonnet (Lanczos) term
    - generalized to higher orders by Lovelock
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string theory behind the scenes?

- effective action obtained from string theories
  - higher derivative corrections to the gravity interactions
- first correction exactly of the form of the Gauss-Bonnet term
  - local field redefinitions

- dilaton: scalar field of the gravitational sector
- $\alpha'$ expansion in the string theories
  - higher order corrections also for the dilaton interactions
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construction of the dilaton gravity equations of motion

- Einstein-Lovelock higher order gravity
  \( \rightarrow \) couple to the dilaton

- \( N \)-th order dilaton gravity equations of motion: \( T^{(N)}_{\mu \nu} = 0 \) & \( W^{(N)} = 0 \)
  \( \rightarrow \) constructed
  - at each order: unique up to a normalization

- \( E-L \) theory generalized \( \rightarrow \) higher order dilaton gravity
some useful notation

- a generalization of the Kronecker delta

\[ \delta_{\rho_1\rho_2\cdots\rho_N}^{\sigma_1\sigma_2\cdots\sigma_N} = \det \begin{vmatrix} \delta_{\rho_1}^{\sigma_1} & \delta_{\rho_2}^{\sigma_1} & \cdots & \delta_{\rho_N}^{\sigma_1} \\ \vdots & \vdots & \cdots & \vdots \\ \delta_{\rho_1}^{\sigma_N} & \delta_{\rho_2}^{\sigma_N} & \cdots & \delta_{\rho_N}^{\sigma_N} \end{vmatrix} \]

- a generalization of the trace operator

\[ T(M) = \delta_{\rho_1\rho_2\cdots\rho_N}^{\sigma_1\sigma_2\cdots\sigma_N} M^{\rho_1\rho_2\cdots\rho_N} \sigma_1\sigma_2\cdots\sigma_N \]

- an extension of the trace operator

\[ \overline{T}_\mu^{\nu}(M) = \delta_{\mu_1\mu_2\cdots\mu_N}^{\nu_1\nu_2\cdots\nu_N} M^{\nu_1\nu_2\cdots\nu_N} \sigma_1\sigma_2\cdots\sigma_N \]
some useful notation → example

▶ and a generalization of the $N$-th power operator

$$T \left( \left[ \frac{1}{2} R_{**}^{*} \oplus 2 (\nabla \nabla)^{*} \phi \right]^{2} \right) =$$

$$= \frac{1}{4} T (R_{**}^{**} R_{**}) + 2 T (R_{**}^{**} (\nabla \nabla)^{*} \phi) + 4 T ((\nabla \nabla)^{*} \phi (\nabla \nabla)^{*} \phi) =$$

$$= \frac{1}{4} \delta_{\rho_{1} \rho_{2} \rho_{3} \rho_{4}}^{\sigma_{1} \sigma_{2} \sigma_{3} \sigma_{4}} \ R^{\rho_{1} \rho_{2}}_{\sigma_{1} \sigma_{2}} \ R^{\rho_{3} \rho_{4}}_{\sigma_{2} \sigma_{4}} +$$

$$+ 2 \delta_{\rho_{1} \rho_{2} \rho_{3}}^{\sigma_{1} \sigma_{2} \sigma_{3}} \ R^{\rho_{1} \rho_{2}}_{\sigma_{1} \sigma_{3}} \ (\nabla^{\rho_{3}} \partial_{\sigma_{3}} \phi) +$$

$$+ 4 \delta_{\rho_{1} \rho_{2}}^{\sigma_{1} \sigma_{2}} \ (\nabla^{\rho_{1}} \partial_{\sigma_{1}} \phi) \ (\nabla^{\rho_{2}} \partial_{\sigma_{2}} \phi)$$

▶ asterisks → tensors ranks

▶ shorthand: $(\nabla \nabla)_{\sigma}^{\rho} \equiv \nabla^{\rho} \partial_{\sigma}$
(just acquired) starting point: \textit{d}-dimensional \textit{higher order dilaton gravity}

- \textit{d}-dimensional tensor $T_{\mu \nu} = 0$ and scalar $W = 0$ equations of motion

\[
- \sum_{N=1}^{N_{\text{max}}} \frac{\alpha N}{2} \bar{T}_{\mu \nu} \left( \left[ \frac{1}{2} R_{*}^{**} \oplus 2(\nabla \nabla)^{*} \phi \oplus (-1)(\partial \phi)^2 \right]^N \right) + g_{\mu \nu} V(\phi) - \tau_{\mu \nu} \delta_B = 0
\]

\[
- \sum_{N=1}^{N_{\text{max}}} \frac{\alpha N}{2} T \left( \left[ \frac{1}{2} R_{*}^{**} \oplus 2(\nabla \nabla)^{*} \phi \oplus (-1)(\partial \phi)^2 \right]^N \right) + V(\phi) - V'(\phi) - \tau_\phi \delta_B = 0
\]

- position of the brane: Dirac delta type distribution $\delta_B$

- brane localized terms: $\tau_{\mu \nu} = h_{\mu \nu} \mathcal{L}_B - 2 \frac{\delta \mathcal{L}_B}{\delta h_{\mu \nu}}$ \& $\tau_\phi = \mathcal{L}_B - \frac{\delta \mathcal{L}_B}{\delta \phi}$

due to the brane interactions given by $\mathcal{L}_B$

- induced brane metric: $h_{\mu \nu} = g_{\mu \nu} - n_\mu n_\nu$
  - $n^\mu$: unit vector field normal to the brane (at the brane)

- corresponding Lagrangian density

\[
\mathcal{L} = e^{-\phi} \left\{ - V(\phi) + \mathcal{L}_B \delta_B + \sum_{N=1}^{N_{\text{max}}} \frac{\alpha N}{2} T \left( \left[ \frac{1}{2} R_{*}^{**} \oplus 2(\nabla \nabla)^{*} \phi \oplus (-1)(\partial \phi)^2 \right]^N \right) \right\}
\]
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where to?

- **brane-world** ideology: standard model localized on a brane
  → embedded in higher dimensional space-time
  ⇝ how will the induced gravity look like on the brane?

- **effective equations of motion**: \((d - 1)\)-dimensional
  ⇝ simply restricting full \(d\)-dimensional equations? NO!
  - certain quantities contributing to \(T^{(N)}_{\mu\nu} = 0\) and \(W^{(N)} = 0 \ldots\)
    - singular: explicit Dirac delta contributions
      or discontinuous functions derivatives

  ⇝ *non-trivial derivation* of effective equations on the brane
  ⇝ will be carried out in **COVARIANT APPROACH**
Where to?

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  - ⇻ will be carried out in **COVARIANT APPROACH**
projecting: “parallel” & “perpendicular” to the brane

- identification in relevant tensors: benefits from \( \mathcal{I} \& \mathcal{I} \) antisymmetrization

\[
\mathcal{R}^{**} \rightarrow \mathcal{R}^{**} - 2K^* K^* - 4(nn)^* \{\mathcal{L}_n K^* - (KK)^*\} - 8(nD)^* K^*
\]
\[
(\nabla \nabla)^* \phi \rightarrow \left[(DD)^* \phi + K^* \mathcal{L}_n \phi\right] + (nn)^* \left\{\mathcal{L}_n^2 \phi - a^e \nabla_e \phi\right\} + \\
+ 2[(nD)^* \mathcal{L}_n \phi - (nKD)^* \phi]
\]
\[
(\partial \phi)^2 = (D\phi)^2 + (\mathcal{L}_n \phi)^2
\]

- \( g_{\mu \nu} \): \( \mathcal{R}^{\mu \nu}_{\rho \sigma} \) & \( \nabla_{\mu} \) vs \( h_{\mu \nu} \): \( R^{\mu \nu}_{\rho \sigma} \) & \( D_{\mu} \)

- \( K_{\mu \nu} \): extrinsic curvature of hypersurfaces orthogonal to \( n^\mu \)

- \( \mathcal{L}_n \): Lie derivative along \( n^\mu \)

- \( a^e \nabla_e \phi = n^a (\nabla_a n^b) (\nabla_b \phi) \) (‘non-typical’; not present in final results)

- shorthand again: \( (nn)^* \equiv n^* n^*, \ (DD)^* \equiv D^* D^*, \ (KK)^* \equiv K^x K^x, \ \ (nD)^* \equiv \frac{1}{2} (n^* D^* + n^* D_*), \ (nKD)^* \equiv \frac{1}{2} (n^* K^x D^x + n^* K^x D_* ) \)
effective gravitational equations on the brane . . . identifying problems

▶ the good

→ $h_{\mu\nu}, R_{\mu\nu}, (DD)_{\mu\nu}\phi, (D\phi)^2, V(\phi)$

⇝ no work needed here, rejoice!

▶ the kind of bad

→ $K_{\mu\nu}, \mathcal{L}_n\phi$

⇝ can be discontinuous when ‘crossing’ the brane

▶ and the slightly ugly

→ $\mathcal{L}_nK_{\mu\nu}, \mathcal{L}_n^2\phi$

⇝ can be singular on the brane

⇝ can have a finite contribution as well

⇝ all this information has to be properly taken into account

& the quest begins
effective gravitational equations on the brane . . . identifying problems

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  $nK_{\mu\nu}, n^2\phi$

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- the good
  \[ h_{\mu \nu}, R_{\mu \nu}, (DD)_{\mu \nu} \phi, (D\phi)^2, V(\phi) \]
  \[ \Rightarrow \text{no work needed here, rejoice!} \]

- the kind of bad
  \[ K_{\mu \nu}, E_n \phi \]
  \[ \Rightarrow \text{can be discontinuous when ‘crossing’ the brane} \]

- and the slightly ugly
  \[ E_n K_{\mu \nu}, E_n^2 \phi \]
  \[ \Rightarrow \text{can be singular on the brane} \]
  \[ \Rightarrow \text{can have a finite contribution as well} \]

\[ \Rightarrow \text{all this information has to be properly taken into account} \]

& the quest begins
effective gravitational equations on the brane . . . addressing problems

- terms **discontinuous** on the brane (leading to singularities)
  
  → $K_{\mu\nu}$, $\mathcal{L}_n \phi$

  $\leadsto$ **junction** (boundary) **conditions**

- terms **singular** on the brane
  
  → $\mathcal{L}_n K_{\mu\nu}$, $\mathcal{L}_n^2 \phi$
  
  - purely singular contributions already addressed by the junction conditions
  - the smooth part has to be determined as well
  - (yields a finite contribution to the effective equations)

$\leadsto$ **“brane limit of bulk equations system”**

- take scalar equation of motion & the trace of tensor equation of motion
  
  → **“bulk equations system”**

- now the brane limit (i.e. evaluate ‘next to the brane’)

- solve it
effective gravitational equations on the brane . . . addressing problems

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\[ K_{\mu\nu} \] , \[ \mathcal{L}_n \phi \]

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effective gravitational equations on the brane . . .

- any order? complexity rather overwhelming
  - \( N = 1 \) & \( d = 5 \) example
    - first order dilaton gravity
    - and a 4-dimensional brane

- higher order dilaton gravity: exactly the same procedure
  - appropriate formulae derived just as well
    - explicit results? even fancy notation not always sufficient
effective gravitational equations . . . bulk equations projected ($N = 1$)

- "parallel-parallel" part $T_{\mu \nu}^{|| ||} = 0$

$$\left\{ R_{\mu \nu} + (DD)_{\mu \nu} \phi - \frac{h_{\mu \nu}}{2} \left( R + 2(DD)\phi - (D\phi)^2 \right) + h_{\mu \nu} \frac{V(\phi)}{\alpha_1} \right\} +$$

$$+ \left\{ (KK)_{\mu \nu} - K_{\mu \nu} (K - \xi_n \phi) \right\} - \frac{h_{\mu \nu}}{2} \left( (KK) - (K - \xi_n \phi)^2 \right) \right\} +$$

$$- \left\{ (\xi_n K_{\mu \nu} - (KK)_{\mu \nu}) - h_{\mu \nu} \left( h\xi_n K - (KK) \right) - h_{\mu \nu} \left( \xi_n^2 \phi - \alpha^2 \nabla e \phi \right) \right\} \right\} \pm = 0$$
addressing the discontinuities’ problem: junction conditions

- junction conditions for a given point $x_0^\mu$ on the brane:
  - integrating the $d$-dimensional equations of motion ‘across-the-brane’
  - i.e. in the direction perpendicular to the brane
  - and shrinking the interval: ‘infinitesimal across-the-brane integration’

- only some terms in the equations of motions $\rightarrow$ zero
  - explicit brane contributions proportional to $\delta_B$
  - terms containing second Lie derivatives: $\mathcal{L}_n^2 \phi$ and $\mathcal{L}_n K_{\mu\nu}$

- useful notation
  - discontinuous at the brane $\rightsquigarrow$ jump: $[f(x_0)]_\pm = [f(x_0)]_+ - [f(x_0)]_-$
  - “brane limits”: $[f(x_0)]_+, [f(x_0)]_-$
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junction conditions: $N = 1 \rightsquigarrow [K_{\mu\nu}]_\pm \& [\mathcal{E}_n\phi]_\pm$ explicitly

- tensor junction condition (effectively from $T_{\mu\nu}^{||} = 0$)
- scalar junction condition

$\rightsquigarrow$ easily solvable, jumps can be determined

$$[K_{\mu\nu}]_\pm = \frac{1}{\alpha_1} \left( h_{\mu\nu} \tau_\phi - \tau_{\mu\nu} \right)$$

$$[\mathcal{E}_n\phi]_\pm = \frac{1}{\alpha_1} \left( (d-2)\tau_\phi - \tau \right)$$

- however, no information whatsoever about the brane limits
  $\rightsquigarrow$ unless …
junction conditions: \[ N = 1 \quad \text{and} \quad \mathbb{Z}_2 \rightsquigarrow [K_{\mu\nu}]_+ \quad \& \quad [\mathcal{E}_n\phi]_+ \text{ explicitly} \]

- \( \mathbb{Z}_2 \) symmetry, brane located at the orbifold fixed point
  \[ \rightsquigarrow [f]_\pm = 2[f]_+ \quad \text{if} \quad f \text{ is} \ \mathbb{Z}_2\text{-odd, i.e.} \quad [f]_- = -[f]_+ \]

\[ \rightsquigarrow \text{brane limits can be determined} \]

\[ [K_{\mu\nu}]_+ = \frac{1}{2\alpha_1} \left( h_{\mu\nu} \tau_\phi - \tau_{\mu\nu} \right) \]

\[ [\mathcal{E}_n\phi]_+ = \frac{1}{2\alpha_1} \left( (d - 2) \tau_\phi - \tau \right) \]
addressing the discontinuities’ problem: bulk equations system

- terms singular on the brane
  → tend to appear in equations of motion in certain combinations
  → \{\mathcal{L}_n K_{\mu\nu} - (KK)_{\mu\nu}\}, \{\mathcal{L}_n^2 \phi - a^e \nabla_e \phi\}
  \sim good occasion to get rid of \(a^e \nabla_e \phi\) as well

- “brane limit of bulk equations system”
  - supposed to yield finite contributions to \(\mathcal{L}_n K_{\mu\nu}\) & \(\mathcal{L}_n^2 \phi\)
  - a system of scalar equations \((\mathcal{W}^{(N)} = 0\) and trace of \(T^{(N)}_{\mu\nu} = 0\))
  - but \(\mathcal{L}_n K_{\mu\nu}\) is a tensor variable... how come it can work?
  → yes, it can

\[
\begin{align*}
\{\mathcal{L}_n K_{\mu\nu} - (KK)_{\mu\nu}\} &= \frac{h_{\mu\nu}}{d-1} \left\{ (h\mathcal{L}_n K) - (KK) \right\} - \frac{1}{d-3} (R_{\mu\nu} - KK_{\mu\nu} + (KK)_{\mu\nu}) + \\
&+ \frac{h_{\mu\nu}}{(d-1)(d-3)} (R - K^2 + (KK)) - \frac{d-2}{d-3} E_{\mu\nu}
\end{align*}
\]
addressing the discontinuities’ problem: bulk equations system

- terms singular on the brane
  - tend to appear in equations of motion in certain combinations
  - \( \{ \nabla_n K_{\mu\nu} - (KK)_{\mu\nu} \}, \{ \nabla^2_n \phi - a^e \nabla_e \phi \} \)
  - \( \sim \) good occasion to get rid of \( a^e \nabla_e \phi \) as well
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  - supposed to yield finite contributions to \( \nabla_n K_{\mu\nu} \) & \( \nabla^2_n \phi \)
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&\quad + \frac{h_{\mu\nu}}{(d-1)(d-3)} \left( R - K^2 + (KK) \right) - \frac{d-2}{d-3} E_{\mu\nu}
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\[
\left\{\mathcal{L}_n K_{\mu\nu} - (KK)_{\mu\nu}\right\} = \frac{h_{\mu\nu}}{d - 1} \left\{ (h \mathcal{L}_n K) - (KK) \right\} - \frac{1}{d - 3} (R_{\mu\nu} - KK_{\mu\nu} + (KK)_{\mu\nu}) + \\
+ \frac{h_{\mu\nu}}{(d - 1)(d - 3)} (R - K^2 + (KK)) - \frac{d - 2}{d - 3} E_{\mu\nu}
\]
addressing the discontinuities’ problem: bulk equations system

\[ E_{\mu\nu} \equiv C_{abcd} h_{\mu}^{a} n_{\nu}^{b} h_{\nu}^{c} n_{\mu}^{d} \]

where \( C_{abcd} \): bulk Weyl tensor

\( E_{\mu\nu} \) enters \( T_{\mu\nu}^{\parallel\parallel} = 0 \)

(so promising as effective gravitational equations on the brane)

\( \rightarrow \) to never leave it

- treating \( T_{\mu\nu}^{\parallel\parallel} = 0 \) as effective gravitational equations on the brane . . .
  - single bulk associated variable \( E_{\mu\nu} \)
  - describes the permanent influence of bulk theory on brane-world gravity
  - not a closed system
  - bulk solutions essential to fully describe the gravity induced on the brane
addressing the discontinuities’ problem: bulk equations system

- \( E_{\mu\nu} \equiv C_{abcd} h^a_{\mu} n^b h^c_{\nu} n^d \), where \( C_{abcd} \): bulk Weyl tensor
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  (so promising as effective gravitational equations on the brane)
  \( \rightarrow \) to *never leave* it

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  - *not a closed system*
    \( \rightarrow \) bulk solutions essential to fully describe the gravity induced on the brane
bulk equations system: \( N = 1 \)

- two linear equations with two variables
  \[ g^{\mu\nu} T_{\mu\nu} = 0 \quad \text{&} \quad W = 0 \]
- non-zero determinant
  \[ \sim \{(he_nK) - (KK)\} \text{ and } \{\mathcal{L}_n^2 \phi - a^e \nabla e \phi\} \text{ can be determined uniquely} \]
- \( \{\mathcal{L}_n K_{\mu\nu} - (KK)_{\mu\nu}\} \) to be calculated subsequently
effective gravitational equations on the brane: \( N = 1, \ d = 5, \mathbb{Z}_2 \)

- 4-dimensional brane
  embedded in a 5-dimensional space-time with \( \mathbb{Z}_2 \) symmetry

\( \Rightarrow \) how shall we do it?

- take \( T_{\mu\nu}^{||} = 0 \)

- enter the solution of the brane limit of bulk equations system
  \( \Rightarrow \) \( \{ (hE_nK) - (KK) \} \ \& \ \{ L_n^2 \phi - a^e \nabla_e \phi \} \ \& \ \{ L_n K_{\mu\nu} - (KK)_{\mu\nu} \} \)
  - slightly readjust the relative \( R_{\mu\nu} \) vs. \( R \) coefficient with \( T_{\perp\perp} = 0 \)
  \( \Rightarrow \) ‘Einstein-like’ form with the Einstein tensor \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R h_{\mu\nu} \)

- enter the result of junction conditions analysis
  \( \Rightarrow \) \( [K_{\mu\nu}]_+ \ \& \ [E_n \phi]_+ \)
effective gravitational equations on the brane: $N = 1, \ d = 5, \ \mathbb{Z}_2$

$\rightarrow$ effective gravitational equations on the brane read

$$
G_{\mu\nu} + E_{\mu\nu} + \frac{2}{3} \left( (DD)_{\mu\nu} \phi - h_{\mu\nu} (DD) \phi \right) + \frac{1}{4} h_{\mu\nu} (D\phi)^2 + h_{\mu\nu} \frac{V(\phi)}{2\alpha_1} + \\
+ \frac{1}{(2\alpha_1)^2} \left[ \frac{1}{3} \tau \tau_{\mu\nu} - (\tau \tau)_{\mu\nu} + h_{\mu\nu} \left( \frac{1}{2} (\tau \tau) - \frac{1}{12} \tau^2 - \frac{1}{2} \tau \tau \phi + \frac{3}{4} \tau^2 \phi \right) \right] = 0
$$
Effective equations of motion on the brane in higher order dilaton gravity
Friedmann equations in first order dilaton gravity

- the effective equations on the brane derived ... so what?
  - let's see what can happen to the physics

- phenomenological applications to cosmology
  - *Friedmann equations modified / generalized*

- standard cosmology \( \rightsquigarrow \) FRW metric tensor ansatz \( \rightarrow \) Friedmann equations are given by gravitational equations of motion
  - trace
  - \((t,t)\) component

- let's just do the very same here
modified Friedmann equations: comments and complaints?

\[
\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\dot{a}}{a} \dot{\phi} - \frac{1}{3} \ddot{\phi} + \frac{1}{6} \dot{\phi}^2 + \frac{1}{3 \alpha_1} V(\phi) + \\
+ \frac{1}{(4 \alpha_1)^2} \left[ - \frac{2}{3} (\tau_1^1)^2 - 2 (\tau_4^4)^2 + \frac{4}{3} \tau_1^1 \tau_\phi + 4 \tau_4^4 \tau_\phi - 2 \tau_\phi^2 \right] = 0
\]

\[
\frac{\ddot{a}^2}{a^2} + \frac{k}{a^2} - \frac{1}{3} E_1 \frac{1}{3} a \dot{\phi} + \frac{1}{12} \dot{\phi}^2 + \frac{1}{6 \alpha_1} V(\phi) + \\
+ \frac{1}{(4 \alpha_1)^2} \left[ \frac{1}{3} (\tau_1^1)^2 - \frac{2}{3} \tau_1^1 \tau_4^4 - (\tau_4^4)^2 + \frac{2}{3} \tau_1^1 \tau_\phi + 2 \tau_4^4 \tau_\phi - \tau_\phi^2 \right] = 0
\]

greater with the scalar factor only: exactly the same

greater, however, there are obviously quite relevant differences

greater terms associated with dilaton appear, as well as mixing terms

greater due to the introduction of the additional field interacting with graviton

greater terms quadratic in the brane localized terms

greater a feature of higher-dimensional theories?

greater \(E_{\mu\nu}\): influence of the original higher-dimensional theory

greater on the effective 4-dimensional phenomenology
Effective equations of motion on the brane in higher order dilaton gravity

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Effective gravitational equations on the brane: arbitrary $N$

- derivation procedure for the effective gravitational equations on the brane
  → established

  ~ works fine, we’ve just seen the example of $N = 1$ & $d = 5$

- and what about higher orders?
  → exactly the same procedure, namely
    - take $T^{\parallel\parallel}_{\mu\nu} = 0$
    - enter the solution of the brane limit of bulk equations system
      ~ $(h\mathcal{E}_n K) - (KK)$ & $(\mathcal{E}_n^2 \phi - a^e \nabla_e \phi)$ & $(\mathcal{E}_n K_{\mu\nu} - (KK)_{\mu\nu})$
    - enter the result of junction conditions analysis
      ~ $[K_{\mu\nu}]_+$ & $[\mathcal{E}_n \phi]_+$
effective gravitational equations on the brane: arbitrary $N$

- derivation procedure for the effective gravitational equations on the brane
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  ⇝ works fine, we’ve just seen the example of $N = 1$ & $d = 5$
- and what about higher orders?
  → exactly the same procedure, namely
    - take $T_{\mu\nu}^{\parallel\parallel} = 0$
    - enter the solution of the brane limit of bulk equations system
      ⇝ $\{(\mathcal{L}^n h - (KK)) \& \{\mathcal{L}^2 \phi - a^\theta \nabla_e \phi\} \& \{\mathcal{L}^n K_{\mu\nu} - (KK)_{\mu\nu}\}$
    - enter the result of junction conditions analysis
      ⇝ $[K_{\mu\nu}]_+ \& [\mathcal{L}^n \phi]_+$
conclusions

- starting point: higher order dilaton gravity
  - natural to consider in higher-dimensional space-times
  - physically viable equations of motion: constructed
  - appropriate lagrangian: presented

- effective gravitational equations on the brane (co-dimension 1)
  - derivation procedure for arbitrary order $N$: established
  - details and results presented explicitly for $N = 1$ & $d = 5$
    - together with a cosmological example
  - modified Friedmann equations on 4d brane