Gluon mass and small-x dynamics in hadrons

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(Dated: Tuesday 3rd March, 2026, 20260303 15:25 wto Pasteura)

Computational access to the logarithmically scale-dependent Hamiltonian eigenstate picture for hadrons in the space of virtual quark and gluon states in the canonical front form of QCD is impeded by the small-x divergences stronger than logarithmic. We propose introducing a gluon mass parameter and an auxiliary color-octet scalar field to regulate and break through this barrier, using the renormalization group procedure for effective particles (RGPEP) [1] in the limit of the gluon mass parameter going to zero at the end of a calculation. At that point the auxiliary field decouples from the finessed effective constituent dynamics, as required in the gauge theory. The same method yields cancellation of the quadratic ultraviolet transverse divergences in the self-interactions. We explain how this approach works in virtual quark and gluon scattering amplitudes and in Hamiltonian eigenvalue problems for bound states. We focus the discussion on the case of heavy quarkonia [2], where one obtains rotationally symmetric effective interactions at small interquark distances, akin to harmonic oscillator potential, and only globally colorless states having finite masses in the limit of the gluon mass going to zero. Previously [3], it was suggested that the QCD vacuum effects, such as in the QCD sum rules [4] and due to the instantons [5], can manifest themselves effectively through the front form counterterms to the small-x regularization singularities. We advance the hypothesis that the corresponding effective interaction terms can instead appear in the finite renormalization-group evolution when the scale parameter is lowered towards $\Lambda_{\rm QCD}$ in the RGPEP scheme.

[1] S. D. Głazek, Eur. Phys. J. Spec. Top. (2026), https://doi.org/10.1140/epjs/s11734-026-02127-y [2] K. Serafin et al., Phys.Rev.D 109, 016017 (2024), https://arxiv.org/abs/2310.00365 [3] K. G. Wilson et al., Phys.Rev.D 49, 6720 (1994), https://arxiv.org/abs/hep-th/9401153 [4] M. Shifman, A. Vainshtein, I. Zakharov, Nucl.Phys.B 147, 385 (1979) [5] E. Shuryak, I. Zahed, https://arxiv.org/abs/2601.15085

Precise derivation of the logarithmically scale-dependent Hamiltonian eigenstate picture for hadrons in the space of virtual quark and gluon states of the canonical front form of QCD cannot be achieved without first addressing the problem of small$x$ divergences stronger than logarithmic. We propose to facilitate cancellation of these strong divergences using a mass parameter for gluons and an auxiliary color-octet scalar field corresponding to the longitudinal polarization. The auxiliary field decouples from the hadronic constituent dynamics in the limit of the gluon mass parameter going to zero, as required in the gauge theory. The same method simultaneously facilitates cancellation of the quadratic ultraviolet transverse divergences in the self-interactions. After explaining how this approach works for virtual quark and gluon scattering amplitudes, we describe how it applies in the Hamiltonian eigenvalue problems for bound states.

Saturation of gluons because small p^+ leads to large s?

Explanation of how Susskind's vacuum freezing occurs, by making small-x gluons freeze into the bound-state eigenvalue shape

$E(P^\perp = 0, P^+ = M) = M$. In exact solution the eigenvalues do not depend on s and therefore do not depend on P^+ and $P^\perp$ Precision of solving the RGPEP equation and the associated eigenvalue problems is reflected by degree of independence of the eigenvalues from P^+.

van der Waals forces?

Instead of seeking vacuum effects in reaction to the cutoff ϵ^+, like Wilson et al., it is proposed to work with m_g and approach the vacuum issues for finite size s of the effective quanta in the limit $m_g \to 0$, the parameter that can be used to describe confinement in the Fock space, including the quantum gluon string formation mechanism. So, instead of studying vacuum effects through regularization, study them using the RGPEP in the theory with $m_g \to 0$. The point is that the parameter m_g regulates infrared singularities and the RGPEP can be used to take care of the ultraviolet effects in the eigenvalue problems.

Small denominators are avoided in computation of H_s.

Rephrase deeply: Advance the hypothesis that the vacuum-like effects develop in the renormalized small-x dynamics and can be found using the RGPEP by matching the parton dynamics [1, 2], QCD sum rules [3, 4], large momentum effective theory methods of lattice approach [5, 6], and including vacuum effects in the parton picture [7].

Quarks and gluons with large p^+ are little evolved in the RGPEP. They closely resemble partons understood as quanta in canonical QCD. By comparison, quarks and gluons with small p^+, on the order of a hadron mass, are evolved a lot in the RGPEP towards the picture of the constituent quarks. This means that the quanta with large p^+, like in the parton model, interact nearly as in the canonical QCD, while the interaction vertices of the quanta of small p^+, such as in the rest frame of a hadron, are tempered by relatively narrow form factors, include terms resembling potentials and are expected to theoretically support the constituent quark models success in the classification of hadrons.

Technical improvements: boost invariant f^r, simple generator $[\mathcal{H}_f, \mathcal{H}_I]$, no need for counterterm $\delta(q_{gluon})$, confinement for $m_g \to 0$.

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II. INTRODUCTION

For precise application to the physics of hadron constituents, the Hamiltonian of QCD needs to be constructed as an operator acting in the space of states of virtual quarks and gluons in a definite form of dynamics. Among the forms identified in [8], the commonly known way of describing the evolution of physical systems using time s of some observer, is called the instant form (IF). The name originates from the space-time hyperplane corresponding to one value of s, an instant. The less familiar form, in which an observer describes the system evolution from one value of $x^+ = t + z$ to another, is called the front form (FF). The name comes from the fact that a space-time hyperplane corresponding to one value of x^+ is swept by the front of a plane wave of light moving against the z-axis. The choice of the FF of dynamics for describing how quarks and gluons form hadrons stems from the desire to describe the hadrons in any state of motion, such as being at rest in a laboratory or moving with a speed nearly equal to the speed of light after leaving an accelerator. In the IF, boosting hadrons is associated with complex dynamical effects because the quantum generators of boosts involve interactions. In the FF, there are 3 independent Lorentz transformations that are free from interaction effects and the dynamics is invariant with respect to them. This feature suggests a simplification of the task of relating the observed parton structure of fast moving hadrons and their spectroscopic classification at rest in terms of states of 3 quarks or a quark-antiquark pair [9]. Besides offering a way to approach the quantum boost problem for bound states, the FF Hamiltonian of QCD naturally provides the eigenvalue equations determining how the physical hadrons, represented by the eigenstates, are built from the quanta of quark and gluon fields.

However, the canonical FF Hamiltonian of QCD is a singular operator. One needs to identify a computational path that leads from the canonical theory to the finite solutions of the bound-state eigenvalue problems. Asymptotic freedom suggests that the binding mechanism for hadronic constituents involves logarithms of their relative momenta. But to reliably compute the logarithmic effects in bound-state dynamics, one first needs to find a way to control the functions more singular than the logarithms. This paper proposes a way to do so. The difficulty to overcome is that the FF of the theory involves singular functions of two different kinds of variables.

In particular, the constituent momenta measured along the z-axis greatly differ from the the momenta in directions perpendicular to that axis. Consequently, the FF Hamiltonians do not possess any explicit three-dimensional rotation symmetry that the IF Hamiltonians have. Instead of the three-dimensional momenta, say $\vec{p}$, one works with the transverse momenta of quanta, denoted by $p^\perp = (p^1, p^2)$, and the longitudinal momenta, denoted by $p^+ = p^0 + p^3$. The corresponding boost-invariant variables of two kinds are the quanta relative transverse momenta $k^\perp = p^\perp - xP^\perp$ and the ratios $x = p^+/P^+$, where P denotes the total momentum of the quanta involved in a Hamiltonian interaction term. Divergences occur due to $k^\perp$ going to ∞ or 0 and due to x going to 0 or 1. For example, there are singular functions that behave as $1/x^2$ or $1/x$. Reliable treatment of $\ln x$ when $x \to 0$ requires a Hamiltonian renormalization procedure for the divergences in x and $k^\perp$ [10]. The first step of any such procedure is to regulate the diverging terms.

Regularization of the Hamiltonian terms can be arranged by limiting the values of $k^\perp$ and x. Suppose one does it for the transverse and longitudinal coordinates separately. In that case, the ultraviolet divergences associated with the longitudinal and transverse directions pile up on each other, which leads to a complex renormalization procedure. One can correlate the regularization cutoffs on $k^\perp$ and x by limiting the total invariant-mass squared, denoted by $\mathcal{M}^2$, of the quanta involved in an interaction term. Each of the quanta contributes to $\mathcal{M}^2$ via P^+ times

$$k^- \;=\; k^0 - k^3 \;=\; \frac{k^{\perp 2} + m^2}{xP^+} \;, \tag{1}$$

with m being a mass assigned to a quantum in the free part of the Hamiltonian. Limiting $\mathcal{M}$ by a cutoff from above, say Λ, one simultaneously limits $|k^\perp|$ from above and prevents x from approaching 0 because m^2/x is limited. The role of a mass parameter as regulator is of interest because mass does not distinguish any direction in the space of four-momenta. The issue with gluons, however, is that they are massless, as befits gauge bosons. For them,

$$P^+ k^- \;=\; \frac{k^{\perp 2}}{x} \; . \tag{2}$$

Small x is not limited from below by the upper limit on the invariant mass $\mathcal{M}$ when $k^\perp$ is correspondingly small. Therefore, the ultraviolet renormalization procedure is interfered with by the infrared issues of the theory. Namely, the small-x region belongs to the ultraviolet regime when $k^\perp$ is sizable and it also contributes to the infrared regime when $k^\perp$ is small. This is where the gluon mass parameter, m_g, can be introduced as a regulator disentangling these regimes. Replacement of $k^{\perp 2}$ in Eq. (2) by $k^{\perp 2} + m_g^2$ makes small x correspond to the ultraviolet regime even if $k^\perp$ vanishes. But a gluon mass term the canonical FF Hamiltonian of QCD for the regularization purposes, the resulting virtual quark and gluon transition amplitudes involve integrals of, in essence,

$$\frac{m_g^2}{k^{\perp 2} + m_g^2} \, \frac{1}{x^2} \; . \tag{3}$$

The small-x singularity is associated with a function of the transverse momentum. The singularity strengthens when $k^\perp$ decreases. We propose to counter this gluon mass effect by adding quanta of an auxiliary scalar octet field $\phi = \phi^a T^a$ to the dynamics. These quanta correspond to the longitudinally polarized gluons. In the limit $m_g \to 0$, the longitudinal quanta decouple from the theory when $m_g \to 0$. Such decoupling is know in case of the FF of QED [11, 12], but as far as the author knows has not been taken advantage of in the FF Hamiltonian of QCD as it is done in this paper.

Since the paper is focused on the use of gluon mass for the purpose of regularization and renormalization of the FF Hamiltonian of QCD, it does not report on or directly refer to most of the rich literature on the FF formulation of quantum field theory (QFT) and its applications. The literature can be traced following [2, 10, 13–20].

In Sec. III . . . Paper outline:

III. BARE HAMILTONIAN

Canonical FF Hamiltonian of QCD is derived [13, 14] from the Lagrangian density

$$\mathcal{L}_{\rm QCD} \;=\; \bar\psi \left(i\partial\!\!\!/ - gA\!\!\!/ - m\right)\psi - \frac{1}{2}\, {\rm Tr}\, G_{\mu\nu}G^{\mu\nu} \; , \tag{4}$$

where the sum over quark flavors is assumed but not indicated. The derivation is recalled below for it is used later to explain our procedure for dealing with the small-x divergences caused by gluons.

A. FF Hamiltonian of QCD

By minimizing action described by $\mathcal{L}_{\rm QCD}$ of Eq. (4), one obtains equations

$$(i\partial\!\!\!/ - gA\!\!\!/ - m)\psi \;=\; 0 \; , \tag{5}$$

$$\partial_\alpha G^{a\alpha\beta} + ig[A_\alpha, G^{\alpha\beta}]^a + J_\psi^{a\beta} \;=\; 0 \; , \tag{6}$$

where the quark current is $J_\psi^{a\beta} = -g\bar\psi T^a \gamma^\beta \psi$. In the gauge $A^+ = 0$, Eq. (5) relates the field $\psi_- = \Lambda_-\psi$ to the fields $\psi_+ = \Lambda_+\psi$ and $A^\perp$, where $\Lambda_\pm = \frac{1}{2}\,\gamma^0\gamma^\pm$. Namely,

$$\psi_- \;=\; \frac{1}{i\partial^+}[\alpha^\perp(i\partial^\perp - gA^\perp) + \beta m]\psi_+ \; . \tag{7}$$

Similarly, Eq. (6) with $\beta = +$, provides the constraint

$$A^- \;=\; \frac{2\partial^\perp}{\partial^+}A^\perp + \frac{2}{\partial^{+2}}(J_A^+ + J_\psi^+) \; , \tag{8}$$

with $J_A^{a\mu} = -ig[\partial^\mu A^\perp, A^\perp]^a$. Inversion of ∂^{+2} is a remote analog of inverting Laplasian in the Poisson equation to obtain the Coulomb potential in the IF of Hamiltonian dynamics. Generally, inversion of the derivative ∂^+ is understood here in terms of the inverse of momentum in the Fourier transform. Field modes constant in x^- will not appear in the regulated theory, see below. When the coupling constant g is set to 0, the constraints simplify to the constraint equations for free particles,

$$\psi_{f-} = \frac{1}{i\partial^+}\left(\alpha^\perp i\partial^\perp + \beta m\right)\psi_+ \ , \tag{9}$$

$$A_f^- = \frac{2\partial^\perp}{\partial^+}A^\perp \ . \tag{10}$$

One uses these equations to define the fields

$$\psi_f = \psi_+ + \psi_{f-} \ , \tag{11}$$

$$A_f = \left(A_f^-, A^+ = 0, A^\perp\right) \ . \tag{12}$$

Canonical formula for the FF Hamiltonian of QCD expressed in terms of the fields ψ_f and A_f, is provided by the component P^- of the total four-momentum carried by the quark and gluon fields. Thus, P^- is an integral of Noether's energy-momentum tensor-density component $+-$ over the front,

$$H_{\rm QCD} = \int_F T^{+-}_{\rm QCD} \ , \tag{13}$$

where $\int_F = \int d^4x\, \delta(x^+)$. The Noether tensor is

$$T^{\mu\nu}_{\rm QCD} = \sum_\chi \frac{\partial \mathcal{L}_{\rm QCD}}{\partial\partial_\mu\chi}\ \partial^\nu\chi - g^{\mu\nu}\mathcal{L}_{\rm QCD} \ . \tag{14}$$

The sum extends over all fields χ in $\mathcal{L}_{\rm QCD}$ with $A^+ = 0$. The Hamiltonian reads

$$H_{\rm QCD} = \int d^2x^\perp dx^-\ \mathcal{H}_{\rm QCD} \ , \tag{15}$$

where the integrated density is, *cf.* [21],

$$\begin{aligned}\mathcal{H}_{\rm QCD} &= \bar\psi_f \frac{\gamma^+(-\partial^{\perp 2}+m^2)}{2i\partial^+}\psi_f + \frac{1}{2}A^a_{f\mu}\partial^{\perp 2}A^{a\mu}_f - (J^{a\mu}_{\psi f} + J^{a\mu}_{Af})A^a_{f\mu} \\ &+ \frac{1}{2}g^2\bar\psi_f \not{A}_f\frac{\gamma^+}{i\partial^+}\not{A}_f\psi_f - \frac{1}{4}g^2[A_{f\mu}, A_{f\nu}]^a[A^\mu_f, A^\nu_f]^a \\ &- \frac{1}{2}(J^{a+}_{\psi f} + J^{a+}_{Af})\frac{1}{\partial^{+2}}(J^{a+}_{Af} + J^{a+}_{\psi f}) \ ,\end{aligned} \tag{16}$$

with currents

$$J^{a\mu}_{\psi f} = -g\bar\psi_f T^a\gamma^\mu\psi_f \ , \tag{17}$$

$$J^{a\mu}_{Af} = ig[\partial^\mu A_{f\nu}, A^\nu_f]^a \ . \tag{18}$$

These currents depend on the fields $A^\perp$ and ψ_+ only, since they include explicit solutions for ψ_{f-} and A_f^-. They differ from the currents

$$J^{a\mu}_\psi = -g\bar\psi T^a\gamma^\mu\psi \ , \tag{19}$$

$$J^{a\mu}_A = ig[A_\alpha, G^{\alpha\mu}]^a \tag{20}$$

B. Gluon mass and auxiliary field

The gluon mass term and the associated term for the auxiliary octet scalar field $\phi = \phi^a T^a$, are added to the density $\mathcal{H}_{\rm QCD}$ of Eq. (16) to form the amended density,

$$\mathcal{H}^{m_g}_{\rm QCD} = \mathcal{H}_{\rm QCD} - \frac{1}{2}m_g^2 A^a_{f\mu}A^{a\mu}_f + \frac{1}{2}\phi^a(-\partial^{\perp 2} + m_g^2)\phi^a + m_g\phi^a\frac{1}{\partial^+}(J^{a+}_{Af} + J^{a+}_{\psi f}) \ . \tag{21}$$

The gluon mass parameter m_g and field ϕ introduced in Eq. (21) suffice for the purpose of regulating the gluon dynamics, see Sec. IV. When m_g is set to 0, the amended density becomes equal to the canonical $\mathcal{H}_{\rm QCD}$ of Eq. (16) plus a density for a decoupled, free massless field ϕ. The latter one can be ignored. Thus the limit $m_g \to 0$ recovers the canonical FF Hamiltonian of QCD.

In Eq. (21), the mass term for the gluon field A involves only the constrained field A_f of Eq. (12). The kinetic term for the field ϕ has the standard form for scalar bosons. The coupling of ϕ to the quark current is similar in structure, besides the color factors, to the couplings introduced in massive QED [11, 22, 23]. The auxiliary field ϕ couples to the color current of gluons. However, it does not couple directly to itself, despite that it carries color. Our scheme differs from the generalized Stueckelberg formalism for massive Yang-Mills fields, in which additional constraints are imposed to eliminate terms inversely proportional to the gauge-boson mass parameter [24]. Such additional constraints are not needed here.

The Fourier modes of the field ϕ with momentum p couple to the color currents proportionally to m_g/p^+. This ratio diverges for $p^+ \to 0$. Precisely such coupling yields the cancellation of the severe small-x divergences due to the assignment of mass m_g to the transverse gluos, see Sec. V.

The density $\mathcal{H}^{m_g}_{\rm QCD}$ can also be derived from the Lagrangian density

$$\mathcal{L}^{m_g}_{\rm QCD} \;=\; \mathcal{L}_{\rm QCD} + \frac{1}{2}\,(m_g A^a_\mu + \partial_\mu \phi^a)^2 \;, \tag{22}$$

proceeding in parallel to Sec. III A using gauge $A^+ = 0$. The term added to $\mathcal{L}_{\rm QCD}$ in Eq. (22) is not gauge invariant, as the regularization purpose it serves is not. Since the field ϕ is assumed very weakly coupled to gluons and since color is confined, any physical effects due to that field, if it existed, would be hard to detect. In principle, ϕ might contribute to the gravitational effects.

Quark fields ψ obey the same equations in the case of $\mathcal{L}^{m_g}_{\rm QCD}$ as in the case of $\mathcal{L}_{\rm QCD}$, for the added term does not depend on the fermion fields. Therefore, the constraint on ψ_- is the same as in QCD with $m_g = 0$, Eq. (7). Keeping $A^+ = 0$, variation of A^- yields the constraint

$$A^- \;=\; A^-_f + \frac{2}{\partial^{+2}}(J^+_{Af} + J^+_{\psi f}) + \frac{2m_g}{\partial^+}\phi \;, \tag{23}$$

which differs from the canonical Eq. (8) by the last term only, given the identities $J^+_A = J^+_{Af}$ and $J^+_\psi = J^+_{\psi f}$ that follow from $\gamma^{+2} = 0$. The field ϕ obeys

$$\Box\phi + m_g \partial_\mu A^\mu \;=\; 0 \;. \tag{24}$$

The constraint Eq. (23) implies

$$\partial_\mu A^\mu \;=\; \frac{1}{\partial^+}(J^+_A + J^+_\psi) + m_g \phi \;, \tag{25}$$

$$(\Box + m_g^2)\phi \;=\; -\frac{m_g}{\partial^+}(J^+_A + J^+_\psi) \;. \tag{26}$$

The latter equation exhibits the so-called "longitudinal" or "good" currents [25] as the only sources of the field ϕ. The amended Hamiltonian density $\mathcal{H}^{m_g}_{\rm QCD}$ is obtained by evaluating P^- that follows from the energy-momentum tensor density implied by $\mathcal{L}^{m_g}_{\rm QCD}$.

C. Quantum Hamiltonian

The Hamiltonian,

$$H^{m_g}_{\rm QCD} \;=\; \int d^2x^\perp dx^- \; \mathcal{H}^{m_g}_{\rm QCD} \;, \tag{27}$$

where the density $\mathcal{H}^{m_g}_{\rm QCD}$ is a function of fields ψ_f, A_f, and ϕ, given in Eq. (21), is turned into a quantum operator $\hat H^{m_g}_{\rm QCD}$ by replacing the fields at $x^+ = 0$ with operators,

$$\hat\psi_f = \sum_{c=1}^{3}\sum_{\sigma=1}^{2}\int[p]\left[u_{p\sigma}\chi_c\hat b_{p\sigma c}e^{-ipx} + v_{p\sigma}\chi_c\hat d^\dagger_{p\sigma c}e^{ipx}\right]_{x^+=0} , \tag{28}$$

$$\hat A^\mu_f = \sum_{c=1}^{8}\sum_{\sigma=1}^{2}\int[p]\left[\varepsilon^\mu_{p\sigma}T^c\hat a_{p\sigma c}e^{-ipx} + \varepsilon^{\mu *}_{p\sigma}T^c\hat a^\dagger_{p\sigma c}e^{ipx}\right]_{x^+=0} , \tag{29}$$

$$\hat\phi = \sum_{c=1}^{8}\int[p]\left[-iT^c\hat a_{p3c}e^{-ipx} + iT^c\hat a^\dagger_{p3c}e^{ipx}\right]_{x^+=0} . \tag{30}$$

In this notation, c refers to colors of $SU(3)$ and σ denotes spin projections on z-axis. Momentum integration measure reads $[p] = dp^+d^2p^\perp/[2p^+(2\pi)^3]$ with $p^+ > 0$, $p^\perp = (p^1, p^2)$, $px = p^-x^+/2 + p^+x^-/2 - p^\perp x^\perp$. The quark spinors, $u_{p\sigma}$, $v_{p\sigma}$, and gluon polarization vectors, $\varepsilon^\mu_{p\sigma}$, are given in [14], App. A. Flavors of quarks are kept in mind. The creation and annihilation operators, $\hat b, \hat b^\dagger, \hat d, \hat d^\dagger, a, a^\dagger$, will be commonly referred to as particle operators. They are assumed to satisfy the anti-commutation relations for fermions and commutation relations for bosons,

$$\left\{\hat b_{p\sigma c}, \hat b^\dagger_{p'\sigma'c'}\right\} = \left\{\hat d_{p\sigma c}, \hat d^\dagger_{p'\sigma'c'}\right\} = \left[\hat a_{p\sigma c}, \hat a^\dagger_{p'\sigma'c'}\right] = 2p^+(2\pi)^3\delta^3(p-p')\delta_{\sigma\sigma'}\delta_{cc'} . \tag{31}$$

Other commutators or anti-commutators vanish. Fermion operators commute with the boson operators. By definition, the annihilation operators b, d, a produce 0 when they act on the vacuum state, denoted by $|0\rangle$. Products of particle operators in all terms of the Hamiltonian $\hat H^{m_g}_{\rm QCD}$ are normal-ordered according to the pattern $a^\dagger b^\dagger d^\dagger dba$. The normal-ordering is indicated by double dots. Thus,

$$\hat H^{m_g}_{\rm QCD} = \int d^2x^\perp dx^- \ : \mathcal{H}^{m_g}_{\rm QCD}(\hat\psi_f, \hat A_f, \hat\phi) : \ . \tag{32}$$

This Hamiltonian is singular and requires regularization as a part of the renormalization procedure described in next sections. The normal ordering will be always present but not indicated. Hats will be omitted to farther simplify the notation.

IV. REGULARIZATION

Regulating the Hamiltonian of Eq. (32) begins with limiting from below the range of momentum p^+ labeling creation and annihilation operators in the quantum fields in Eqs. (28), (29) and (30). The lower bound on p^+ is denoted by $\epsilon^+ > 0$. This lower bound is set to be smaller than any p^+ in any physical process.

A. Particle momenta and cutoff ϵ^+

To introduce the cutoff ϵ^+, we define the integrals over the momenta that label creation and annihilation operators in Eqs. (28), (29) and (30) as limits of discrete sums. The continuous notation is meant to correspond to the discrete momentum spacing that is too small to notice in physical applications. Care needs to be exercised because there are two limits to consider in discussion of physical processes, a large-volume limit and a long-time limit, *cf.* [26]. We introduce a large cuboid extending from $-2L$ to $2L$ in x^- and from $-L$ to L in x^1 and x^2. The cuboid is assumed large enough to easily contain all processes of physical interest. These processes are meant to develop over front "time" x^+ much shorter than $4L$.

The discrete plane-wave momenta in the quark and gluon fields are defined using the periodic boundary conditions on the cuboid walls at $x^+ = 0$,

$$p_n = \epsilon\, n \ , \ n = (n^+, n^1, n^2) , \tag{33}$$

where $\epsilon = \pi/L$, n^+ is a natural number while n^1 and n^2 are integers. The number n^+ is positive because a free particle of positive mass and finite energy E has $p^+ = E + p^z > 0$ no matter how fast and in what direction the particle moves. The assumption is that, in the absence of interaction, the theory describes the energy of free particles, which we consider to be the field quanta, see *e.g.*, [27], p. 297. Since the box can be arbitrarily large, ϵ can be arbitrarily small.

Suppose one sets an ultraviolet cutoff Δ on the particle energy [28–31],

$$E_n \;=\; (p_n^+ + p_n^-)/2 < \Delta \;, \tag{34}$$

where $p_n^- = (p_n^{\perp 2} + \mu^2)/p_n^+$. Such cutoff implies that

$$\Delta - \Delta_n \;<\; p_n^+ < \Delta + \Delta_n \;, \tag{35}$$

with $\Delta_n = \sqrt{\Delta^2 - p_n^{\perp 2} - \mu^2}$. Thus, $p_n^+ > \epsilon^+$, where $\epsilon^+ = (\mu^2 + p_n^{\perp 2})/(2\Delta) > 0$ for $\Delta \gg \mu^2 \gg \epsilon^2$. Thus, the energy cutoff implies that the momentum p_n^+ for particles with positive masses is separated by the positive gap ϵ^+ from zero. In the infrared box limit, $L \to \infty$, and the ultraviolet cutoff limit, $\Delta \to \infty$, the question of which quantity is greater, ϵ or ϵ^+, has an answer depending on the order of limits,

$$\frac{\epsilon^+}{\epsilon} \;=\; \frac{\mu L}{2\pi}\,\frac{\mu}{\Delta} \;. \tag{36}$$

This formula relates the ratio ϵ^+/ϵ to the ratio of the number of the particle Compton wavelengths that fit in the cube edge and the number of free particles at rest whose total energy equals the cutoff Δ. We assume that the infrared limit $L \to \infty$ is taken first for a fixed mass μ. In that case, $\epsilon^+ \gg \epsilon$. In the field expansions, the sum over momenta p_n involves p_n^+ separated from 0 by the gap $\mu^2/(2\Delta)$. Therefore, the integrals in Eqs. (28), (29), and (30), do not extend down to $p^+ = 0$. Consequently, the Hamiltonian $H_{\rm QCD}^{m_g}$ does not contain any terms that are made only of creation or only of annihilation operators. Such terms would imply that a natural multiple of ϵ^+ equals 0.

The largest value of p^+ allowed in the operator labels in Eqs. (28), (29) and (30), say Q^+, is assumed large enough to consider all physical processes of interest. Since every physical process is characterized by some conserved value of its total plus momentum, say P^+, and since the plus momentum of each participating field quantum is smaller than P^+, it is sufficient to assume that Q^+ is finite but greater than the total P^+ in any process of physical interest.

B. The weak-coupling issue of vacuum

The Hamiltonian of Eq. (32) is normal-ordered and, according to Sec. IV A, all its terms contain at least one annihilation operator. Such terms annihilate the bare vacuum state, denoted by $|0\rangle$. Therefore, the state $|0\rangle$ is an eigenstate of H with an eigenvalue 0. It is also an eigenstate with eigenvalue 0 of the normal-ordered operators P^+ and $P^\perp$, which makes it a candidate for the theory vacuum state.

The issue is that the interactions can cause formation of Hamiltonian eigenstates with eigenvalues smaller than 0, for any finite eigenvalues of P^+ and $P^\perp$. But when the regulated interaction terms are made arbitrarily small, by making the bare coupling constant g arbitrarily small, all eigenstates with positive P^+ must have eigenvalues P^- greater than 0. It has to be so because all the quanta of the regulated theory are assigned masses squared greater than 0. Therefore, all quanta necessarily contribute only positive amounts to the total value of free P^-. The interaction energy is too small to cancel that free positive P^- when g tends to 0. Hadrons are then considered to be states obtained by acting with the creation operators on $|0\rangle$.

The situation changes when the coupling constant is increased and the Hamiltonian develops negative eigenvalues due to the interactions. Such eigenvalues would invalidate the assumption that the eigenstate $|0\rangle$ of eigenvalue 0 is the theory's ground state. Therefore, we assume that no such strong binding effect occurs in QCD with physically justified values of the coupling constant and masses of the quanta. The issue becomes whether the squares of masses in the renormalized effective-particle FF kinetic energies, including the adjustment of the mass-squared counterterms to be discussed later, are large enough to prevent interactions from generating negative eigenvalues P^- for bound states of quarks and gluons. The author does not know any example of such negative eigenvalues in any realistic theory.

C. Regulating interaction terms

Interaction terms in $H_{\rm QCD}^{m_g}$ do not converge as functions of the particle momenta. Such interactions cause arbitrarily large changes of momenta and produce diverging transition amplitudes. The pertinent measure of momentum is p^-, an eigenvalue of the free part of the Hamiltonian, obtained from $H_{\rm QCD}^{m_g}$ by setting the bare coupling constant g to 0. The free part is denoted by H_f,

$$H_f \;=\; \sum_{\sigma=1}^{2} \int [p]\, \frac{p^{\perp 2} + m^2}{p^+} \left(b_{p\sigma}^\dagger b_{p\sigma} + d_{p\sigma}^\dagger d_{p\sigma} \right) + \sum_{\sigma=1}^{3} \int [p]\, \frac{p^{\perp 2} + m_g^2}{p^+}\, a_{p\sigma}^\dagger a_{p\sigma} \;. \tag{37}$$

It results from the first two density terms on the right-hand side of Eq. (16) and from the terms second and third on the right-hand side of Eq. (21). Quanta of the auxiliary gluon field ϕ are accounted for by summing over 3 instead of only 2 polarizations. The interaction part of $H_{\rm QCD}^{m_g}$ is denoted by H_I,

$$H_{\rm QCD}^{m_g} \;=\; H_f + H_I \; . \tag{38}$$

With all annihilation operators commonly denoted by a, the structure of H_I reads

$$H_I \;=\; \sum_i H_i \; , \tag{39}$$

$$H_i \;=\; \left[\prod_{m=1}^{c_i} \int [p_m] a^\dagger_{p_m}\right] \left[\prod_{n=1}^{a_i} \int [q_n] a_{q_n}\right] \tilde\delta_{c.a} \; h_i(\bar p, \bar q) \; , \tag{40}$$

where c_i and a_i are the numbers of creation and annihilation operators in the term i, respectively. The coefficient $h_i(\bar p, \bar q)$ depends on the set of created particles' quantum numbers, $\bar p$, which includes p_m with $m = 1, ..., c_i$, and the set of annihilated particles' quantum numbers, $\bar q$, which includes q_n with $n = 1, ..., a_i$. Canonical coefficients h_i are proportional to the coupling constant g or its square. The symbol $\tilde\delta_{c.a}$ stands for the δ-function that secures conservation of the total kinematic momentum of quanta involved in a term,

$$\tilde\delta_{c.a} \;=\; 2(2\pi)^3 \delta\left(p_c^+ - p_a^-\right) \; \delta^2\left(p_c^\perp - p_a^\perp\right) \; , \tag{41}$$

where the subscripts c and a refer to quanta created and annihilated by a term, respectively,

$$p_c \;=\; \sum_{m=1}^{c_i} p_m \; , \; p_a \;=\; \sum_{n=1}^{a_i} q_n \; . \tag{42}$$

Our regularization scheme sets a limit on the magnitude of change of the particle momenta that the interaction terms can cause. We first describe the regularization of the interaction terms that are linear in g and then of the terms that are quadratic in g.

1. *Terms linear in g*

Regularization of terms linear in g amounts to inserting in them regulating factors $f^r_{\bar p, \bar q}$,

$$T(\bar p, \bar q) \;\to\; f^r_{\bar p, \bar q} \; T(\bar p, \bar q) \; , \tag{43}$$

where, using notation of Eq. (42),

$$f^r_{\bar p, \bar q} \;=\; e^{-[r\,(p_c^- - p_a^-)]^2} \;=\; e^{-[r\,(\mathcal{M}_c^2 - \mathcal{M}_a^2)/p_c^+]^2} \; , \tag{44}$$

with $\mathcal{M}_c^2 = p_c^2$ and $\mathcal{M}_a^2 = p_a^2$. The specific form of $f^r_{\bar p, \bar q}$, Eq. (44), is chosen to match the solutions of the homogeneous renormalization group equations for Hamiltonians, described in Sec. VI.

The insertion of regulating factors in terms linear in g, is illustrated below using the three-gluon interaction term. The term originates from the density term $-J^{a\mu}_{Af} A^a_{f\mu}$ for transverse gluons in Eq. (16). Before regularization, the canonical term reads

$$H_{A^3} \;=\; g \sum_{c_1, c_2, c_3 = 1}^{8} \; \sum_{\sigma_1, \sigma_2, \sigma_3 = 1}^{2} \int [123] \, a^\dagger_1 a^\dagger_2 a_3 \; Y_{123} \, \tilde\delta_{12.3} + h.c. \; , \tag{45}$$

where

$$Y_{123} \;=\; i f^{c_1 c_2 c_3} \left[\varepsilon_1^* \varepsilon_2^* \, \varepsilon_3 k_{12} - \varepsilon_1^* \varepsilon_3 \, \varepsilon_2^* k_{12}/x_2 - \varepsilon_2^* \varepsilon_3 \, \varepsilon_1^* k_{12}/x_1\right] \; . \tag{46}$$

The symbols ε denote the gluon polarization four-vectors. The three-gluon vertex function Y_{123} grows proportionally to $k^\perp_{12}$, the relative transverse momentum of the created gluons 1 and 2, $k^\perp_{12} = x_2 p^\perp_1 - x_1 p^\perp_2$, $x_1 = p^+_1/p^+_3$, $x_2 = p^+_2/p^+_3 = 1 - x_1$, $k^+_{12} = 0$. This growth leads to the ultraviolet transverse divergences in transition amplitudes. The vertex also grows as $1/x_1$ or $1/x_2$ when x_1 or x_2 tend to 0, which leads to the small-x divergences. Additional complexity results from the ratios $k^\perp_{12}/x_1$ and $k^\perp_{12}/x_2$. For example, a term with an infrared $k^\perp$ may become ultraviolet when $x \to 0$.

We use the gluon mass parameter m_g to untangle the mixture of ultraviolet and infrared divergences. The regulated three-gluon interaction term reads

$$H^r_{A^3} \;=\; g \sum_{\sigma_1,\sigma_2,\sigma_3=1}^{2} \int [123]\, a_1^\dagger a_2^\dagger a_3 \; Y_{123}\, \tilde\delta_{12.3}\; f^r_{12,3} + h.c.\ , \tag{47}$$

where

$$f^r_{12,3} \;=\; e^{-[r\,(\mathcal{M}^2_{12}-m_g^2)/p_3^+]^2}\ . \tag{48}$$

The invariant mass $\mathcal{M}_{12}$ only depends on the relative motion variables of gluons 1 and 2 , $x = x_1$ and $k^\perp = k^\perp_{12}$,

$$\mathcal{M}^2_{12} \;=\; \frac{k^{\perp 2} + m_g^2}{x(1-x)}\ . \tag{49}$$

When the invariant mass is finite and $r \to 0$, the regulating factor $f^r_{12,3}$ tends to 1. When the invariant mass diverges and r is fixed, the factor tends to 0. The regulating suppression of the vertex occurs when $\mathcal{M}^2_{12}$ exceeds the magnitude of p_3^+/r or, for $|k^\perp|$ smaller than m_g, when x decreases below rm_g^2/p_3^+. The presence of m_g guarantees that $\mathcal{M}_{12} \to \infty$ when $x \to 0$. Thus the small-x gluon divergences are turned into ultraviolet divergences. Without m_g, the region of small x would not be regulated by the factor $f^r_{12,3}$, for one could have $k^\perp \sim x^\alpha$ with $\alpha \geq 1/2$. The non-locality of vertices regulated using factors $f^r_{12,3}$ is analyzed in [32, 33].

All terms linear in g in the Hamiltonian $\hat H^{m_g}_{\rm QCD}$ are regulated using factors $f^r_{\tilde p,\tilde q}$ as in the above example. Such terms contain two creation operators, one for a quantum of some mass m_1 and another one for a quantum of some mass m_2, and an annihilation operator for some quantum of mass m_3, in correspondence to $a_1^\dagger$, $a_2^\dagger$ and a_3 in Eq. (47). One uses the invariant mass of quanta 1 and 2, squared,

$$\mathcal{M}^2_{12} \;=\; \frac{k^{\perp 2}_{12} + m_1^2}{x_1} + \frac{k^{\perp 2}_{12} + m_2^2}{x_2}\ . \tag{50}$$

The regulating factor is

$$f^r_{12,3} \;=\; e^{-[r\,(\mathcal{M}^2_{12}-m_3^2)/p_3^+]^2}\ . \tag{51}$$

2. *Terms quadratic in g*

Regularization of the canonical terms proportional to g^2 concerns products of four fields. There are two kinds of them. One kind results from the constraint Eqs. (7) and (23). These terms involve an inverse of ∂^+. They are traditionally called seagulls. The other kind originates from the square of $G^{a\mu\nu}$ in the Lagrangian density. Both kinds are treated in the same way, described below using the example of a seagull.

Constraint Eqs. (8) and (23) relate gauge field component A^- to $1/\partial^{+2}$ acting on the color current J. The latter involves products of two fields. Therefore, when a color current is multiplied by the constrained A^-, the resulting density term is of the form $J(1/\partial^{+2})J$. Each J can be though of as creating or annihilating a *gedanken* gluon that carries the momentum associated with $1/\partial^{+2}$. The imagined gluon together with the current J form a vertex that is regulated by the factor $f^r_{12,3}$ in the same way as the three-gluon vertex is in Eq. (47). In the fermion seagulls, the *gedanken* quantum is a fermion rather than a boson. In the Hamiltonian term due to the product of commutators, $g^2[A^i, A^j]^a[A^i, A^j]^a$, each commutator is treated as a current, even though the inverse of ∂^+ is absent.

The rgularization of terms proportional to g^2 is illustrated using the gluon seagull term

$$H_{JJ} \;=\; \int dx^- d^2x^\perp\ \frac{1}{2}\,(J^{a+}_{\psi f} + J^{a+}_{Af})\frac{-1}{\partial^{+2}}(J^{a+}_{Af} + J^{a+}_{\psi f})\ , \tag{52}$$

originating from the last density term in Eq. (16). It involves four terms, each of which is a product of four fields. All these products are regulated according to the same rule. It suffices to describe one product. We describe in detail the regularization of the term that involves only quark and antiquark operators,

$$H_{J_\psi J_\psi} \;=\; \int dx^- d^2x^\perp\ \frac{1}{2}\, J^{a+}_{\psi f}\frac{1}{(i\partial^+)^2} J^{a+}_{\psi f}\ . \tag{53}$$

The current at $x^+ = 0$ is

$$J^{a+}_{\psi f}(x) \;=\; -g \sum_{c_1,c_2}\sum_{\sigma_1,\sigma_2} \chi^\dagger_{c_1} T^a \chi_{c_2} \int [p_1 p_2]$$
$$\times\; \left[\bar u_{p_1\sigma_1} b^\dagger_{p_1\sigma_1 c_1} e^{ip_1 x} + \bar v_{p_1\sigma_1} d_{p_1\sigma_1 c_1} e^{-ip_1 x}\right]$$
$$\times\; \gamma^+ \;\left[u_{p_2\sigma_2} b_{p_2\sigma_2 c_2} e^{-ip_2 x} + v_{p_2\sigma_2} d^\dagger_{p_2\sigma_2 c_2} e^{ip_2 x}\right] \;. \tag{54}$$

The spinor products are

$$\bar u_1 \gamma^+ u_2 \;=\; \bar v_1 \gamma^+ v_2 \;=\; 2\sqrt{p_1^+ p_2^+}\;\delta_{\sigma_1,\sigma_2} \;, \tag{55}$$

$$\bar u_1 \gamma^+ v_2 \;=\; \bar v_1 \gamma^+ u_2 \;=\; -2\sqrt{p_1^+ p_2^+}\;\delta_{\sigma_1,-\sigma_2} \;. \tag{56}$$

Hence the normal-ordered, regulated current is

$$:J^{a+}_{\psi f}: \;=\; -g \sum_{c_1,c_2}\sum_{\sigma_1,\sigma_2} \chi^\dagger_{c_1} T^a \chi_{c_2} \int [p_1 p_2]\; 2\sqrt{p_1^+ p_2^+}\;\; [:\{\;\;\}:]^r \;, \tag{57}$$

where the bracket,

$$[:\{\;\;\}:]^r \;=\; b_1^\dagger b_2 e^{i(p_1-p_2)x}\left[\theta_{1-2} f^r_{1,2(1-2)} + \theta_{2-1} f^r_{2,1(2-1)}\right]\delta_{\sigma_1,\sigma_2}$$
$$-\; d_2^\dagger d_1 e^{i(p_2-p_1)x}\left[\theta_{2-1} f^r_{2,1(2-1)} + \theta_{1-2} f^r_{1,2(1-2)}\right]\delta_{\sigma_1,\sigma_2}$$
$$-\; b_1^\dagger d_2^\dagger e^{i(p_1+p_2)x} f^r_{12,(1+2)}\delta_{-\sigma_1,\sigma_2}$$
$$-\; d_1 b_2 e^{-i(p_1+p_2)x} f^r_{12,(1+2)}\delta_{-\sigma_1,\sigma_2} \;, \tag{58}$$

carries the superscript r to indicate the insertion of the regulating factors. The *gedanken* gluon is assigned the mass m_g for evaluating its minus momentum. The Heaviside's θ-function $\theta_{i-j} = \theta(p_i^+ - p_j^+)$. Thus the regulated quark seagull is

$$H^r_{J_\psi J_\psi} \;=\; \int dx^- d^2x^\perp \;\left[\frac{1}{2}\, J^{a+}_{\psi f} \frac{1}{(i\partial^+)^2} J^{a+}_{\psi f}\right]^r \tag{59}$$

$$=\; \frac{1}{2}\int dx^- d^2x^\perp \;:\left[:J^{a+}_{\psi f}:\right]^r \frac{1}{(i\partial^+)^2}\left[:J^{a+}_{\psi f}:\right]^r: \;. \tag{60}$$

The colon, a symbol for the normal-ordering, is dropped below. For the purpose of illustration of Eq. (60). We display details of the term that transforms a quark-antiquark pair into another such pair.

$$H^{r\,q\bar q}_{J_\psi J_\psi} \;=\; \frac{1}{2}\, g^2 \sum_{1234} T^{\tilde c}_{12} T^{\tilde c}_{34} \int [1234]\;\tilde\delta_{c.a}\;\{\;\;\}\, 4\sqrt{p_1^+ p_2^+ p_3^+ p_4^+} \;, \tag{61}$$

where one sums over the color superscripts $\tilde c$ from 1 to 8, and

$$\{\;\;\} \;=\; b_1^\dagger d_4^\dagger d_3 b_2\;\theta_{1-2} f^r_{1,2(1-2)}\delta_{\sigma_1,\sigma_2}\frac{-2}{(p_3^+ - p_4^+)^2}\theta_{3-4} f^r_{3,4(3-4)}\delta_{\sigma_3,\sigma_4}$$
$$+\; b_1^\dagger d_4^\dagger d_3 b_2\;\theta_{2-1} f^r_{2,1(2-1)}\delta_{\sigma_1,\sigma_2}\frac{-2}{(p_3^+ - p_4^+)^2}\theta_{4-3} f^r_{4,3(4-3)}\delta_{\sigma_3,\sigma_4}$$
$$+\; b_1^\dagger d_2^\dagger d_3 b_4\;f^r_{12,(1+2)}\delta_{-\sigma_1,\sigma_2}\frac{2}{(p_3^+ + p_4^+)^2} f^r_{34,(3+4)}\delta_{-\sigma_3,\sigma_4} \;. \tag{62}$$

3. *Fully regulated bare Hamiltonian*

Fully regulated QCD Hamiltonian with the gluon mass m_g and field ϕ, denoted by $H^{m_g\,r}_{\rm QCD}$, reads

$$H^{m_g\,r}_{\rm QCD} \;=\; \int d^2x^\perp dx^-\;\mathcal{H}^{m_g\,r}_{\rm QCD}(\psi_f, A_f, \phi) \;, \tag{63}$$

where

$$\mathcal{H}^{m_g\,r}_{\rm QCD} \;=\; \mathcal{H}^{r}_{\rm QCD} - \frac{1}{2}\, m_g^2 A^a_{f\mu} A^{a\mu}_f + \frac{1}{2}\, \phi^a(-\partial^{\perp 2} + m_g^2)\phi^a + \left[m_g \phi^a \frac{1}{\partial^+}(J^{a+}_{Af} + J^{a+}_{\psi f}) \right]^r \;, \tag{64}$$

and

$$\mathcal{H}^{r}_{\rm QCD} \;=\; \bar\psi_f \frac{\gamma^+(-\partial^{\perp 2} + m^2)}{2i\partial^+}\psi_f + \frac{1}{2}\, A^a_{f\mu}\partial^{\perp 2} A^{a\mu}_f - \left[(J^{a\mu}_{\psi f} + J^{a\mu}_{Af}) A^a_{f\mu} \right]^r + \frac{1}{2}\, g^2 \left[\bar\psi_f \not{A}_f \frac{\gamma^+}{i\partial^+} \not{A}_f \psi_f \right]^r - \frac{1}{4}\, g^2 \left[[A_{f\mu}, A_{f\nu}]^a [A^\mu_f, A^\nu_f]^a \right]^r - \frac{1}{2} \left[(J^{a+}_{\psi f} + J^{a+}_{Af}) \frac{1}{\partial^{+2}} (J^{a+}_{Af} + J^{a+}_{\psi f}) \right]^r \;. \tag{65}$$

The regularization brackets $[\quad]^r$ embrace all interaction terms. The regularization brackets may be omitted to simplify notation.

V. GLUON MASS AND SMALL x IN SCATTERING AMPLITUDES

Insertion of a mass term for gluons in the FF Hamiltonian of QCD leads to severe small-x divergences due to inverse of ∂^{+2}, see, *e.g.*, [10], Sec. IX A. In contrast, the Hamiltonian $H^{m_g\,r}_{\rm QCD}$ of Eq. (63), with density $\mathcal{H}^{m_g\,r}_{\rm QCD}$ given in Eq. (64), leads to the cancellation of these singularities despite that $m_g > 0$. The reason is that $H^{m_g\,r}_{\rm QCD}$ includes also a kinetic term for a color octet of scalar fields of the same mass and an interaction term that couples that octet field to the "good" color currents of quarks and transverse gluons, proportionally to the gluon mass.

A. Severe small-x singularity

To identify the source of the gluon severe small-x divergences, the current-gluon coupling term, *i.e.*, the Hamiltonian term resulting from the third density term on the right-hand side of Eq. (65), is separated into its transverse and longitudinal parts,

$$-(J^{a\mu}_{\psi f} + J^{a\mu}_{Af}) A^a_{f\mu} \;=\; (J^{aj}_{\psi f} + J^{aj}_{Af}) A^{aj}_f - \frac{1}{2}\, (J^{a+}_{\psi f} + J^{a+}_{Af}) A^{a-}_f \;. \tag{66}$$

Only the longitudinal part involves the inverse of ∂^+. Using Eq. (10) for A^{a-}_f and integrating by parts, the Hamiltonian density of Eq. (64) for fields that vanish at large distances is transformed to the equivalent form

$$\mathcal{H}^{m_g\,r}_{\rm QCD} \;=\; \bar\psi_f \frac{\gamma^+(-\partial^{\perp 2} + m^2)}{2i\partial^+}\psi_f + \frac{1}{2}\, A^{aj}_f(-\partial^{\perp 2} + m_g^2) A^{aj}_f + (J^{aj}_{\psi f} + J^{aj}_{Af}) A^{aj}_f + \frac{1}{2}\, g^2 \bar\psi_f A^j_f \gamma^j \frac{\gamma^+}{i\partial^+} \gamma^k A^k_f \psi_f - \frac{1}{4}\, g^2 [A^j_f, A^k_f]^a [A^j_f, A^k_f]^a - \frac{1}{2}\, (J^{a+}_{\psi f} + J^{a+}_{Af}) \frac{1}{\partial^{+2}} (J^{a+}_{Af} + J^{a+}_{\psi f}) + \frac{1}{2}\, \phi^a(-\partial^{\perp 2} + m_g^2)\phi^a + \left(\partial^\perp A^{a\perp}_f + m_g \phi^a\right) \frac{1}{\partial^+} \left(J^{a+}_{Af} + J^{a+}_{\psi f} \right) \;. \tag{67}$$

The superscripts j and k take only two values, 1 and 2, for the front transverse directions. The only terms leading to $1/\partial^{+2}$, or $1/x^2$ for gluons, are

$$\mathcal{H}_{+1} \;=\; \left(\partial^\perp A^{a\perp}_f + m_g \phi^a\right) \frac{1}{\partial^+} \eta J^a_f \;, \tag{68}$$

$$\mathcal{H}_{+2} \;=\; -\frac{1}{2}\, \eta J^a_f \frac{1}{\partial^{+2}} \eta J^a_f \;, \tag{69}$$

where $\eta J^a_f = J^{a+}_{\psi f} + J^{a+}_{Af}$. The four-vector η has components $\eta^- = 2$, $\eta^+ = 0$, and $\eta^\perp = 0$. The term $\mathcal{H}_{+1}$ is proportional to g and the term $\mathcal{H}_{+2}$ to g^2. Integration of these two densities over the front yields the Hamiltonian terms denoted by H_{+1} and H_{+2}, respectively. Regularization factors in these terms are introduced according to the rules described in the previous section. The gluon severe small-x singularities occur due to H_{+2} and square of H_{+1}. It is shown in the next section that they cancel out in the transition amplitudes for quarks and gluons despite the presence of the mass parameter m_g.

B. Cancellation of $1/x^2$ in scattering amplitudes

Scattering of quarks and gluons in the femtouniverse [34] is described assuming that in the first approximation they propagate as free. Interactions are accounted for using an expansion in a series of powers of the coupling constant g. Our discussion begins with the small-x divergences in the second-order scattering matrix operator [26],

$$T^{(2)} \;=\; H_I^{(1)} \frac{1}{P^- - H_f + i\epsilon} H_I^{(1)} + H_I^{(2)} \; . \tag{70}$$

Symbols $H_I^{(1)}$ and $H_I^{(2)}$ denote the Hamiltonian terms of order g and g^2, respectively. P^- denotes the initial-state eigenvalue of the free Hamiltonian H_f.

Consider the quark-antiquark scattering in which a pair $q\bar q$ turns into $q'\bar q'$. Dropping the overall momentum conservation $\tilde\delta$-function, using j_q and $j_{\bar q}$ to denote the quark currents, which are free from the gluon mall-x singularities, denoting the four-momentum transfer from the quark to the antiquark by $k = q - q' = \bar q' - \bar q$, $k^+ > 0$, and omitting $i\epsilon$, one obtains the transition amplitude on-shell of total P^- in the form

$$\langle q'\bar q'|T^{(2)}|q\bar q\rangle \;=\; j_{q\alpha} T^{\alpha\beta} j_{\bar q\beta} \; , \tag{71}$$

$$T^{\alpha\beta} \;=\; \frac{1}{k^+(P^- - H_f)} \sum_{\sigma=1}^{3} \epsilon^\alpha_{k\sigma}\epsilon^{*\beta}_{k\sigma} + \frac{\eta^\alpha\eta^\beta}{k^{+2}} \; . \tag{72}$$

The first term on the right-hand side of Eq. (72) comes from the exchange of gluons between the quarks. The second term, proportional to $\eta^\alpha\eta^\beta$, is contributed by the seagull term, or H_{+2} corresponding to the density $\mathcal{H}_{+2}$ in Eq. (69). Both, the exchange and seagull contributions diverge as $1/k^{+2}$ when $k^+ \to 0$. The sum over gluon polarizations extends from 1 to 3 because it includes the contribution of the longitudinal gluons described by the field operator $\hat\phi$ in Eq. (30).

$$\sum_{\sigma=1}^{3} \epsilon^\alpha_{k\sigma}\epsilon^{*\beta}_{k\sigma} \;=\; -g^{\alpha\beta} + (k_0^\alpha\eta^\beta + \eta^\alpha k_0^\beta)/k^+ + m_g^2\eta^\alpha\eta^\beta/k^{+2} \; . \tag{73}$$

The first two terms above come from the transverse gluons, with $k_0^- = k^{\perp 2}/k^+$, and the third term comes from the longitudinal gluons. So, the amplitude is

$$\langle q'\bar q'|T^{(2)}|q\bar q\rangle \;=\; j_{q\alpha}\left[\frac{-g^{\alpha\beta} + \Pi^{\alpha\beta}}{k^2 - m_g^2}\right] j_{\bar q\beta} \; , \tag{74}$$

where

$$\Pi^{\alpha\beta} \;=\; (k_0^\alpha\eta^\beta + \eta^\alpha k_0^\beta)/k^+ + m_g^2\eta^\alpha\eta^\beta/k^{+2} + (k^2 - m_g^2)\,\eta^\alpha\eta^\beta/k^{+2} \; . \tag{75}$$

The quark and anti-quark currents in Eq. (74) are conserved, $k_\alpha\, j^\alpha_q = k_\alpha\, j^\alpha_{\bar q} = 0$. The formula $k_0^\alpha = k^\alpha + (k_0^- - k^-)\eta^\alpha/2$ implies

$$\Pi^{\alpha\beta} \;\equiv\; \Pi\;\eta^\alpha\eta^\beta \; , \tag{76}$$

$$\Pi \;=\; (k_0^- - k^-)/k^+ + m_g^2/k^{+2} + (k^2 - m_g^2)/k^{+2} \; . \tag{77}$$

The key result is that Π vanishes. The terms $\sim 1/x^2$ cancel out completely. The regularization factors f^r do not interfere with the cancellation because they are the same in the gluon exchange and the seagull term. The resulting scattering amplitude has the co-variant form of Eq. (74) with $\Pi^{\alpha\beta} = 0$. Moreover, the amplitude is free from any small-x singularity, not only the severe $1/x^2$. An alternative way of stating this result is that the diagrammatic rule (R8) of perturbative QCD in App. A of [14], continues to be valid even though the gluons are assigned the mass m_g.

In order to extend the above reasoning to the transition amplitudes between intermediate states that are off-shell of total P^-, we first note that the severe gluon singularities appear in perturbative calculations due to the operator

$$T_{\rm singular} \;=\; H_{+1} \frac{1}{P^- - H_f + i\epsilon} H_{+1} + H_{+2} \;, \tag{78}$$

which sums the exchange of a gluon between two color currents and the seagull. The divergences occur in the coefficients of the tensor $\eta^\alpha \eta^\beta$, contracted with the quark or gluon currents. The sum over transverse polarizations of the exchanged gluon contributes $k^{\perp 2}/k^{+2}$, as dictated by H_{+1}. The intermediate quanta of field ϕ contribute m_g^2/k^{+2}, also dictated by H_{+1}. Since the eigenvalues k^- of H_f for the quanta of fields A and ϕ are the same, $k_g^- = (k^{\perp 2} + m_g^2)/k^+$, the most singular contribution to the transition amplitude due to $k^+ \to 0$, takes the form

$$\langle T_{\rm singular} \rangle \;=\; -j_1 \left[\frac{k^{\perp 2}/k^{+2} + m_g^2/k^{+2}}{k^+(P^- - Q^- - k_g^-)} + \frac{1}{k^{+2}} \right] j_2 \;, \tag{79}$$

where the second term in the bracket, $1/k^{+2}$, comes from H_{+2}. The sum of eigenvalues of H_f for other quanta in the same intermediate state, in which the exchanged gluons appear, is denoted by Q^-. Symbols j_1 and j_2 stand for the contractions of the quark or gluon currents with η. The three-momentum conservation $\tilde{\delta}$-function and $i\epsilon$ are omitted. The term $k^{\perp 2}/k^{+2}$ in the numerator in Eq. (79) is provided by the transverse gluons and the term m_g^2/k^{+2} by the longitudinal ones. The inverse of k^{+2} would produce the severe singularity for $k^+ \to 0$ if the free Hamiltonian H_f would include the mass m_g for transverse gluons and the longitudinal gluons were absent. When their contribution is included,

$$\langle T_{\rm singular} \rangle \;=\; j_1 \frac{Q^- - P^-}{k^+(P^- - Q^-) - (k^{\perp 2} + m_g^2)} \, j_2 \, \frac{1}{k^+} \;. \tag{80}$$

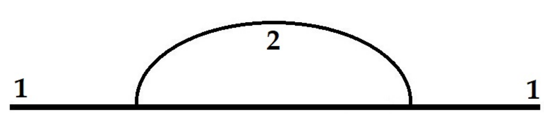
The divergence $\sim 1/k^{+2}$ for $k^+ \to 0$ cancels out despite that $m_g > 0$. At the same time, $m_g^2 > 0$ regulates the infrared divergence due to $k^\perp \to 0$ when $k^+ \to 0$.

The same mechanism works to all orders of perturbation theory. Our demonstration begins with Eq. (68). Every numerator term associated with propagation of a transverse gluon, $(k^\perp/k^+)^2$, is accompanied by a term due to the propagation of a longitudinal gluon, $(m_g/k^+)^2$. The transverse-gluon exchange and the longitudinal-gluon exchange involve the same denominator factor with the same value of the gluon k_g^-. The exchange extending over one intermediate state always appears in addition to a contribution of the Hamiltonian term H_{+2}. Therefore, the cancellation in exchanges extending over one intermediate state occurs as described above. The cancellation holds no matter how many additional interactions precede or follow the exchange. When the gluon exchange extends over more than one intermediate state, the contribution of H_{+2} is absent. However, in that case, the gluon k_g^- appears in more than one denominator. The additional denominators provide additional powers of k^+ in the numerator, which eliminate the severe singularity $1/k^{+2}$.

C. Small x in self-interactions

In the amplitudes discussed in Sec. V B, the severe small-x divergences of the gluon exchange and seagull terms cancel each other. But in the quark and gluon self-interactions, schematically illustrated in Fig. 1, the normal-ordered seagulls do not contribute. The second-order self-interaction sketched in Fig. 1 requires a counterterm. Including the

FIG. 1: Self-interaction of particle of type 1 via emission and absorption of particle of type 2.



counterterm, the self-interaction reads

$$\langle p'|p\rangle \, \frac{\Sigma^{(2)}}{p^+} \;=\; \langle p'| H_{+1} \frac{1}{P^- - Q^- - H_f} H_{+1} |p\rangle + \langle p'|p\rangle \, \frac{C^{(2)}}{p^+} \;, \tag{81}$$

where the counterterm contribution is denoted by $C^{(2)}$. The factor $\langle p'|p\rangle$ accounts for the state normalization. We obtain

$$\frac{\Sigma_q^{(2)}}{p^+} = \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3} \left[\frac{N}{D} + 4\sigma_q^{(2)}\right] f^{r2}_{qg,p} \ , \tag{82}$$

$$N = 4[k^{\perp 2} + (1-x)m_g^2]/x^2 + 2(k^{\perp 2} + x^2 m_i^2)/(1-x) \ , \tag{83}$$

$$D = x(1-x)p^+(P^- - Q^- - p_q^- - p_g^-) \ , \tag{84}$$

where $4\sigma_q^{(2)}$ stands for the integrand of the counterterm. x is the fraction of the quark momentum p^+, carried by the emitted gluon, ranging between 0 and 1. $k^\perp = p_g^\perp - xp^\perp$. For N colors of quarks, $C_F = (N^2-1)(2N)$. The squared regularization-factor, $f^{r2}_{qg,q}$, comes from the double action of the Hamiltonian term H_{+1}. The severe small-x singularities are contained in

$$\frac{\Sigma_{q\,\mathrm{sing}}^{(2)}}{p^+} = \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3} \left[\frac{4p_g^-/x}{x(P^- - Q^- - p_q^- - p_g^-)} + 4\sigma_q^{(2)}\right] f^{r2}_{qg,q} \ . \tag{85}$$

In our notation, p_q denotes the free four-momentum of a quark of mass m, after emitting and before absorbing a gluon. p_g denotes the free four momentum of the gluon, of mass m_g. It is visible that to counter the singularity $\sim 1/x^2$, one needs the counterterm integrand

$$\sigma_q^{(2)} = 1/x^2 \ . \tag{86}$$

Inclusion of the less singular terms than $4p_g^-/x$, shows that the small-x divergence $1/x^2$ and the ultraviolet quadratic transverse divergence in $\Sigma_q^{(2)}/p^+$ are simultaneously countered by the counterterm whose integrand is

$$\sigma_q^{(2)} = 1/x^2 + 1/[2(1-x)] \ . \tag{87}$$

Similarly, the second-order transverse-gluon self-interaction reads,

$$\frac{\Sigma_{g\,\mathrm{sing}}^{(2)}}{p^+} = \frac{2g^2 C_A}{p^+} \int \frac{dx d^2k^\perp}{2(2\pi)^3} \left[\frac{N}{D} + \sigma_g^{(2)}\right] f^{r2}_{gg,g} \ , \tag{88}$$

$$N = (k^{\perp 2} + m_g^2)\left[1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}\right] - m_g^2\left[\frac{1}{2} + \frac{1}{x(1-x)}\right] \ , \tag{89}$$

$$D = x(1-x)p^+(P^- - Q^-) - (k^{\perp 2} + m_g^2) \ . \tag{90}$$

For N colors $C_A = N$. It is visible that the counterterm integrand

$$\sigma_g^{(2)} = 1 + \frac{1}{x^2} + \frac{1}{(1-x)^2} \ , \tag{91}$$

removes the severe divergences of type $1/x^2$ for $x \to 0$. Inclusion of 1 in Eq. (91), secures simultaneous cancellation of the ultraviolet quadratic transverse divergence. The gluon self-interaction due to the intermediate quark pairs does not involve divergences $\sim 1/x^2$. The self-interaction of longitudinal gluons is free from such divergences as well. The remaining singularities are only logarithmic, as promised. In orders higher than second, the one-particle irreducible self-interactions require more complex expressions for the counterterms. Self-interactions in the renormalized bound-state eigenvalue problems of the Hamiltonian $H_{\mathrm{QCD}}^{m_g\, r}$ are discussed in the following sections.

VI. HAMILTONIAN EIGENVALUE PROBLEM

Eigenstates of the Hamiltonian $H_{\mathrm{QCD}}^{m_g\, r}$ of Eq. (63) represent hadrons of QCD in the limit of lifting the regularization, $r \to 0$ and $m_g \to 0$. The eigenstates are the combinations

$$|h\rangle = \sum_i \psi_i |i\rangle, \tag{92}$$

where $|i\rangle$ are the basis states obtained by acting on the state $|0\rangle$ with products of creation operators introduced in Eqs. (28), (29) and (30). To describe a hadron, one needs to compute the relevant eigenvalue of the matrix

$H^{m_g\,r}_{ij} = \langle i|H^{m_g\,r}_{\rm QCD}|j\rangle$ and evaluate the corresponding set of coefficients ψ_i. Analytic calculations appear not feasible because of the matrix complexity. To use computers, one needs to overcome several difficulties.

The Hamiltonian matrix has infinite size. One must replace it by some equivalent finite matrix. To begin with, one could limit the momenta and number of quarks and gluons in the basis states. However, setting such limits is not straightforward because the matrix elements $H^{m_g\,r}_{ij}$ diverge as functions of the momenta, *e.g.,* see Eq. (45) for $k^\perp \to \infty$ or $x \to 0$. The diverging matrix elements dominate the dynamics. They need to be dealt with first thing to gain a computational access to the finite quantities of physical interest [10, 35, 36], including the issue of explaining the relativistic quantum-binding mechanism for quarks and gluons. The singular interactions can change the number of quanta. Therefore, the states with quarks and gluons in fast relative motion are degenerate with states of less rapid but more numerous quanta. Traces of these complexities appear in QCD phenomenology. For example, applications of the parton model suggest that the gluon distribution in proton grows for small x and requires some mechanism of saturation. Eventually, one needs a method of estimating the accuracy of a computation.

To overcome the difficulties mentioned above, the QCD bound-state eigenvalue problems are formulated not in terms of the canonical creation and annihilation operators introduced in Eqs. (28), (29) and (30). Instead of b, d, and a, commonly denoted below by q, one uses operators b_s, d_s, and a_s, denoted by q_s. The latter are called the operators for effective particles. The parameter s can be thought about as a size of the particles. The canonical operators q correspond to the point-like quanta, $q = q_{s=0}$. When $s \sim 1/m$, where m is a hadron mass, the effective particles are meant to correspond to the constituent quarks and gluons. Evaluation of the QCD Hamiltonian in terms of operators q_s is carried out using the renormalization group procedure for effective particles (RGPEP) [37]. The procedure is designed to rewrite the canonical FF of QCD bound-state eigenvalue equations in an equivalent way using the effective particles, whereby the singular canonical interactions are replaced by the smoothed ones, which depend on the parameter s and do not create the eigenstate components with unlimited numbers and momenta of the constituents. To describe how the gluon mass m_g and field ϕ contribute to the effective bound-state dynamics, we need to recall the elements of the RGPEP.

A. Elements of the RGPEP

Quantum numbers of the phenomenological constituent quarks [9] are the same as those of the bare quarks in the canonical QCD, with the exception of the mass parameters. Therefore, the RGPEP for QCD assumes that the effective particles are unitarily related to the bare particles,

$$q_s \;=\; \mathcal{U}_s^\dagger q \,\mathcal{U}_s \ , \tag{93}$$

and the quantum numbers of q_s and q are the same. The Hamiltonian remains unchanged. Consequently, the products of operators q_s in the Hamiltonian have different coefficients, say c_s, from the coefficients c of the corresponding products of operators q, but

$$H(c_s, q_s) \;=\; H(c, q) \ . \tag{94}$$

The renormalized expressions for the coefficients c_s, are obtained by solving a first-order differential equation in s, called here the RGPEP equation, see Eq. (99) below. The initial condition at $s = 0$ is provided by the canonical coefficients, c, including the modifications resulting from the counterterms determined while solving the RGPEP equation, according to the rules of the similarity renormalization group procedure (SRG) [38, 39]. Computation of the coefficients c_s is carried out using the operator

$$\mathcal{H} \;=\; H(c_s, q) \;=\; \mathcal{U}_s H(c_s, q_s)\,\mathcal{U}_s^\dagger \ , \tag{95}$$

whose structure implies

$$\mathcal{H}' \;=\; [\mathcal{G}, \mathcal{H}] \ , \tag{96}$$

where prime denotes the derivative d/ds^2; the differentiation with respect to s^2 will be needed for dimensional reasons. The anti-Hermitian operator

$$\mathcal{G} \;=\; \mathcal{U}'\mathcal{U}^\dagger \tag{97}$$

is called the generator of the RGPEP transformation of the Hamiltonian. Whole class of generators can be considered [40, 41]. Hamiltonian operators we discuss have the form $\mathcal{H} = \mathcal{H}_f + \mathcal{H}_I$, where $\mathcal{H}_f$ is given in Eq. (37). The anti-Hermitian commutator $[\mathcal{H}_f, \mathcal{H}_I]$ can be used as a generator of a unitary transformation,

$$\mathcal{G} \;=\; [\mathcal{H}_f, \mathcal{H}_I] \ . \tag{98}$$

Evolution in s with such generator tends to bring the matrix of $\mathcal{H}_I$ to a band diagonal form in the basis of space of states provided by the eigenstates of $\mathcal{H}_f$. The width of the band decreases as s increases. With this generator, Eq. (96) takes the double-commutator form,

$$\mathcal{H}' = [[\mathcal{H}_f, \mathcal{H}_I], \mathcal{H}] \ , \tag{99}$$

developed by Wegner for Hamiltonians in condensed matter physics [42], see also [43–49]. The double commutator structure implies that $\mathcal{H}$ fulfills the cluster decomposition principle [27], there are no disconnected interactions. Model studies of the RGPEP equations with various generators show a need for alterations of the generator $\mathcal{G}$ of Eq. (98) when one seeks to achieve convergence of solutions for $\mathcal{H}$ obtained using the weak-coupling expansion [50, 51]. Here we are concerned with the lowest orders of the expansion and no need arises for altering Eqs. (98) and (99).

B. Renormalized effective eigenvalue problem

Assuming a solution for $\mathcal{H}$ as a function of s is available, the QCD Hamiltonian is obtained in the form

$$H_s \equiv H(c_s, q_s) = \mathcal{U}_s^\dagger [\mathcal{H} = H(c_s, q_0)] \mathcal{U}_s \ . \tag{100}$$

The eigenvalue problem for hadrons as bound states of effective quarks and gluons reads

$$H_s|\psi\rangle = P^-|\psi\rangle \ , \tag{101}$$

$$|\psi\rangle = \sum_i \psi_i(s)|i\rangle_s \ , \tag{102}$$

where P^- is the eigenvalue. The basis states $|i\rangle_s$ are created by applying products of the creation operators q_s to the state $|0\rangle$. If one solved the RGPEP equation for H_s exactly and found exact eigenvalues and eigenstates of H_s, the coefficients $\psi_i(s)$ and basis states $|i\rangle_s$ would depend on s but the eigenvalues and corresponding eigenstates $|\psi\rangle$ would not [52]. Approximate calculations yield results varying with s. The magnitude of such variation indicates how large the theoretical error of an approximate calculation may be. Another measure is provided by the accuracy of obeying symmetries, such as the rotation symmetry.

VII. WEAK-COUPLING EXPANSION FOR H_s

Given the initial condition at $s = 0$ in the form of the regulated Hamiltonian $H^{m_g\, r}_{\rm QCD}$ of Eq. (63), the differential Eq. (99) can be solved order-by-order in the expansion of the coefficients c_s in series of powers of the coupling constant g. Such expansion is valid when the coupling constant is made so small that all computed interaction terms are small. But solutions for the coefficients c_s may contain inverse powers or logarithms of the regularization parameter r. When one lifts the regularization, taking the limit $r \to 0$, such coefficients become infinite despite that s is finite and g small. Consequently, the corresponding matrix elements of H_s between the effective basis states of finite momenta diverge and the eigenvalue problem is ill-defined. To formulate a soluble theory, the initial condition at $s = 0$, given by $H^{m_g\, r}_{\rm QCD}$ in Eq. (63), needs to be modified by adding terms, called counterterms, determined by the condition that the divergences in c_s are eliminated. Thus, the initial condition for solving the RGPEP equation is changed from $H^{m_g\, r}_{\rm QCD}$ of Eq. (63) to

$$H^{m_g\, r}_{{\rm QCD}\, \mathcal{C}} = \int dx^- d^2x^\perp \left(\mathcal{H}^{m_g\, r}_{\rm QCD} + \mathcal{C}^{m_g\, r}_{\rm QCD} \right) \ . \tag{103}$$

The subscript $\mathcal{C}$ indicates the inclusion of the counterterms, denoted by $\mathcal{C}^{m_g\, r}_{\rm QCD}$. Perturbative solution of the RGPEP equation for H_s, including the counterterms in the initial condition at $s = 0$, becomes calculable order-by-order for very small coupling constants, though the radius of convergence is not known. To describe hadrons, one needs to extrapolate the weak-coupling expansion to the realistic values of the interaction strength [10]. The realistic magnitudes ought to match the running interaction strength fitted to high-energy data within the same scheme. Outlines of the entire procedure, using simple models, are available in [50–52].

A. Solutions of lowest orders

This section illustrates the weak-coupling approach to solving the RGPEP equation for terms up to the third order. The second-order formulas are used later on to discuss the bound-state eigenvalue problems in QCD of heavy quarks including the mass m_g and field ϕ. The third-order solution shows how the RGPEP computations proceed beyond the second order, *cf.* [53].

In the series expansion of the operator $\mathcal{H}$ in powers of g,

$$\mathcal{H} \;=\; \mathcal{H}_f + \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + \mathcal{H}^{(3)} + O(g^4) \;, \tag{104}$$

where the terms $\mathcal{H}^{(n)}$ are proportional to g^n. The operator $\mathcal{H}^{(n)}$ obeys the equation obtained by equating coefficients of g^n on both sides of Eq. (99). For the terms of first 3 orders, besides $\mathcal{H}^{(0)} = \mathcal{H}_f$, the equations read

$$\mathcal{H}^{(1)'} \;=\; \left[\left[\mathcal{H}_f, \mathcal{H}^{(1)}\right], \mathcal{H}_f\right] \;, \tag{105}$$

$$\mathcal{H}^{(2)'} \;=\; \left[\left[\mathcal{H}_f, \mathcal{H}^{(2)}\right], \mathcal{H}_f\right] + \left[\left[\mathcal{H}_f, \mathcal{H}^{(1)}\right], \mathcal{H}^{(1)}\right] \;, \tag{106}$$

$$\mathcal{H}^{(3)'} \;=\; \left[\left[\mathcal{H}_f, \mathcal{H}^{(3)}\right], \mathcal{H}_f\right] + \left[\left[\mathcal{H}_f, \mathcal{H}^{(2)}\right], \mathcal{H}^{(1)}\right] + \left[\left[\mathcal{H}_f, \mathcal{H}^{(1)}\right], \mathcal{H}^{(2)}\right] \;. \tag{107}$$

In general, the derivative of $\mathcal{H}^{(n)}$ only involves operators $\mathcal{H}^{(k)}$ with $k < n$, which facilitates solving for $\mathcal{H}$ order-by-order to all orders.

The solution for terms of the 1st order reads

$$\mathcal{H}^{(1)} \;=\; f_{LR}\, \mathcal{H}_0^{(1)} \;, \tag{108}$$

where the subscript 0 refers to the initial condition at $s = 0$, and

$$f_{LR} \;=\; e^{-(s\Delta_{LR})^2} \;, \tag{109}$$

$$\Delta_{LR} \;=\; p_L^- - p_R^- \;. \tag{110}$$

The subscript L refers to the product of creation operators in a term in $\mathcal{H}$, standing on its left-hand side, and the subscript R refers to the product of annihilation operators in a term, standing on its right-hand side. Using the notation introduced in Eqs. (40) - (44) for an interaction term $\mathcal{H}_i$, action of f_{LR} on any term in Eq. (108), and on any other operator at any value of s, is defined by

$$f_{LR}\mathcal{H}_i \;=\; \left[\prod_{m=1}^{c_i} \int [p_m] a^\dagger_{p_m}\right] \left[\prod_{n=1}^{a_i} \int [q_n] a_{q_n}\right] \tilde{\delta}_{c.a}\; T_i(\bar{p}, \bar{q})\, f^s_{\bar{p},\bar{q}} \;. \tag{111}$$

In words, the action of f_{LR} on $\mathcal{H}_i$ implies insertion of the factor $f^s_{\bar{p},\bar{q}}$ in the integrand of $\mathcal{H}_i$. Correct for the difference between f^r and f^s: The first-order solution explains our earlier adoption of the factor $f^r_{\bar{p},\bar{q}}$ as a regulator in Eqs. (43) and (44), since $p_L^- = p_c^-$ and $p_R^- = p_a^-$. Namely, the regularization factors match the 1st-order solution of the RGPEP equation with s set to r.

The solution for terms of the second order, obtained using solutions for the first-order terms, reads

$$\mathcal{H}^{(2)} \;=\; f_{LR}\, \mathcal{H}_0^{(2)} + (f_{LR} - f_{LI} f_{IR})\, \Delta_{LIR}\, \mathcal{H}_0^{(1)}\, \mathcal{H}_0^{(1)} \;. \tag{112}$$

The subscript I refers to the labels of the annihilation operators in $f_{LI}\mathcal{H}_0^{(1)}$, or to the labels of the creation operators in $f_{IR}\mathcal{H}_0^{(1)}$.

$$\Delta_{LIR} \;=\; \frac{\Delta_{LI} - \Delta_{IR}}{\Delta_{LI}^2 + \Delta_{IR}^2 - \Delta_{LR}^2} \;. \tag{113}$$

When the denominator in Δ_{LIR} vanishes, the difference of the form factors, $f_{LR} - f_{LI} f_{IR}$, vanishes, too.

The solution for the third-order terms reads

$$\begin{aligned}
\mathcal{H}^{(3)} \;=\;& (f_{LR} - f_{LI} f_{IR})\, \Delta_{LIR} \left[\Delta_{IJR}\, \mathcal{H}_0^{(1)}\, \mathcal{H}_0^{(1)}\, \mathcal{H}_0^{(1)} + \mathcal{H}_0^{(1)}\, \mathcal{H}_0^{(2)}\right] \\
+\;& (f_{LR} - f_{LJ} f_{JR})\, \Delta_{LJR} \left[\Delta_{LIJ}\, \mathcal{H}_0^{(1)}\, \mathcal{H}_0^{(1)}\, \mathcal{H}_0^{(1)} + \mathcal{H}_0^{(2)}\, \mathcal{H}_0^{(1)}\right] \\
-\;& (f_{LR} - f_{LI} f_{IJ} f_{JR})\, \Delta_{LIJR}\, \mathcal{H}_0^{(1)}\, \mathcal{H}_0^{(1)} \mathcal{H}_0^{(1)} \;,
\end{aligned} \tag{114}$$

where

$$\Delta_{LIJR} \;=\; \frac{(\Delta_{LJ} - \Delta_{JR})\Delta_{LIJ} + (\Delta_{LI} - \Delta_{IR})\Delta_{IJR}}{\Delta_{LI}^2 + \Delta_{IJ}^2 + \Delta_{JR}^2 - \Delta_{LR}^2} \;. \tag{115}$$

Subscripts I and J refer to the intermediate configurations of the particle operators involved in the interaction terms. Solutions for the Hamiltonian terms of orders higher than 3rd are obtained following the pattern shown by these examples.

B. Counterterms

Computation of the counterterms in the initial Hamiltonian, $H^{m_g\, r}_{\mathrm{QCD}\,\mathcal{C}}$, is outlined above Eq. (103). It is illustrated below in the case of the quark self-interaction, which contributes to the quark bound-state dynamics. The self-interaction appears first in the second-order Hamiltonian in Eq. (112). The right-hand side contains a product of two first-order terms, $\mathcal{H}_0^{(1)}\,\mathcal{H}_0^{(1)}$. Consider the part of that product corresponding to Fig. 1, where a quark emits and absorbes gluons. The part involves the product $b_1^\dagger a_3 b_2\, b_{\tilde 2}^\dagger a_{\tilde 3}^\dagger b_{\tilde 1}$. Commuting b_2 to the right of $b_{\tilde 2}^\dagger$ and a_3 to the right of $a_{\tilde 3}^\dagger$ yields $\delta_{2\tilde 2}\delta_{3\tilde 3}$, according to the commutation rules specified in Eq. (31). Integration $\int[\tilde 1\tilde 2\tilde 3]$ results in the operator $\hat C = \int[1]\, c_1\, \hat b_1^\dagger \hat b_1$, with the coefficient

$$c_1 \;=\; \frac{g^2 C_F}{p_1^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\, \left(f_{LR} - f_{LI} f_{IR}\right) \frac{N}{D}\, f^{r2}_{qg,p} \; . \tag{116}$$

The numerator N and denominator D are given in Eqs. (83) and (84), *cf.* Eq. (82). Integration variables x and $k^\perp$ are defined below Eq. (85). The color factor C_F equals 4/3 for $SU(3)$. The operators $\hat b_1^\dagger$ and $\hat b_1$ correspond to the same quark p_1^-. Therefore, Eq. (109) impllies $f_{LR}=1$. In contrast, the four-momenta p_2 and p_3, corresponding to the labels of the contracted quark and gluon operators 2 and 3, yield $f_{LI} = f_{IR} = \exp\left\{-(s/p_1^+)^2[m_1^2 - (p_2+p_3)^2]^2\right\}$, which results in

$$f_{LI} f_{IR} \;=\; \exp\left\{-2(sD/p_1^+)^2/[x(1-x)]^2\right\} \; . \tag{117}$$

This factor vanishes for small x and large $k^\perp$. Therefore, the integrand involving $f_{LI} f_{IR}$ contributes a finite quantity to the coefficient $c_1(s>0)$ in the limit $r\to 0$.

The only source of severe small-x singularity in the coefficient c_1 is the term $4(k^{\perp 2}+m_g^2)/x^2$ in the numerator N, in the integrand with $f_{LR}=1$. This term is also singular as a function of $k^\perp \to \infty$. Both singularities are regulated by the factor $f^{r2}_{qg,p}$. When the regularization is lifted by going to the limit $r\to 0$, the quark self-interaction diverges to negative infinity proportionally to $1/r$. But the solution for $\mathcal{H}^{(2)}$ in Eq. (112) contains also the term $f_{LR}\,\mathcal{H}_0^{(2)}$, where $\mathcal{H}_0^{(2)}$ is the initial condition for solving the RGPEP equation. The initial condition can be set to contain a part canceling the divergent dependence of the self-interaction on r. It is visible that the integrand $f_{LR}N/D$ needs to be replaced by $f_{LR}[N/D + 4\sigma_q^{(2)}]$, where $\sigma_q^{(2)}$ is given in Eq. (87). Indeed, the RGPEP explains the guess made in the scattering theory. The resulting integrand,

$$f_{LR}[N/D + 4\sigma_q^{(2)}] \;=\; -(4m_1^2 + 2m_g^2)/D \; , \tag{118}$$

integrates in the limit $r\to 0$ to a logarithmically divergent term in the quark self-interaction,

$$-\delta m^2_{1\,\ln} \;=\; -\frac{g^2 C_F}{(2\pi)^2} m_1^2 \left[\ln(\sqrt{2}\, r_p\, m_1^2) + \gamma_E/2\right] \; . \tag{119}$$

Consequently, the Hamiltonian $\mathcal{H}_0^{(2)}$ is further supplied with a mass-squared counterterm $\delta m^2_{1\,\ln}$. The result of including these quark self-interaction counterterms is that the quark masses appearing in the eigenvalue problems for effective Hamiltonian matrices, such as in Eq. (125), are the ones introduced in H_f in Eq. (37). Similar analysis can be carried out for the gluon self-interaction counterterms.

VIII. GLUON MASS AND SMALL x IN HEAVY QUARKONIA

To discuss the cancellation of gluon singularities $\sim 1/x^2$ in the QCD eigenvalue problems with $m_g>0$, we consider the theory with only heavy quarks, *i.e.*, the theory in which Λ_{QCD} of the RGPEP scheme is set to values much smaller than the quark masses, *cf.* [54]. In such a theory, the running coupling constant for the parameter $s \lesssim 1/m$, where m is the quark mass, is sufficiently small for using the weak-coupling expansion of Sec. VII to approximately compute the Hamiltonian H_s in Eq. (100).

A. Eigenvalue problem for the dominant component

According to Eqs. (101) and 102), the quarkonium eigenstate of the Hamiltonian H_s has infinitely many components with various numbers of the effective quarks, antiquarks, and gluons, *cf.* [52]. However, the interaction terms that change the number of effective particles include a small coupling constant and the vertex factors f^s, which implies that a quarkonium eigenstate of the lowest mass is dominated by its quark-antiquark component,

$$|P\rangle \; = \; \int [12]\, \psi_P(1,2)\, |12\rangle_s \; , \tag{120}$$

where $|12\rangle_s = b^\dagger_{s1} d^\dagger_{s2} |0\rangle$ and P indicates that $|P\rangle$ is an eigenstate of the operators $\hat{P}^+$ and $\hat{P}^\perp$ with the eigenvalues P^+ and $P^\perp$, respectively. The assumption of dominance of the two-body component in heavy quarkonia is successfully used in interpreting data [9, 55–60]. In our approach, the two-body component can, as a first approximation, be described by solutions to the eigenvalue problem for the part of the operator H_s whose structure has the form

$$H_{Q\bar{Q}} \; = \; \int [122'1']\, {}_s\langle 12|H_s|1'2'\rangle_s \; b^\dagger_{s1} d^\dagger_{s2} d_{s2'} b_{s1'} \; . \tag{121}$$

The integration includes summing over the discrete quantum numbers. Without losing generality, the wave function is given the form

$$\psi_P(1,2) \; = \; P^+ \tilde{\delta}_{12.P} \; \psi_{12}(x_1, k^\perp_{12}) \; , \tag{122}$$

where $\tilde{\delta}_{12.P}$ constrains the quarks to carry together P^+ and $P^\perp$, see Eq. (41). The relative momentum variables x_1 and $k^\perp_{12}$ are defined as in the case of an interaction term in Eq. (45). The eigenvalue equation for the wave function reads

$$\int [1'2']\, {}_s\langle 12|H_s|1'2'\rangle_s\, \psi_P(1',2') \; = \; E\, \psi_P(1,2) \; , \tag{123}$$

where $E(P^+, P^\perp) = (P^{\perp 2} + M^2)/P^+$ and M denotes the quarkonium mass, $E(M, 0^\perp) = M$. In the leading order of the weak-coupling expansion, $H_s = H_f + H^{(2)}_s$, where $H^{(2)}_s$ is obtained from $\mathcal{H}^{(2)}$ of Eq. (112) using Eq. (100). Hence,

$${}_s\langle 12|H^{(2)}_s|1'2'\rangle_s \; = \; \langle 12| \left[\mathcal{H}_f + \mathcal{H}^{(2)} \right] |1'2'\rangle \; , \tag{124}$$

which allows one to directly obtain the interaction matrix elements in ${}_s\langle 12|H^{(2)}_s|1'2'\rangle_s$ from Eq. (112) for $\mathcal{H}^{(2)}$. The direct relationship between matrix elements of H_s and $\mathcal{H}$ and is a general feature of the RGPEP. Also, notice that there are no first-order terms in Eq. (124), corresponding to $\mathcal{H}^{(1)}$ in Eq. (104), because such terms change the number of quanta.

B. Hamiltonian matrix for dominant component

According to Eq. (112), the Hamiltonian matrix of Eq. (124) reads

$${}_s\langle 12|H_s|1'2'\rangle_s \; = \; \langle 12| \left\{ \mathcal{H}_f + f_{LR}\, \mathcal{H}^{(2)}_r - f_{LI} f_{IR}\, \Delta_{LIR}\, \mathcal{H}^{(1)}_0\, \mathcal{H}^{(1)}_0 \right\} |1'2'\rangle \; , \tag{125}$$

where

$$\mathcal{H}^{(2)}_r \; = \; \mathcal{H}^{(2)}_0 + \Delta_{LIR}\; \mathcal{H}^{(1)}_0\, \mathcal{H}^{(1)}_0 \; . \tag{126}$$

Quark self-interaction counterterms in $\mathcal{H}^{(2)}_0$ cancel the divergent self-interaction terms resulting from $\Delta_{LIR}\; \mathcal{H}^{(1)}_0\, \mathcal{H}^{(1)}_0$, as described in Sec. VII B. The term $\Delta_{LIR}\; \mathcal{H}^{(1)}_0\, \mathcal{H}^{(1)}_0$ in $\mathcal{H}^{(2)}_r$ in Eq. (126) contains also the exchange terms diverging like $1/x^2$.

C. Cancellation of $1/x^2$ in bound states

Besides the gluon exchange terms $\sim 1/x^2$, contained in $\Delta_{LIR}\; \mathcal{H}^{(1)}_0\, \mathcal{H}^{(1)}_0$, Eq. (126) includes, in $\mathcal{H}^{(2)}_0$, the equally singular seagull term. We now describe how these two terms cancel each other in the bound-state eigenvalue problem for the matrix ${}_s\langle 12|H_s|1'2'\rangle_s$ despite that m_g is not zero. We focus on the matrix elements $\langle 12|\mathcal{H}_r|1'2'\rangle$, which is where the small-x singularities reside.

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$${}_s\langle 12|H_s|1'2'\rangle_s \; = \; \langle 12| \left\{ \mathcal{H}_f + f_{LR}\, \mathcal{H}_r - f_{LI} f_{IR}\, \Delta_{LIR}\, \mathcal{H}^{(1)}_0 \mathcal{H}^{(1)}_0 \right\} |1'2'\rangle \; , \tag{127}$$

$$\mathcal{H}_r \; = \; \mathcal{H}^{(2)}_0 + \Delta_{LIR}\; \mathcal{H}^{(1)}_0\, \mathcal{H}^{(1)}_0 \; . \tag{128}$$

1. $\mathcal{H}_r$: small-x divergences in the seagull and exchange, and the ultraviolet divergences in the quark self-interactions.
2. $\mathcal{H}_0^{(2)}$: the seagull, the seagull counterterm, the counterterm to gluon xchange, the quark self-interaction counterterms.
3. $\Delta_{LIR}\,\mathcal{H}_0^{(1)}\,\mathcal{H}_0^{(1)}$ in $\mathcal{H}_r$: the gluon exchange, the quark self-interactions.
4. $f_{LI}f_{IR}\,\Delta_{LIR}\,\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$: the finite gluon exchange between the quarks and the finite quark self-interactions, depending on the gluon mass m_g.

The terms in $\mathcal{H}_r$ are independent of the gluon mass $m_g \to 0$.

The terms in $f_{LI}f_{IR}\,\Delta_{LIR}\,\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$ depend on $m_g \to 0$ in a useful way.

1. The Coulomb and other terms in $\mathcal{H}_r$

The quark self-interactions are canceled by the self-interaction counterterms. The only quark masses in the projected 2-body dynamics are the ones introduced in the eigenvalues of $\mathcal{H}_f$. The seagull and exchange counterterms in $\mathcal{H}_r$ cancel each other. The gluon exchange and seagull together contribute the kernel

$$f_{LR}(-1)_{jj}\left\{\left[-g^{\mu\nu}+\eta^\mu\eta^\nu(\rho_1+\rho_2)/(2q^{+2})\right]\,\frac{1}{2}\left(\frac{-1}{\rho_1}+\frac{-1}{\rho_2}\right)+\frac{\eta^\mu\eta^\nu}{q^{+2}}\right\}f^r_{1,\underline{1}\mathrm{g}}f^r_{\underline{2},\underline{2}\mathrm{g}}\;. \tag{129}$$

The term with $-g^{\mu\nu}$ provides the Coulomb potential term in the Hamiltonian for the 2-body component in the limit $m_g \to 0$, with the kernel

$$-f_{LR}\;g^{\mu\nu}\;\frac{1}{2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\;. \tag{130}$$

This kernel is contracted with the quark and antiquark currents providing spin effects. The factor f_{LR} deviates from 1 in the non-relativistic limit only by small amounts discussed in [61]. In that limit, $\rho_1=\rho_2=\vec q^{\,2}+m_g^2$. The terms with η sum up to

$$f_{LR}\left[(\rho_1+\rho_2)\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-4\right]\;\frac{\eta^\mu\eta^\nu}{4q^{+2}}\;=\;f_{LR}\;\frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2}\;\frac{\eta^\mu\eta^\nu}{4q^{+2}}\;. \tag{131}$$

Using Eq. (**??**), one obtains

$$f_{LR}\;\frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2}\;\frac{\eta^\mu\eta^\nu}{4q^{+2}}\;=\;f_{LR}\;\frac{(\mathcal{M}_{12}^2-\mathcal{M}_{1'2'}^2)^2}{\rho_1\rho_2}\;\frac{\eta^\mu\eta^\nu}{4P_{12}^{+2}}\;, \tag{132}$$

which is a spin-independent interaction kernel integrated with the measure $d^3k_{1'2'}$ and the wave function $\psi_{1'2'}(\vec k_{1'2'})$ in the 2-body eigenvalue equation. This term vanishes on shell of the free 2-body P^- and hence has no counterpart in any classical approximation for the quarkonium dynamics. Its magnitude can be illustrated with the case of equal quark masses, in which $\mathcal{M}_{12}^2-\mathcal{M}_{1'2'}^2=4(\vec k_{12}^2-\vec k_{1'2'}^2)$. In the non-relativistic limit, $\rho_1=\rho_2=(\vec k_{12}-\vec k_{1'2'})^2$ for $m_g\to 0$. The term is suppressed by the factor $\sim K^2c^2/(2m)^2$ in comparison with the Coulomb term, where $\vec K=\vec k_{12}+\vec k_{1'2'}$ and c denotes the cosine of the angle between $\vec q$ and $\vec K$. Its quantitative role in the eigenvalue equation requires a study. For example, it contributes to the quarkonium mass splittings between states with the same spins.

2. Contribution of $-f_{LI}f_{IR}\,\Delta_{LIR}\,\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$ to Eq. (125)

The last term in the 2-body Hamiltonian matrix in Eq. (125) is

$$H_{\rm 2body}\;=\;-f_{LI}f_{IR}\,\Delta_{LIR}\,\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}\;. \tag{133}$$

It includes the effective self-interaction of quarks and an exchange of an effective gluon between them. These terms are all finite because the vertex form factors prevent production of singular configurations from regular ones. But they do depend on the gluon mass m_g. The question is what happens in the limit $m_g \to 0$.

The exchange terms

The exchange-term kernel has necessarily the form shared with the exchange term in Eq. (129). To interpret these interaction terms, I need the operator $-f_{LI} f_{IR}\, \Delta_{LIR}\, \mathcal{H}_0^{(1)} \mathcal{H}_0^{(1)}$. Using Eqs. (16) and (21), I have

$$\mathcal{H}_0^{(1)} = \int d^2x^\perp dx^- \left[: A^a_{f\mu}(-J^{a\mu}_{\psi f}) + i m_g \phi^a \frac{1}{i\partial^+} J^{a+}_{\psi f} : \right]^r \tag{134}$$

$$= \int d^2x^\perp dx^- \left[: \left(A^a_{f\mu} + i m_g \frac{\eta_\mu}{i\partial^+}\phi^a \right) g\bar{\psi}_f T^a \gamma^\mu \psi_f : \right]^r \tag{135}$$

and according to Eqs. (28), (29), and (30), I insert

$$\hat{\psi}_f = \sum_{c=1}^{3}\sum_{\sigma=1}^{2}\int [p] \left[u_{p\sigma}\chi_c \hat{b}_{p\sigma c} e^{-ipx} + v_{p\sigma}\chi_c \hat{d}^\dagger_{p\sigma c} e^{ipx} \right]_{x^+=0} , \tag{136}$$

$$\hat{A}^\mu_f = \sum_{c=1}^{8}\sum_{\sigma=1}^{2}\int [p] \left[\varepsilon^\mu_{p\sigma} T^c \hat{a}_{p\sigma c} e^{-ipx} + \varepsilon^{\mu *}_{p\sigma} T^c \hat{a}^\dagger_{p\sigma c} e^{ipx} \right]_{x^+=0} , \tag{137}$$

$$i m_g \frac{\eta_\mu}{i\partial^+}\hat{\phi} = \sum_{c=1}^{8}\int [p] \frac{\eta^\mu m_g T^c}{p^+} \left[\hat{a}_{p3c} e^{-ipx} + \hat{a}^\dagger_{p3c} e^{ipx} \right]_{x^+=0} . \tag{138}$$

I can introduce the third polarization vector for the longitudinal gluons, by writing

$$\hat{A}^\mu_f = \sum_{c=1}^{8}\sum_{\sigma=1}^{3}\int [p] \left[\varepsilon^\mu_{p\sigma} T^c \hat{a}_{p\sigma c} e^{-ipx} + \varepsilon^{\mu *}_{p\sigma} T^c \hat{a}^\dagger_{p\sigma c} e^{ipx} \right]_{x^+=0} , \quad \epsilon^\mu_{p3} = m_g \eta^\mu / p^+ . \tag{139}$$

The current is

$$-J^{a\mu}_{\psi f} = g\bar{\psi}_f T^a \gamma^\mu \psi_f \tag{140}$$

$$= \int [pp'] \left[\bar{u}_p \chi^\dagger_p b^\dagger_p e^{ipx} T^q \gamma^\mu u_{p'} \chi_{p'} b_{p'} e^{-ip'x} + \bar{v}_p \chi^\dagger_p d_p e^{-ipx} T^q \gamma^\mu v_{p'} \chi_{p'} d^\dagger_{p'} e^{ip'x} \right] . \tag{141}$$

The Hamiltonian term is

$$\mathcal{H}_0^{(1)} = g\int [pp'q]\ \tilde{\delta}_{c.a}\, f^t \left(\bar{u}_p \gamma_\mu u_{p'}\ \chi^\dagger_p T^q \chi_{p'}\ b^\dagger_p b_{p'} - \bar{v}_{p'}\gamma_\mu v_p\ \chi^\dagger_{p'} T^q \chi_p\ d^\dagger_p d_{p'} \right) \left(\varepsilon^\mu_q a_q + \varepsilon^{\mu *}_q a^\dagger_q \right) . \tag{142}$$

Consider only real polarization vectors for gluons; I sum over their polarization anyway.

$$\mathcal{H}_0^{(1)} = g\int [pp'q]\ \tilde{\delta}_{c.a}\, f^r \left[\bar{u}_p \gamma_\mu \varepsilon^\mu_q u_{p'}\ \chi^\dagger_p T^q \chi_{p'}\ (a_q + a^\dagger_q) b^\dagger_p b_{p'} - \bar{v}_{p'}\gamma_\mu \varepsilon^\mu_q v_p\ \chi^\dagger_{p'} T^q \chi_p\ (a_q + a^\dagger_q) d^\dagger_p d_{p'} \right] . \tag{143}$$

So,

$$H_{2\text{body}} = -f_1^t f_2^t\, \Delta_{12g2'1'}\, \mathcal{H}_0^{(1)} \mathcal{H}_0^{(1)} \tag{144}$$

involves the product $\mathcal{H}_0^{(1)} \mathcal{H}_0^{(1)}$ reading

$$\begin{aligned} & g\int [pp'q]\ \tilde{\delta}_{c.a}\, f^r \left[\bar{u}_p \gamma_\mu \varepsilon^\mu_q u_{p'}\ \chi^\dagger_p T^q \chi_{p'}\ (a_q + a^\dagger_q) b^\dagger_p b_{p'} - \bar{v}_{p'}\gamma_\mu \varepsilon^\mu_q v_p\ \chi^\dagger_{p'} T^q \chi_p\ (a_q + a^\dagger_q) d^\dagger_p d_{p'} \right] \\ \times\ & g\int [pp'q]\ \tilde{\delta}_{c.a}\, f^r \left[\bar{u}_p \gamma_\mu \varepsilon^\mu_q u_{p'}\ \chi^\dagger_p T^q \chi_{p'}\ (a_q + a^\dagger_q) b^\dagger_p b_{p'} - \bar{v}_{p'}\gamma_\mu \varepsilon^\mu_q v_p\ \chi^\dagger_{p'} T^q \chi_p\ (a_q + a^\dagger_q) d^\dagger_p d_{p'} \right] . \end{aligned} \tag{145}$$

The gluon operators need to be contracted and only terms with $b^\dagger b$, $d^\dagger d$, and $b^\dagger d^\dagger d b$ are kept. The products that contribute are

$$\begin{aligned} & g\int [pp'q]\ \tilde{\delta}_{c.a}\, f^r \left[\bar{u}_p \gamma_\mu \varepsilon^\mu_q u_{p'}\ \chi^\dagger_p T^q \chi_{p'}\ b^\dagger_p b_{p'} a_q - \bar{v}_{p'}\gamma_\mu \varepsilon^\mu_q v_p\ \chi^\dagger_{p'} T^q \chi_p\ d^\dagger_p d_{p'} a_q \right] \\ \times\ & g\int [pp'q]\ \tilde{\delta}_{c.a}\, f^r \left[\bar{u}_p \gamma_\mu \varepsilon^\mu_q u_{p'}\ \chi^\dagger_p T^q \chi_{p'}\ a^\dagger_q b^\dagger_p b_{p'} - \bar{v}_{p'}\gamma_\mu \varepsilon^\mu_q v_p\ \chi^\dagger_{p'} T^q \chi_p\ a^\dagger_q d^\dagger_p d_{p'} \right] . \end{aligned} \tag{146}$$

The exchange terms between a quark and antiquark come from

$$g \int [pp'q]\ \tilde{\delta}_{c.a}\, f^r \left[\bar{u}_p \gamma_\mu \varepsilon^\mu_q u_{p'}\ \chi^\dagger_p T^q \chi_{p'}\ b^\dagger_p b_{p'} a_q\right]\ g \int [pp'q]\ \tilde{\delta}_{c.a}\, f^r \left[-\bar{v}_{p'} \gamma_\mu \varepsilon^\mu_q v_p\ \chi^\dagger_{p'} T^q \chi_p\ a^\dagger_q d^\dagger_p d_{p'}\right]$$
$$+\ g \int [pp'q]\ \tilde{\delta}_{c.a}\, f^r \left[-\bar{v}_{p'} \gamma_\mu \varepsilon^\mu_q v_p\ \chi^\dagger_{p'} T^q \chi_p\ d^\dagger_p d_{p'} a_q\right]\ g \int [pp'q]\ \tilde{\delta}_{c.a}\, f^r \left[\bar{u}_p \gamma_\mu \varepsilon^\mu_q u_{p'}\ \chi^\dagger_p T^q \chi_{p'}\ a^\dagger_q b^\dagger_p b_{p'}\right]\ . \quad (147)$$

In the notation 123, the exchange terms read

$$g \int [11'3]\ \tilde{\delta}_{1.1'3}\, f^r \left[\bar{u}_1 \gamma_\mu \varepsilon^\mu_3 u_{1'}\ \chi^\dagger_1 T^3 \chi_{1'}\ b^\dagger_1 b_{1'} a_3\right]\ g \int [22'\tilde{3}]\ \tilde{\delta}_{2\tilde{3}.2'}\, f^r \left[-\bar{v}_{2'} \gamma_\mu \varepsilon^\mu_{\tilde{3}} v_2\ \chi^\dagger_{2'} T^{\tilde{3}} \chi_2\ a^\dagger_{\tilde{3}} d^\dagger_2 d_{2'}\right]$$
$$+\ g \int [22'3]\ \tilde{\delta}_{2.2'3}\, f^r \left[-\bar{v}_{2'} \gamma_\mu \varepsilon^\mu_3 v_2\ \chi^\dagger_{2'} T^3 \chi_2\ d^\dagger_2 d_{2'} a_3\right]\ g \int [11'\tilde{3}]\ \tilde{\delta}_{1\tilde{3}.1'}\, f^r \left[\bar{u}_1 \gamma_\mu \varepsilon^\mu_{\tilde{3}} u_{1'}\ \chi^\dagger_1 T^{\tilde{3}} \chi_{1'}\ a^\dagger_{\tilde{3}} b^\dagger_1 b_{1'}\right]\ . \quad (148)$$

Contraction of $\tilde{3}$ with 3 yields the exchange terms

$$-g^2 \int [11'3]\ \tilde{\delta}_{1.1'3}\, f^r\ \bar{u}_1 \gamma_\mu \varepsilon^\mu_3 u_{1'}\ \chi^\dagger_1 T^3 \chi_{1'}\ \int [22']\ \tilde{\delta}_{23.2'}\, f^r\ \bar{v}_{2'} \gamma_\nu \varepsilon^\nu_3 v_2\ \chi^\dagger_{2'} T^3 \chi_2\ b^\dagger_1 d^\dagger_2 d_{2'} b_{1'}$$
$$-\ g^2 \int [22'3]\ \tilde{\delta}_{2.2'3}\, f^r\ \bar{v}_{2'} \gamma_\mu \varepsilon^\mu_3 v_2\ \chi^\dagger_{2'} T^3 \chi_2\ \int [11']\ \tilde{\delta}_{13.1'}\, f^r\ \bar{u}_1 \gamma_\nu \varepsilon^\nu_3 u_{1'}\ \chi^\dagger_1 T^3 \chi_{1'}\ b^\dagger_1 d^\dagger_2 d_{2'} b_{1'}\ . \quad (149)$$

Integration over 3 yields $\tilde{\delta}_{12.2'1'}/p^+_g$ and $\theta_{1-1'}$ in the first term and $\theta_{1'-1}$ in the second term.

$$-g^2 \sum_3 \int [122'1']\ \tilde{\delta}_{12.1'2'}\ \theta_{1-1'}/p^+_g\ f^r\ \bar{u}_1 \gamma_\mu \varepsilon^\mu_3 u_{1'}\ \chi^\dagger_1 T^3 \chi_{1'}\ f^r\ \bar{v}_{2'} \gamma_\nu \varepsilon^\nu_3 v_2\ \chi^\dagger_{2'} T^3 \chi_2\ b^\dagger_1 d^\dagger_2 d_{2'} b_{1'}$$
$$-\ g^2 \sum_3 \int [122'1']\ \tilde{\delta}_{12.1'2'}\ \theta_{1'-1}/p^+_g\ f^r\ \bar{v}_{2'} \gamma_\mu \varepsilon^\mu_3 v_2\ \chi^\dagger_{2'} T^3 \chi_2\ \int [11']\ \tilde{\delta}_{13.1'}\, f^r\ \bar{u}_1 \gamma_\nu \varepsilon^\nu_3 u_{1'}\ \chi^\dagger_1 T^3 \chi_{1'}\ b^\dagger_1 d^\dagger_2 d_{2'} b_{1'}\ . \quad (150)$$

Denoting

$$j_\mu \ =\ \bar{u}_1 \gamma_\mu u_{1'}\ ,\ \bar{j}_\nu\ =\ \bar{v}_{2'} \gamma_\nu v_2\ , \quad (151)$$

I obtain

$$H_{\rm 2body\ exch} \ =\ -f^t_1 f^t_2\, \Delta_{12q2'1'}\, \mathcal{H}^{(1)}_0 \mathcal{H}^{(1)}_0 \quad (152)$$
$$=\ g^2 f^t_1 f^t_2\, \Delta_{12q2'1'}\ \sum_3 \int [122'1']\ \tilde{\delta}_{12.1'2'}\ \theta_{1-1'}/p^+_g\ f^r\ \bar{u}_1 \gamma_\mu \varepsilon^\mu_3 u_{1'}\ \chi^\dagger_1 T^3 \chi_{1'}\ f^r\ \bar{v}_{2'} \gamma_\nu \varepsilon^\nu_3 v_2\ \chi^\dagger_{2'} T^3 \chi_2\ b^\dagger_1 d^\dagger_2 d_{2'} b_{1'}$$
$$+\ g^2 f^t_1 f^t_2\, \Delta_{12q2'1'}\ \sum_3 \int [122'1']\ \tilde{\delta}_{12.1'2'}\ \theta_{1'-1}/p^+_g\ f^r\ \bar{v}_{2'} \gamma_\mu \varepsilon^\mu_3 v_2\ \chi^\dagger_{2'} T^3 \chi_2\, f^r\ \bar{u}_1 \gamma_\nu \varepsilon^\nu_3 u_{1'}\ \chi^\dagger_1 T^3 \chi_{1'}\ b^\dagger_1 d^\dagger_2 d_{2'} b_{1'} \quad (153)$$

Introducing the color factor and sum over gluon polarization,

$$C \ =\ \sum_3 \chi^\dagger_1 T^3 \chi_{1'}\ \chi^\dagger_{2'} T^3 \chi_2\ , \quad (154)$$
$$d^{\mu\nu} \ =\ -g^{\mu\nu} + (p^\mu_g \eta^\nu + \eta^\mu p^\nu_g)/p^+_g\ , \quad (155)$$
$$(156)$$

and neglecting factors f^r in the presence of the factors f^t, I have

$$H_{\rm 2body} \ =\ -f^t_1 f^t_2\, \Delta_{12g2'1'}\, \mathcal{H}^{(1)}_0 \mathcal{H}^{(1)}_0 \quad (157)$$
$$=\ g^2 f^t_1 f^t_2\, \Delta_{12g2'1'} \int [122'1']\ C\ \theta_{1-1'}/p^+_g\ j_\mu d^{\mu\nu} \bar{j}_\nu\ \tilde{\delta}_{12.1'2'}\ b^\dagger_1 d^\dagger_2 d_{2'} b_{1'}$$
$$+\ g^2 f^t_1 f^t_2\, \Delta_{12g2'1'} \int [122'1']\ C\ \theta_{1'-1}/p^+_g\ j_\mu d^{\mu\nu} \bar{j}_\nu\ \tilde{\delta}_{12.1'2'}\ b^\dagger_1 d^\dagger_2 d_{2'} b_{1'}\ . \quad (158)$$

The two terms can be written together as

$$H_{\rm 2body} = -f_1^t f_2^t \,\Delta_{12q2'1'}\,\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)} \tag{159}$$

$$= g^2 C \int [122'1'] \, f^t_{\underline{1}g,\bar{1}} f^t_{\bar{2},\underline{2}g} \,\Delta_{12q2'1'}/p_g^+ \; j_\mu d^{\mu\nu} \bar{j}_\nu \; \tilde{\delta}_{12.1'2'} \; b_1^\dagger d_2^\dagger d_{2'} b_{1'} \; . \tag{160}$$

Contribution of this term to the eigenvalue Eq. (123), *cf.* Eq. **??**), in the form

$$\langle 12 | H_{\rm 2body\,exch} | P_\phi \rangle = \int [2'1'] \, K_{\rm exch}(12,2'1') \; P_\phi^+ \tilde{\delta}_{P_\phi.1'2'} \; \phi_{1'2'}(x_{1'}, k^\perp_{1'2'}) \tag{161}$$

with the kernel of dimension $\frac{1}{+\perp^2}$,

$$K_{\rm exch}(12,2'1') = g^2 C_{|2\rangle} \, f^t_{\underline{1}g,\bar{1}} f^t_{\bar{2},\underline{2}g} \,\Delta_{12q2'1'}/p_g^+ \; j_\mu d^{\mu\nu} \bar{j}_\nu \, \tilde{\delta}_{12.1'2'} \; , \tag{162}$$

$$\Delta_{12q2'1'}/p_g^+ = -\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \; , \tag{163}$$

$$d^{\mu\nu} = -g^{\mu\nu} + \eta^\mu \eta^\nu (\rho_1 + \rho_2)/(2p_g^{+2}) \; . \tag{164}$$

The exchange interaction is finite for finite s despite that it includes the factor $1/q^{+2}$. It is finite because of the presence of $m_g > 0$ in the factors $f^t_{\underline{1}g,\bar{1}} f^t_{\bar{2},\underline{2}g}$. Discussion of the 2-body eigenvalue equation in the limit $m_g \to 0$ requires inclusion of the self-interaction terms, which also diverge in the limit $m_g \to 0$.

The self-interaction terms

The self-interactions are one-body operators which I reevaluate below.

$$H_{\rm 2body\,\Sigma} = \left[-f_{LI} f_{IR} \, \Delta_{LIR} \, \mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)} \right]_\Sigma \; . \tag{165}$$

The products contributing are

$$g\int [pp'q] \; \tilde{\delta}_{c.a} \, f^t \left[\bar{u}_p \gamma_\mu \varepsilon_q^\mu u_{p'} \;\, \chi_p^\dagger T^q \chi_{p'} \;\, a_q^\dagger b_p^\dagger b_{p'} - \bar{v}_{p'} \gamma_\mu \varepsilon_q^\mu v_p \;\, \chi_{p'}^\dagger T^q \chi_p \;\, a_q^\dagger d_p^\dagger d_{p'} \right]$$
$$\times \; g\int [pp'q] \; \tilde{\delta}_{c.a} \, f^t \left[\bar{u}_p \gamma_\mu \varepsilon_q^\mu u_{p'} \;\, \chi_p^\dagger T^q \chi_{p'} \;\, a_q^\dagger b_p^\dagger b_{p'} - \bar{v}_{p'} \gamma_\mu \varepsilon_q^\mu v_p \;\, \chi_{p'}^\dagger T^q \chi_p \;\, a_q^\dagger d_p^\dagger d_{p'} \right] \; , \tag{166}$$

but for the self-interaction count only

$$g\int [pp'q] \; \tilde{\delta}_{c.a} \, f^t \left[\bar{u}_p \gamma_\mu \varepsilon_q^\mu u_{p'} \;\, \chi_p^\dagger T^q \chi_{p'} \;\, b_p^\dagger b_{p'} a_q \right] \; g\int [pp'q] \; \tilde{\delta}_{c.a} \, f^t \left[\bar{u}_p \gamma_\mu \varepsilon_q^\mu u_{p'} \;\, \chi_p^\dagger T^q \chi_{p'} \;\, a_q^\dagger b_p^\dagger b_{p'} \right]$$
$$+ \; g\int [pp'q] \; \tilde{\delta}_{c.a} \, f^t \left[-\bar{v}_{p'} \gamma_\mu \varepsilon_q^\mu v_p \;\, \chi_{p'}^\dagger T^q \chi_p \;\, d_p^\dagger d_{p'} a_q \right] \; g\int [pp'q] \; \tilde{\delta}_{c.a} \, f^t \left[-\bar{v}_{p'} \gamma_\mu \varepsilon_q^\mu v_p \;\, \chi_{p'}^\dagger T^q \chi_p \;\, a_q^\dagger d_p^\dagger d_{p'} \right] \; . \tag{167}$$

Using the parallel with the exchange calculation in Eq. (149), the notation in terms of 123 yields

$$g\int [11'3] \; \tilde{\delta}_{1.1'3} \, f^t \left[\bar{u}_1 \gamma_\mu \varepsilon_3^\mu u_{1'} \;\, \chi_1^\dagger T^3 \chi_{1'} \;\, b_1^\dagger b_{1'} a_3 \right] \; g\int [22'\tilde{3}] \; \tilde{\delta}_{2\tilde{3}.2'} \, f^t \left[\bar{u}_2 \gamma_\mu \varepsilon_{\tilde{3}}^\mu u_{2'} \;\, \chi_2^\dagger T^{\tilde{3}} \chi_{2'} \;\, a_{\tilde{3}}^\dagger b_2^\dagger b_{2'} \right]$$
$$+ \; g\int [22'3] \; \tilde{\delta}_{2.2'3} \, f^t \left[-\bar{v}_{2'} \gamma_\mu \varepsilon_3^\mu v_2 \;\, \chi_{2'}^\dagger T^3 \chi_2 \;\, d_2^\dagger d_{2'} a_3 \right] \; g\int [11'\tilde{3}] \; \tilde{\delta}_{1\tilde{3}.1'} \, f^t \left[-\bar{v}_{1'} \gamma_\mu \varepsilon_{\tilde{3}}^\mu v_1 \;\, \chi_{1'}^\dagger T^{\tilde{3}} \chi_1 \;\, a_{\tilde{3}}^\dagger d_1^\dagger d_{1'} \right] \; . \tag{168}$$

Contraction of $\tilde{3}$ with 3 and ignoring $(-1)^2$ for antiquarks,

$$g\int [11'3] \; \tilde{\delta}_{1.1'3} \, f^t \left[\bar{u}_1 \gamma_\mu \varepsilon_3^\mu u_{1'} \;\, \chi_1^\dagger T^3 \chi_{1'} \;\, b_1^\dagger b_{1'} \right] \; g\int [22'] \; \tilde{\delta}_{23.2'} \, f^t \left[\bar{u}_2 \gamma_\mu \varepsilon_3^\mu u_{2'} \;\, \chi_2^\dagger T^3 \chi_{2'} \;\, b_2^\dagger b_{2'} \right]$$
$$+ \; g\int [22'3] \; \tilde{\delta}_{2.2'3} \, f^t \left[\bar{v}_{2'} \gamma_\mu \varepsilon_3^\mu v_2 \;\, \chi_{2'}^\dagger T^3 \chi_2 \;\, d_2^\dagger d_{2'} \right] \; g\int [11'] \; \tilde{\delta}_{13.1'} \, f^t \left[\bar{v}_{1'} \gamma_\mu \varepsilon_3^\mu v_1 \;\, \chi_{1'}^\dagger T^3 \chi_1 \;\, d_1^\dagger d_{1'} \right] \; . \tag{169}$$

Contractions of 2 with 1', and 1 with 2', produce

$$g\int [11'3] \; \tilde{\delta}_{1.1'3} \, f^t \left[\bar{u}_1 \gamma_\mu \varepsilon_3^\mu u_{1'} \;\, \chi_1^\dagger T^3 \chi_{1'} \;\, b_1^\dagger \right] \; g\int [2'] \; \tilde{\delta}_{1'3.2'} \, f^t \left[\bar{u}_{1'} \gamma_\mu \varepsilon_3^\mu u_{2'} \;\, \chi_{1'}^\dagger T^3 \chi_{2'} \;\, b_{2'} \right]$$
$$+ \; g\int [22'3] \; \tilde{\delta}_{2.2'3} \, f^t \left[\bar{v}_{2'} \gamma_\mu \varepsilon_3^\mu v_2 \;\, \chi_{2'}^\dagger T^3 \chi_2 \;\, d_2^\dagger \right] \; g\int [1'] \; \tilde{\delta}_{2'3.1'} \, f^t \left[\bar{v}_{1'} \gamma_\nu \varepsilon_3^\nu v_{2'} \;\, \chi_{1'}^\dagger T^3 \chi_{2'} \;\, d_{1'} \right] \; . \tag{170}$$

Integrations over 1'3 and 2'3 yield, replacing the second μ by ν

$$g^2 \sum_{31'} \int [12'] \int [x_3 k_{31'}] \ \tilde{\delta}_{1.2'}/p_1^+ \ f^t \left[\bar{u}_1 \gamma_\mu \varepsilon_3^\mu u_{1'} \ \chi_1^\dagger T^3 \chi_{1'} \ b_1^\dagger \right] \ f^t \left[\bar{u}_{1'} \gamma_\mu \varepsilon_3^\mu u_{2'} \ \chi_{1'}^\dagger T^3 \chi_{2'} \ b_{2'} \right]$$
$$+ \ g^2 \sum_{32'} \int [21'] \int [x_3 k_{32'}] \ \tilde{\delta}_{2.1'}/p_2^+ \ f^t \left[\bar{v}_{2'} \gamma_\mu \varepsilon_3^\mu v_2 \ \chi_{2'}^\dagger T^3 \chi_2 \ d_2^\dagger \right] \ f^t \left[\bar{v}_{1'} \gamma_\nu \varepsilon_3^\nu v_{2'} \ \chi_{1'}^\dagger T^3 \chi_{2'} \ d_{1'} \right] . \tag{171}$$

Sum over colors, integrations over 2' and 1', and rearrangement, produce

$$g^2 C_F \sum_{31'12'} \int [1] \int [x_3 k_{31'}] \ \frac{1}{p_1^{+2}} \ f^{2t} \ \bar{u}_1 \gamma_\mu \varepsilon_3^\mu u_{1'} \ \bar{u}_{1'} \gamma_\nu \varepsilon_3^\nu u_{2'} \ b_1^\dagger b_1$$
$$+ \ g^2 C_F \sum_{32'21'} \int [2] \int [x_3 k_{32'}] \ \frac{1}{p_2^{+2}} \ f^{2t} \ \bar{v}_{2'} \gamma_\mu \varepsilon_3^\mu v_2 \ \bar{v}_{1'} \gamma_\nu \varepsilon_3^\nu v_{2'} \ d_2^\dagger d_2 \ . \tag{172}$$

Sum over intermediate spins yields

$$g^2 C_F \sum_{11'2'} \int [1] \int [x_3 k_{31'}] \ \frac{1}{p_1^{+2}} \ f^{2t} \ j_{11'\mu} d^{\mu\nu} j_{1'2'\nu} \ b_1^\dagger b_1$$
$$+ \ g^2 C_F \sum_{22'1'} \int [2] \int [x_3 k_{32'}] \ \frac{1}{p_2^{+2}} \ f^{2t} \ \bar{j}_{1'2'\nu} d^{\mu\nu} \bar{j}_{2'2} \ d_2^\dagger d_2 \ . \tag{173}$$

With this result for the product $\mathcal{H}_0^{(1)} \mathcal{H}_0^{(1)}$, the self-interaction operator is

$$H_{2\mathrm{body}\,\Sigma} \ = \ \left[-f_{LI} f_{IR} \, \Delta_{LIR} \, \mathcal{H}_0^{(1)} \mathcal{H}_0^{(1)} \right]_\Sigma \tag{174}$$
$$= \ -g^2 C_F \sum_{11'2'} \int [1] \int [x_3 k_{31'}] \ \frac{1}{p_1^{+2}} \ f_1^{2t} \ \frac{j_{11'\mu} d^{\mu\nu} j_{1'2'\nu}}{p_1^- - p_{1'}^- - p_3^-} \ b_1^\dagger b_1$$
$$- \ g^2 C_F \sum_{22'1'} \int [2] \int [x_3 k_{32'}] \ \frac{1}{p_2^{+2}} \ f_2^{2t} \ \frac{\bar{j}_{1'2'\nu} d^{\mu\nu} \bar{j}_{2'2}}{p_2^- - p_{2'}^- - p_3^-} \ d_2^\dagger d_2 \ . \tag{175}$$

The matrix element of quark self-interactions in the eigenvalue equation is

$$\langle 12 | H_{2\mathrm{body}\,\Sigma} | P_\phi \rangle \ = \ \langle 12 | \left[-g^2 C_F \sum_{11'2'} \int [1] \int [x_3 k_{31'}] \ \frac{1}{p_1^{+2}} \ f_1^{2t} \ \frac{j_{11'\mu} d^{\mu\nu} j_{1'2'\nu}}{p_1^- - p_{1'}^- - p_3^-} \ b_1^\dagger b_1 \right.$$
$$\left. - \ g^2 C_F \sum_{22'1'} \int [2] \int [x_3 k_{32'}] \ \frac{1}{p_2^{+2}} \ f_2^{2t} \ \frac{\bar{j}_{1'2'\nu} d^{\mu\nu} \bar{j}_{2'2}}{p_2^- - p_{2'}^- - p_3^-} \ d_2^\dagger d_2 \right]$$
$$\times \ \int [\tilde{2}' \tilde{1}'] \ P_\phi^+ \tilde{\delta}_{P_\phi.\tilde{1}'\tilde{2}'} \ \phi_{\tilde{1}'\tilde{2}'}(x_{\tilde{1}'}, k_{\tilde{1}'\tilde{2}'}^\perp) | \tilde{1}' \tilde{2}' \rangle \ . \tag{176}$$

Evaluation of the matrix element proceeds by

$$\langle 12 | H_{2\mathrm{body}\,\Sigma} | P_\phi \rangle \ = \ \left[-g^2 C_F \sum_{1'2'} \int [x_3 k_{31'}] \ \frac{1}{p_1^{+2}} \ f_1^{2t} \ \frac{j_{11'\mu} d^{\mu\nu} j_{1'2'\nu}}{p_1^- - p_{1'}^- - p_3^-} - g^2 C_F \sum_{2'1'} \int [x_3 k_{32'}] \ \frac{1}{p_2^{+2}} \ f_2^{2t} \ \frac{\bar{j}_{1'2'\nu} d^{\mu\nu} \bar{j}_{2'2}}{p_2^- - p_{2'}^- - p_3^-} \right]$$
$$\times \ \langle 12 | \int [\tilde{2}' \tilde{1}'] \ P_\phi^+ \tilde{\delta}_{P_\phi.\tilde{1}'\tilde{2}'} \ \phi_{\tilde{1}'\tilde{2}'}(x_{\tilde{1}'}, k_{\tilde{1}'\tilde{2}'}^\perp) | \tilde{1}' \tilde{2}' \rangle \tag{177}$$
$$= \ \left[-g^2 C_F \sum_{1'2'} \int [x_3 k_{31'}] \ \frac{1}{p_1^{+2}} \ f_1^{2t} \ \frac{j_{11'\mu} d^{\mu\nu} j_{1'2'\nu}}{p_1^- - p_{1'}^- - p_3^-} - g^2 C_F \sum_{2'1'} \int [x_3 k_{32'}] \ \frac{1}{p_2^{+2}} \ f_2^{2t} \ \frac{\bar{j}_{1'2'\nu} d^{\mu\nu} \bar{j}_{2'2}}{p_2^- - p_{2'}^- - p_3^-} \right]$$
$$\times \ P_\phi^+ \tilde{\delta}_{P_\phi.12} \ \phi_{1'2'}(x_1, k_{12}^\perp) \ . \tag{178}$$

The self-interaction factors preserve spin. The quark contribution of m_1 is multiplied by $\bar{u}_1 u_1 = 1$ and the antiquark contribution of $-m_2$ is multiplied by $\bar{v}_2 v_2 = -1$. Therefore, the self-interactions of the quark and antiquark differ only by the mass.

$$\langle 12 | H_{2\mathrm{body}\,\Sigma} | P_\phi \rangle \ = \ \left(\frac{\Sigma_1}{p_1^+} + \frac{\Sigma_2}{p_2^+} \right) \ P_\phi^+ \tilde{\delta}_{P_\phi.12} \ \phi_{12}(x_1, k_{12}^\perp) \ , \tag{179}$$

where for the quark or antiquark of mass m and momentum p the self-iteraction contribution reads

$$\Sigma \;=\; -g^2 C_F \int [xk] \; e^{-2t^2(p^- - p_q^- - p_g^-)^2} \; \frac{j_\mu d^{\mu\nu} j_\nu}{p^+(p^- - p_q^- - p_g^-)} \; . \tag{180}$$

Summary of the 2-body exchange and self-interaction terms

$$\langle 12|H_{2\text{body exch}}|P_\phi\rangle \;=\; \int [2'1'] \, K_{\rm exch}(12,2'1') \; P_\phi^+ \tilde\delta_{P_\phi.1'2'} \; \phi_{1'2'}(x_{1'}, k_{1'2'}^\perp) \tag{181}$$

with the kernel of dimension $\frac{1}{+\perp^2}$,

$$K_{\rm exch}(12,2'1') \;=\; g^2 C_{|2\rangle} \, f^t_{\underline{1}g,\bar 1} f^t_{\bar 2,\underline{2}g} \, \Delta_{12q2'1'}/p_g^+ \; j_\mu d^{\mu\nu} \bar j_\nu \; \tilde\delta_{12.1'2'} \; , \tag{182}$$

$$\Delta_{12q2'1'}/p_g^+ \;=\; -\,\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \; , \tag{183}$$

$$d^{\mu\nu} \;=\; -g^{\mu\nu} + \eta^\mu\eta^\nu(\rho_1+\rho_2)/(2p_g^{+2}) \; . \tag{184}$$

and

$$\langle 12|H_{2\text{body}\,\Sigma}|P_\phi\rangle \;=\; \left(\frac{\Sigma_1}{p_1^+} + \frac{\Sigma_2}{p_2^+}\right) \; P_\phi^+ \tilde\delta_{P_\phi.12} \; \phi_{1'2'}(x_1, k_{12}^\perp) \; , \tag{185}$$

where for the quark of mass m and momentum p the self-iteraction contribution reads

$$\Sigma \;=\; -g^2 C_F \int [xk] \; e^{-2t^2(p^- - p_q^- - p_g^-)^2} \; \frac{j_\mu d^{\mu\nu} j_\nu}{p^+(p^- - p_q^- - p_g^-)} \; . \tag{186}$$

The self-interaction Σ is positive and kernel $K_{\rm exch}$ is negative. Both diverge in the limit $m_g \to 0$, see below. The question is if the divergences can cancel out in the eigenvalue equation.

Using total-momentum conservation in the eigenvalue equation

First I eliminate the total eigenstate momentum from everywhere except factors f^t where it is stuck in ratio t/P_ϕ^+. I divide the matrix elements by the factor $P_\phi^+ \tilde\delta_{P_\phi.12}$ so that the eigenvalue-side of the 2-body eigenvalue equation is $(P_\phi^{\perp 2} + M^2)/P_\phi^+$, and

$$\frac{\langle 12|H_{2\text{body exch}}|P_\phi\rangle}{P_\phi^+ \tilde\delta_{P_\phi.12}} \;=\; \frac{1}{P_\phi^+} \int [x_{1'} k_{1'2'}] \, K^{xk}_{\rm exch}(12,2'1') \; \phi_{1'2'}(x_{1'}, k_{1'2'}^\perp) \; , \tag{187}$$

$$\frac{\langle 12|H_{2\text{body}\,\Sigma}|P_\phi\rangle}{P_\phi^+ \tilde\delta_{P_\phi.12}} \;=\; \left(\frac{\Sigma_{m_1}}{p_1^+} + \frac{\Sigma_{m_2}}{p_2^+}\right) \; \phi_{12}(x_1, k_{12}^\perp) \; , \tag{188}$$

where

$$\begin{aligned} K^{xk}_{\rm exch}(12,2'1') \;&=\; -g^2 C_{|2\rangle} \; e^{-t^2(p_{\bar 1}^- - p_{\underline{1}}^- - p_g^-)^2} e^{-t^2(p_{\bar 2}^- - p_{\underline{2}}^- - p_g^-)^2} \; \left(\frac{1}{2\rho_1} + \frac{1}{2\rho_2}\right) \\ &\quad\times\; j_\mu \left[-g^{\mu\nu} + \eta^\mu\eta^\nu(\rho_1+\rho_2)/(2p_g^{+2})\right] \bar j_\nu \; , \end{aligned} \tag{189}$$

$$\Sigma_m \;=\; g^2 C_F \int [xk] \; e^{-2t^2(p^- - p_q^- - p_g^-)^2} \; \frac{x}{\rho} \; j_\mu \left[-g^{\mu\nu} + \eta^\mu\eta^\nu \rho/p_g^{+2}\right] j_\nu \; . \tag{190}$$

I use identities

$$p^- - p_q^- - p_g^- \;=\; \frac{-\rho}{p_g^+} \; , \; p_{\bar 1}^- - p_{\underline{1}}^- - p_g^- \;=\; \frac{-\rho_1}{p_g^+} \; , \; p_{\bar 2}^- - p_{\underline{2}}^- - p_g^- \;=\; \frac{-\rho_2}{p_g^+} \; , \tag{191}$$

to obtain

$$K^{xk}_{\rm exch}(12,2'1') \;=\; -g^2 C_{|2\rangle} \; e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}} \; \frac{\rho_1+\rho_2}{2\rho_1\rho_2} \; j_\mu \left[-g^{\mu\nu} + \eta^\mu\eta^\nu(\rho_1+\rho_2)/(2p_g^{+2})\right] \bar j_\nu \; , \tag{192}$$

$$\Sigma_m \;=\; g^2 C_F \int [xk] \; e^{-2t^2\rho^2/p_g^{+2}} \; \frac{x}{\rho} \; j_\mu \left[-g^{\mu\nu} + \eta^\mu\eta^\nu \rho/p_g^{+2}\right] j_\nu \; . \tag{193}$$

I introduce

$$\Sigma_{m_i} = \Sigma_{m_i}^{-g} + \Sigma_{m_i}^{\eta} \ , \tag{194}$$

$$K_{\text{exch}}^{xk} = K_{\text{exch}}^{xk\,-g} + K_{\text{exch}}^{xk\,\eta} \ , \tag{195}$$

where terms with superscript $-g$ are the ones containing $-g^{\mu\nu}$ and terms with the superscript η are the ones containing $\eta^\mu \eta^\nu$.

Analysis of the terms with $-g^{\mu\nu}$

The terms with $-g^{\mu\nu}$ are

$$\left(\frac{\Sigma_{m_1}^{-g}}{p_1^+} + \frac{\Sigma_{m_2}^{-g}}{p_2^+} \right) \ \phi_{12}(x_1, k_{12}^\perp) + \frac{1}{P_\phi^+} \int [x_{1'} k_{1'2'}] \, K_{\text{exch}}^{xk\,-g}(12, 2'1') \ \phi_{1'2'}(x_{1'}, k_{1'2'}^\perp) \ , \tag{196}$$

where

$$\Sigma_m^{-g} = -g^2 C_F \int [xk] \ e^{-2t^2\rho^2/p_g^{+2}} \ \frac{x}{\rho} \ j^\mu j_\mu \ , \tag{197}$$

$$K_{\text{exch}}^{xk\,-g}(12, 2'1') = g^2 C_{|2\rangle} \ e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}} \ \frac{\rho_1+\rho_2}{2\rho_1\rho_2} \ j^\mu \bar{j}_\mu \ . \tag{198}$$

The products of quark currents are finite for quark states with finite invariant masses. The ρs depend only on the quark momenta. Large values of ρs are exponentially suppressed. The issue is what happens when ρ get small, or when $q^\perp$ or z get small. I need to express ρ and p_g^+ in terms of useful integration variables.

The case of Σ_m^{-g}

$$\frac{m^2 - \mathcal{M}_{qg}^2}{p^+} = p_g^+(p^- - p_q^- - p_g^-)/p_g^+ = \frac{q^2 - m_g^2}{p_g^+} = -\rho/p_g^+ \ , \tag{199}$$

$$\mathcal{M}_{qg}^2 - m^2 = \rho/x \ . \tag{200}$$

$$x = p_g^+/p^+ \ , \ k^\perp = p_g^\perp - x p^\perp \ . \tag{201}$$

In the quark-gluon rest frame,

$$p_g^+ = E_g + k^z \ , \ p_q^+ = E_q - k^z \ , \ E = E_g + E_q \ , \tag{202}$$

$$E_g = \sqrt{m_g^2 + k^{\perp 2} + k^{z2}} \ , \ E_q = \sqrt{m^2 + k^{\perp 2} + k^{z2}} \ , \tag{203}$$

$$\int [xk] = \int \frac{dx\, d^2k^\perp}{2(2\pi)^3 x(1-x)} = \int \frac{d^3k}{2(2\pi)^3} \left(\frac{1}{E_g} + \frac{1}{E_q} \right) \ . \tag{204}$$

The $g^{\mu\nu}$ terms contribute to the quark self-interaction

$$\Sigma_m^{-g} = -g^2 C_F \int \frac{d^3k}{2(2\pi)^3} \left(\frac{1}{E_g} + \frac{1}{E_q} \right) \ e^{-2(t/p^+)^2(\mathcal{M}_{qg}^2 - m^2)^2} \ \frac{j^\mu j_\mu}{\mathcal{M}_{qg}^2 - m^2} \ . \tag{205}$$

The fermion and antifermion factors are the same,

$$j^\mu j_\mu = \bar{u}_p \gamma^\mu (\not{p}_q + m) \gamma_\mu u_p = \bar{u}_p(-2\not{p}_q + 4m) u_p = -4pp_q + 8m^2 \tag{206}$$

$$= -2(2m^2 - q^2) + 8m^2 = 2q^2 + 4m^2 = -2(m_g^2 - q^2) + 2m_g^2 + 4m^2 \tag{207}$$

$$= -2\rho + 2m_g^2 + 4m^2 = -2x(E^2 - m^2) + 2m_g^2 + 4m^2 \ . \tag{208}$$

Therefore,

$$\Sigma_m^{-g} = -g^2 C_F \int \frac{d^3k}{2(2\pi)^3} \left(\frac{1}{E_g} + \frac{1}{E_q} \right) \ e^{-2(t/p^+)^2(E^2-m^2)^2} \ \frac{-2x(E^2 - m^2) + 2m_g^2 + 4m^2}{E^2 - m^2} \ . \tag{209}$$

The term with $-2x(E^2-m^2)$ provides in the limit $m_g \to 0$ the angle dependent contribution,

$$-2x \;=\; -2(E_g+k^z)/E \;\to\; -2(k+k^z)/E \;=\; -2k(1+\cos\theta)/E \;, \tag{210}$$

implying

$$\Sigma_m^{-g} \;=\; -g^2 C_F \int \frac{d^3k}{2(2\pi)^3}\left(\frac{1}{E_g}+\frac{1}{E_q}\right) e^{-2(t/p^+)^2(E^2-m^2)^2}\left[-2k(1+\cos\theta)/E+\frac{2m_g^2+4m^2}{E^2-m^2}\right] . \tag{211}$$

Integration over angle θ eliminates the term with $\cos\theta$. Therefore,

$$\Sigma_m^{-g} \;=\; \frac{-g^2 C_F 4\pi}{2(2\pi)^3}\int k^2 dk \left(\frac{1}{E_g}+\frac{1}{E_q}\right) e^{-2(t/p^+)^2(E^2-m^2)^2}\left[-2k/E+\frac{2m_g^2+4m^2}{E^2-m^2}\right] \tag{212}$$

$$\;=\; \frac{-g^2 C_F 4\pi}{2(2\pi)^3}\,\frac{1}{2}\int 2kdE\; e^{-2(t/p^+)^2(E^2-m^2)^2}\left[-2k/E+\frac{2m_g^2+4m^2}{E^2-m^2}\right] , \tag{213}$$

and, using the trick in Eq. (**??**),

$$2k \;=\; (E^2-m^2)/E \;, \tag{214}$$

I get

$$\Sigma_m^{-g} \;=\; \frac{-g^2 C_F 4\pi}{2(2\pi)^3}\,\frac{1}{2}\int \frac{EdE}{E^2}(E^2-m^2)\; e^{-2(t/p^+)^2(E^2-m^2)^2}\left[-\frac{E^2-m^2}{E^2}+\frac{2m_g^2+4m^2}{E^2-m^2}\right] , \tag{215}$$

or, using $u=E^2-m^2$,

$$\Sigma_m^{-g} \;=\; \frac{-g^2 C_F 4\pi}{2(2\pi)^3}\,\frac{1}{4}\int_0^\infty \frac{du}{u+m^2}\; e^{-2(t/p^+)^2u^2}\left[-\frac{u^2}{u+m^2}+2m_g^2+4m^2\right] , \tag{216}$$

which is finite when $m_g \to 0$.

The case of $\frac{1}{P_\phi^+}\int [x_{1'}k_{1'2'}]\, K_{\rm exch}^{xk\,-g}(12,2'1')\; \phi_{1'2'}(x_{1'},k_{1'2'}^\perp)$

The kernel of Eq. (198) involves spin dependence, which contributes to the result of integration with an unknown eigenfunction. But we can assume that the wave function falls off exponentially for extreme quark momenta, because the RGPEP factors force it to have such behavior. The integral to inspect is

$$\frac{1}{P_\phi^+} I_{\rm exch} \;=\; \frac{1}{P_\phi^+}\int [x_{1'}k_{1'2'}] K_{\rm exch}^{xk\,-g}(12,2'1')\; \phi_{1'2'}(x_{1'},k_{1'2'}^\perp) \tag{217}$$

$$\;=\; \frac{1}{P_\phi^+}\int [x_{1'}k_{1'2'}]\; g^2 C_{|2\rangle}\; e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}}\;\frac{1}{2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\; j_{11'}^\mu \phi_{1'2'}(x_{1'},k_{1'2'}^\perp)\bar{j}_{2'2\mu} \; . \tag{218}$$

The gluon mass m_g appears explicitly in ρ_1 and ρ_2 only,

$$\rho_1 \;=\; m_g^2 - q_1^2 \;=\; m_g^2 + \frac{1}{x_{1'}x_1}\left[\left(x_1 k'^\perp - x_{1'}k^\perp\right)^2 + m_1^2\,(x_{1'}-x_1)^2\right] , \tag{219}$$

$$\rho_2 \;=\; m_g^2 - q_2^2 \;=\; m_g^2 + \frac{1}{x_{2'}x_2}\left[\left(x_{2'} k^\perp - x_2 k'^\perp\right)^2 + m_2^2\,(x_2-x_{2'})^2\right] \tag{220}$$

The question is if the integral is finite for $m_g=0$. There is no issue for large ρs, given the exponential damping. The issue is what happens for small ρs appearing in the denominator. For $m_g \to 0$, using

$$x_{1'} \;=\; x_1 + z \;,\; k_{1'2'}^\perp = k_{12}^\perp + q^\perp \;, \tag{221}$$

I get

$$\rho_1 \;=\; \frac{m_1^2(z/x_1)^2+(q^\perp - k_{12}^\perp z/x_1)^2}{1+z/x_1} \;, \tag{222}$$

$$\rho_2 \;=\; \frac{m_2^2(z/x_2)^2+(q^\perp + k_{12}^\perp z/x_2)^2}{1-z/x_2} \; . \tag{223}$$

In the integration region where ρ_i is small, z/x_i is small and $q^\perp$ is close to the small vectors $\pm z k_{12}^\perp/x_i$. I neglect z/x_1 in denominator. For small z, $q^\perp$ is short, since it is close to a short vector. The integral is

$$\begin{aligned} I_{\rm exch} &= \frac{g^2 C_{|2\rangle}}{4(2\pi)^3} \int_{-x_1}^{x_2} \frac{dz}{(x_1+z)(x_2-z)} \int d^2q^\perp \; e^{-(t/P_\phi^+)^2(\rho_1^2+\rho_2^2)/z^2} \; \left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) \\ &\times \; j_{11'}^\mu \phi_{1'2'}(x_1+z, k_{12}^\perp + q^\perp) \bar{j}_{2'2\mu} \; . \end{aligned} \tag{224}$$

In the small-ρ regions, the factor $j_{11'}^\mu \phi_{1'2'}(x_1+z, k_{12}^\perp + q^\perp)\bar{j}_{2'2\mu}$ is a finite 4×4 table of spin amplitudes. Consider small ρ_1. Substitute

$$u^3 \;=\; z\, m_1/x_1 \;,\; u^1 \;=\; q^1 - z k_{12}^1/x_1 \;,\; u^2 \;=\; q^2 - z k_{12}^2/x_1 \;, \tag{225}$$

$$z \;=\; u^3\, x_1/m_1 \;,\; q^1 \;=\; u^1 + z k_{12}^1/x_1 \;,\; q^2 \;=\; u^2 + z k_{12}^2/x_1 \;, \tag{226}$$

$$dz\, d^2q \;=\; \frac{x_1}{m1}\, d^3u \;,\; \rho_1 \sim \vec{u}^{\,2} \; . \tag{227}$$

Thus, for small ρ_1, the essential form of the integral is

$$\int_{-x_1}^{x_2} dz \int d^2q^\perp \; \frac{1}{\rho_1} \;\sim\; \int \frac{d^3u}{\vec{u}^{\,2}} \; . \tag{228}$$

Such integral is finite in the limit $m_g \to 0$. Its limiting value does not depend on m_g. The only terms that are sensitive to $m_g \to 0$ are the terms with $\eta^\mu\eta^\nu$.

There is a strong argument for q_i^2 to never approach 0 if the momenta of fermions are limited and q_i itself is not 0. $q = p' - p$ is a space-like four-vector that can be looked at in any frame. Let

$$p \;=\; \left[\frac{p^{\perp 2}+m^2}{p^+}, p^+, p^\perp\right] \;\;,\;\; p' \;=\; \left[m, m, 0^\perp\right] \;\; . \tag{229}$$

Then,

$$q^2 \;=\; (p-p')^2 \;=\; (p^+ - m)\left(\frac{p^{\perp 2}+m^2}{p^+} - m\right) - p^{\perp 2} \;=\; p^+\frac{p^{\perp 2}+m^2}{p^+} - m\frac{p^{\perp 2}+m^2}{p^+} - p^+ m + m^2 - p^{\perp 2} \tag{230}$$

$$=\; m^2 - m\frac{p^{\perp 2}+m^2}{p^+} - p^+ m + m^2 \;=\; m\left[2m - \frac{p^{\perp 2}+m^2}{p^+} - p^+\right] \;=\; m(2m - 2E) \;=\; -2m(E-m) \tag{231}$$

$$=\; -2m(E^2-m^2)/E+m) \;=\; -2m\,\vec{q}^{\,2}/(E+m) < -\frac{m\vec{q}^{\,2}}{\sqrt{m^2+\vec{q}^{\,2}}} \; . \tag{232}$$

Hence q^2 approaches 0 from below only when $\vec{q}$ approaches 0. For $|\vec{q}| \ll m$,

$$q^2 < -\vec{q}^2 \tag{233}$$

for a particle kicked from rest. For $|\vec{q}| \gg m$,

$$q^2 < -m|\vec{q}| \; . \tag{234}$$

Analysis of the terms with $\eta^\mu\eta^\nu$

The terms with $\eta^\mu\eta^\nu$ are

$$\left(\frac{\Sigma_{m_1}^\eta}{p_1^+} + \frac{\Sigma_{m_2}^\eta}{p_2^+}\right) \; \phi_{12}(x_1, k_{12}^\perp) + \frac{1}{P_\phi^+}\int [x_{1'}k_{1'2'}]\, K_{\rm exch}^{xk\,\eta}(12, 2'1') \; \phi_{1'2'}(x_{1'}, k_{1'2'}^\perp) \; , \tag{235}$$

where

$$\Sigma_m^\eta \;=\; g^2 C_F \int [xk] \; e^{-2t^2\rho^2/p_g^{+2}} \; \frac{x\rho}{\rho p_g^{+2}} \; j^+ j^+ \; , \tag{236}$$

$$K_{\rm exch}^{xk\,\eta}(12, 2'1') \;=\; -g^2 C_{|2\rangle} \; e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}} \; \frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2 p_g^{+2}} \; j^+ \bar{j}^+ \; . \tag{237}$$

The currents j^+ and $\bar{j}^+$ conserve spin and provide p^+-dependent factors, $j_1^+ j_2^+ = 2\sqrt{p_1^+ p_{1'}^+} 2\sqrt{p_2^+ p_{2'}^+}$. Therefore,

$$\Sigma_m^\eta \;=\; g^2 C_F \int [xk] \; e^{-2t^2\rho^2/p_g^{+2}} \; \frac{x\rho}{\rho p_g^{+2}} \; 4p_m^+ p_{\underline{m}}^+ \;, \tag{238}$$

$$K_{\rm exch}^{xk\,\eta}(12,2'1') \;=\; -g^2 C_{|2\rangle} \; e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}} \; \frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2 p_g^{+2}} \; 4\sqrt{p_1^+ p_{1'}^+ p_2^+ p_{2'}^+} \;. \tag{239}$$

Simplifying,

$$\Sigma_m^\eta \;=\; g^2 C_F \int [xk] \; e^{-2t^2\rho^2/p_g^{+2}} \; \frac{4(1-x)}{x} \;, \tag{240}$$

$$K_{122'1'} \;=\; K_{\rm exch}^{xk\,\eta}(12,2'1') \;=\; -g^2 C_{|2\rangle} \; e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}} \; \frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2 x_g^{+2}} \; 4\sqrt{x_1 x_{1'} x_2 x_{2'}} \;, \tag{241}$$

with the potentially diverging terms, in the limit $m_g \to 0$,

$$\left(\frac{\Sigma_{m_1}^\eta}{p_1^+} + \frac{\Sigma_{m_2}^\eta}{p_2^+}\right) \; \phi(x_1, k_{12}^\perp) + \frac{1}{P_\phi^+} \int [x_{1'} k_{1'2'}] \, K_{122'1'} \; \phi(x_{1'}, k_{1'2'}^\perp) \;, \tag{242}$$

where spin labels are dropped because they are conserved by the singular terms. The eigenvalue-equation terms are of the form

$$\Sigma^\eta \phi + \frac{1}{P_\phi^+} \int K^\eta \phi' \;=\; \left(\Sigma^\eta + \frac{1}{P_\phi^+} \int K^\eta\right) \phi + \frac{1}{P_\phi^+} \int K^\eta \, (\phi' - \phi) \;. \tag{243}$$

The case of $\int K^\eta \, (\phi' - \phi)$

The first question is if the integral $I = \int K^\eta \, (\phi' - \phi)$ depends on m_g when $m_g \to 0$. For fixed x_1 and $k_{12}^\perp$, I change variables according to

$$x_{1'} \;=\; x_1 + z \;, \; k_{1'2'}^\perp = k_{12}^\perp + q^\perp \;, \tag{244}$$

and obtain

$$I \;=\; \frac{-g^2 C_{|2\rangle}}{2(2\pi)^3} \int_{-x_1}^{x_2} dz \int d^2 q^\perp \; e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}} \; \frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2 \, z^2} \; \frac{4\sqrt{x_1 x_{1'} x_2 x_{2'}}}{x_{1'} x_{2'}} \; (\phi' - \phi) \;. \tag{245}$$

Using $p_1 = x_1 P + k$, $p_{1'} = x_{1'} P + k'$ etc., and introducing

$$q^\perp \;=\; \sqrt{|z|} \, k^\perp \;, \tag{246}$$

according to Eq. (848), I get, following Eqs. (850) and (851),

$$\rho_1 \;=\; m_g^2 + \frac{m_1^2 (z/x_1)^2 + (\sqrt{|z|} k^\perp - k_{12}^\perp z/x_1)^2}{1 + z/x_1} \;, \tag{247}$$

$$\rho_2 \;=\; m_g^2 + \frac{m_2^2 (z/x_2)^2 + (\sqrt{|z|} k^\perp + k_{12}^\perp z/x_2)^2}{1 - z/x_2} \;. \tag{248}$$

Then,

$$\rho_1 \;=\; m_g^2 + |z| \frac{(m_1/x_1)^2 |z| + (k^\perp - s_z \sqrt{|z|} \, k_{12}^\perp/x_1)^2}{1 + z/x_1} \;, \tag{249}$$

$$\rho_2 \;=\; m_g^2 + |z| \frac{(m_2/x_2)^2 |z| + (k^\perp + s_z \sqrt{|z|} \, k_{12}^\perp/x_2)^2}{1 - z/x_2)} \;. \tag{250}$$

It is visible that ρ_1 and ρ_2 are non-negative, they can be 0 only when $m_g = 0$ and $z = 0$, and they both approach m_g^2 plus terms that tend to 0 as $|z|k^{\perp 2}$ when $z \to 0$ and $|k^\perp| > 0$ is kept fixed. Consequently,

$$\rho_1/|z| \;=\; \frac{m_g^2}{|z|} + \frac{m_1^2|z|/x_1^2 + (k^\perp - k_{12}^\perp s_z\sqrt{|z|}/x_1)^2}{1 + z/x_1} \;, \tag{251}$$

$$\rho_2/|z| \;=\; \frac{m_g^2}{|z|} + \frac{m_2^2|z|/x_2^2 + (k^\perp + k_{12}^\perp s_z\sqrt{|z|}/x_2)^2}{1 - z/x_2} \;. \tag{252}$$

It is also visible that $\rho_1/|z|$ and $\rho_2/|z|$ behave for fixed z and large $k^\perp$ as $k^{\perp 2}/(1+z/x_1)$ and $k^{\perp 2}/(1-z/x_2)$, respectively. The range of integration over $k^{\perp 2}$ is limited from above by a finite value when s is finite and x_1 and $x_{1'}$ are limited, which is the case in the Hamiltonian matrix elements between effective 2-body states of finite invariant mass. **Is the inspection and conclusions below correct?** I inspect the integral in Eq. (245),

$$I \;=\; \frac{-g^2 C_{|2\rangle}}{2(2\pi)^3} \int_{-x_1}^{x_2} dz \int |z|\, d^2k^\perp \; e^{-(t/P_\phi^+)^2(\rho_1^2+\rho_2^2)/z^2} \; \frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2\, z^2} \; \frac{4\sqrt{x_1 x_{1'} x_2 x_{2'}}}{x_{1'} x_{2'}} \; (\phi' - \phi) \;. \tag{253}$$

The singularity for small z is changed from $1/z^2$ to $1/|z|$. But the difference $\phi' - \phi$ is expandable in the Taylor series and the leading terms are

$$\phi(x_{1'}, k_{1'2'}^\perp) - \phi(x_1, k_{12}^\perp) \;=\; \phi(x_1 + z, k_{12}^\perp + q^\perp) - \phi(x_1, k_{12}^\perp) \;=\; z\phi_z + q^\perp \phi_\perp + O(z^2, zq^\perp, q^{\perp 2}) \tag{254}$$

$$\;=\; z\phi_z + \sqrt{|z|}k^\perp \phi_\perp + O(z^2, |z|^{3/2}k^\perp, |z|k^{\perp 2}) \;=\; O(\sqrt{|z|}) \;, \tag{255}$$

which means that the integrand behaves as $1/\sqrt{|z|}$. The integral is finite for $m_g \to 0$. Therefore, in Eq. 243), repeated here,

$$\Sigma^\eta \phi + \frac{1}{P_\phi^+}\int K^\eta \phi' \;=\; \left(\Sigma^\eta + \frac{1}{P_\phi^+}\int K^\eta\right)\phi + \frac{1}{P_\phi^+}\int K^\eta\,(\phi' - \phi) \;\;, \tag{256}$$

the second term on the right-hand side is finite and independent of m_g in the limit $m_g \to 0$. **The question of what to do with $1/(\rho_1\rho_2)$ that leads to the 4th power of momentum transfer in denominator requires inclusion of the RGPEP form factors. 20250906 20:24 sob San Dimas: the reasoning and conclusion are correct.**

Subtraction of wave function with the full kernel

The purpose of the subtraction is to reason in terms of the potential, which applies only in the non-relativistic systems. In Eq. (192), repeated below,

$$K_{\rm exch}^{xk}(12, 2'1') \;=\; -g^2 C_{|2\rangle}\; e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}} \; \frac{\rho_1+\rho_2}{2\rho_1\rho_2} \; j_\mu \left[-g^{\mu\nu} + \eta^\mu\eta^\nu(\rho_1+\rho_2)/(2p_g^{+2})\right] \bar{j}_\nu \;, \tag{257}$$

the kernel can be written as

$$K_{\rm exch}^{xk}(12, 2'1') \;=\; -g^2 C_{|2\rangle}\; e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}} \; \frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2} \; j_\mu \left[\frac{-2g^{\mu\nu}}{\rho_1+\rho_2} + \frac{\eta^\mu\eta^\nu}{p_g^{+2}}\right] \bar{j}_\nu \;. \tag{258}$$

In the non-relativistic limit, valid because the RGPEP form factors limit the quark relative momenta in the bound-state rest frame, the current factors j reduce to $2m_1$ and $2m_2$, preserving spin projections on the z-axis,

$$\rho_1 \;=\; m_g^2 + z^2 m_1^2/x_1^2 + q^{\perp 2} \;,\;\; \rho_2 \;=\; m_g^2 + z^2 m_2^2/x_2^2 + q^{\perp 2} \;,\;\; p_g^+ \;=\; z(m_1+m_2) \;, \tag{259}$$

$$x_i \;=\; \frac{m_i}{m_1+m_2} \;,\;\; \rho_1 \;=\; \rho_2 \;=\; m_g^2 + z^2(m_1+m_2)^2 + q^{\perp 2} \;=\; \rho \;=\; m_g^2 + \vec{q}^{\,2} \;, \tag{260}$$

$$q_z \;=\; z(m_1+m_2) \;, \tag{261}$$

and the kernel becomes

$$K_{\rm exch}^{xk}(12, 2'1') \;=\; -g^2 C_{|2\rangle}\; e^{-2(t\vec{q}^{\,2}/q_z)^2}\; 4m_1 m_2 \left[\frac{-1}{\vec{q}^{\,2}} + \frac{1}{q_z^2}\right] \;. \tag{262}$$

The right-hand side of the eigenvalue equation is M^2/P_ϕ^+. The exchange interaction term can thus be written in the non-relativistic limit as

$$\frac{1}{P_\phi^+}\int K\,(\phi'-\phi) \;=\; \frac{1}{P_\phi^+}\int [x_{1'}k_{1'2'}]\,K_{\rm exch}^{xk\,m_g}(12,2'1')\;\left[\phi_{1'2'}(\vec k+\vec q)-\phi_{1'2'}(\vec k\,)\right] \tag{263}$$

$$= -\frac{g^2C_{|2\rangle}}{P_\phi^+}\int \frac{d^3q}{2(2\pi)^3}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)e^{-2(t\vec q^{\,2}/q_z)^2}\;4m_1m_2$$

$$\times\;\left[\frac{1}{q_z^2}-\frac{1}{\vec q^{\,2}}\right]\;\left[\frac{1}{2}\,q^iq^j\partial_i\partial_j\phi_{12}(\vec k\,)+O(q^4\partial_k^4\phi)\right]\;, \tag{264}$$

where ∂ denotes the gradient in $\vec k$. This result matches Eqs. (67) in [54] and (95) in [62].

The case with $\left(\Sigma^\eta+\frac{1}{P_\phi^+}\int K^\eta\right)\phi$

The sum of integrals that multiply the wave function $\phi_{12}(x_1,k_{12}^\perp)$ is

$$S^\eta \;=\; \frac{g^2C_F}{p_2^+}\int [xk]\;e^{-2t^2\rho_2^2/p_g^{+2}}\;\frac{4(1-x)}{x}+\frac{g^2C_F}{p_1^+}\int [xk]\;e^{-2t^2\rho_1^2/p_g^{+2}}\;\frac{4(1-x)}{x}$$

$$-\;\frac{g^2C_{|2\rangle}}{P_\phi^+}\int [x_{1'}k_{1'2'}]e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}}\;\frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2x_g^{+2}}\;4\sqrt{x_1x_{1'}x_2x_{2'}}\;. \tag{265}$$

The range of integration variables is illustrated in Fig. 2. (20250824 09:10 nie Bukowina) I take advantage of Sec. **??**, beginning with Eq. (**??**). The antiquark, 2, self-interaction term and the corresponding exchange term in the considered sum, using $z=x_{1'}-x_1$, are $g^2(C_F$ or $C_{|2\rangle})/[2(2\pi)^3]$ times

$$\Sigma_{m_2}^\eta \;=\; \frac{1}{p_2^+}\int_0^1\frac{dx\,d^2k^\perp}{x(1-x)}\;\;e^{-2t^2\rho_2^2/p_g^{+2}}\;\frac{4(1-x)}{x}\;, \tag{266}$$

$$K_2^\eta \;=\; \frac{-1}{P^+}\int_{x_1}^1\frac{dx_{1'}d^2k_{1'2'}^\perp}{x_{1'}x_{2'}}\;e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}}\;\frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2z^{+2}}\;4\sqrt{x_1x_{1'}x_2x_{2'}}\;. \tag{267}$$

In the integral K_2^η, one can equally well integrate over $x_{2'}=1-x_{1'}$, with $dx_{1'}=-dx_{2'}$, and

$$\int_{x_1}^1\frac{dx_{1'}d^2k_{1'2'}^\perp}{x_{1'}x_{2'}}\;=\;\int_0^{x_2}\frac{dx_{2'}d^2k_{1'2'}^\perp}{x_{1'}x_{2'}}\;. \tag{268}$$

The momenta in K_2^η are

$$x_g \;=\; x_2-x_{2'} \;=\; xx_2\;,\;\; p_g^\perp \;=\; x(-k_{12}^\perp)+k^\perp\;, \tag{269}$$

$$x_q \;=\; x_{2'} \;=\; (1-x)x_2\;,\;\; p_q^\perp \;=\; p_{2'}^\perp \;=\; -k_{1'2'}^\perp \;=\; (1-x)(-k_{12}^\perp)-k^\perp \tag{270}$$

The Jacobian is

$$\frac{\partial(x_{2'},k_{1'2'}^1,k_{1'2'}^2)}{\partial(x,k^1,k^2)}=-x_2\;. \tag{271}$$

Therefore,

$$\int_{x_1}^1\frac{dx_{1'}d^2k_{1'2'}^\perp}{x_{1'}x_{2'}}\;=\;\int_0^{x_2}\frac{dx_{2'}d^2k_{1'2'}^\perp}{x_{1'}x_{2'}}\;=\;x_2\int_0^1\frac{dx\,d^2k^\perp}{(x_1+xx_2)(1-x)x_2}\;=\;\int_0^1\frac{dx\,d^2k^\perp}{(x_1+xx_2)(1-x)}\;. \tag{272}$$

The integrals to compare are

$$\Sigma_{m_2}^\eta \;=\; \frac{1}{p_2^+}\int_0^1\frac{dx\,d^2k^\perp}{x(1-x)}\;\;e^{-2t^2\rho_2^2/p_g^{+2}}\;\frac{4(1-x)}{x}\;, \tag{273}$$

$$K_2^\eta \;=\; \frac{-1}{P^+}\int_0^1\frac{dx\,d^2k^\perp}{(x_1+xx_2)(1-x)}\;e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}}\;\frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2x^2x_2^2}\;4\sqrt{x_1(x_1+xx_2)x_2(1-x)x_2}\;. \tag{274}$$

Simplifying,

$$\Sigma^{\eta}_{m_2} \;=\; \frac{1}{p_2^+}\int_0^1 dx\, d^2k^{\perp} \;\; e^{-2t^2\rho_2^2/p_g^{+2}} \;\frac{4}{x^2} \;, \tag{275}$$

$$K_2^{\eta} \;=\; \frac{-1}{P^+x_2}\int_0^1 \frac{dx\, d^2k^{\perp}\,\sqrt{x_1}}{\sqrt{(x_1+xx_2)(1-x)}} \; e^{-t^2(\rho_1^2+\rho_2^2)/p_g^{+2}} \;\frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2}\;\frac{4}{x^2} \;. \tag{276}$$

In both integrals,

$$p_g^+ \;=\; xp_2^+ \;. \tag{277}$$

So,

$$\Sigma^{\eta}_{m_2} \;=\; \frac{1}{p_2^+}\int_0^1 dx\, d^2k^{\perp} \;\; e^{-2(t/p_2^+)^2\rho_2^2/x^2} \;\frac{4}{x^2} \;, \tag{278}$$

$$K_2^{\eta} \;=\; \frac{-1}{p_2^+}\int_0^1 \frac{dx\, d^2k^{\perp}\,\sqrt{x_1}}{\sqrt{(x_1+xx_2)(1-x)}} \; e^{-(t/p_2^+)^2(\rho_1^2+\rho_2^2)/x^2} \;\frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2}\;\frac{4}{x^2} \;. \tag{279}$$

I have to work out ρs in both integrals. In Σ, I have

$$\rho_2 \;=\; m_g^2-(p_2-p_{2'})^2 \;=\; m_g^2-p_g^+(p_2^- - p_{2'}^-)+p_g^{\perp 2} \;=\; m_g^2+\frac{m_2^2x^2+k^{\perp 2}}{1-x} \;. \tag{280}$$

In K, I have

$$\rho_2 \;=\; m_g^2-(p_2-p_{2'})^2 \;=\; m_g^2+\frac{m_2^2x^2+k^{\perp 2}}{1-x} \;. \tag{281}$$

$$\begin{aligned}
\rho_1 \;&=\; m_g^2-(p_{1'}-p_1)^2 \;=\; m_g^2-p_g^+(p_{1'}^- - p_1^-)+p_g^{\perp 2} && (282)\\
&=\; m_g^2 - x[p_2^+=(x_2/x_1)p_1^+]\left[\frac{p_{1'}^{\perp 2}+m_1^2}{p_{1'}^+}-\frac{p_1^{\perp 2}+m_1^2}{p_1^+}\right]+p_g^{\perp 2} && (283)\\
&=\; m_g^2-(xx_2/x_1)\left[\frac{k_{1'2'}^{\perp 2}+m_1^2}{(x_{1'}/x_1)}-k_{12}^{\perp 2}-m_1^2\right]+(xk_{12}^{\perp}-k^{\perp})^2 && (284)\\
&=\; m_g^2-(xx_2/x_1)\left[\frac{((1-x)k_{12}^{\perp}+k^{\perp})^2+m_1^2}{(x_{1'}/x_1)}-k_{12}^{\perp 2}-m_1^2\right]+(xk_{12}^{\perp}-k^{\perp})^2 \;. && (285)
\end{aligned}$$

m_1^2:

$$\begin{aligned}
-(xx_2/x_1)\left[\frac{1}{(x_{1'}/x_1)}-1\right] \;&=\; -(xx_2/x_1)\left(\frac{x_1}{x_{1'}}-1\right) \;=\; (xx_2/x_1)z/x_{1'} \;=\; (xx_2/x_1)xx_2/(x_1+xx_2) && (286)\\
&=\; \frac{x^2x_2^2}{x_1(x_1+xx_2)} \;. && (287)
\end{aligned}$$

$k^{\perp 2}$:

$$-(xx_2/x_1)\left[\frac{x_1}{x_{1'}}\right]+1 \;=\; 1-xx_2\frac{1}{x_1+xx_2} \;=\; \frac{x_1}{x_1+xx_2} \;. \tag{288}$$

$k_{12}^{\perp 2}$:

$$\begin{aligned}
-(xx_2/x_1)\left[\frac{(1-x)^2}{(x_{1'}/x_1)}-1\right]+x^2 \;&=\; xx_2\left[1/x_1-(1-x)^2/x_{1'}\right]+x^2 \;=\; xx_2\left[1/x_1-(1-x)^2/(x_1+xx_2)\right]+x^2 && (289)\\
&=\; xx_2\frac{x_1+xx_2-x_1(1-x)^2}{x_1(x_1+xx_2)}+x^2 \;=\; x^2\left[x_2\frac{x_2+2x_1-x_1x}{x_1(x_1+xx_2)}+1\right] && (290)\\
&=\; x^2\frac{x_2^2+2x_1x_2-x_1x_2x+x_1^2+xx_1x_2}{x_1(x_1+xx_2)} \;=\; \frac{x^2}{x_1(x_1+xx_2)} \;. && (291)
\end{aligned}$$

$k^\perp k_{12}^\perp$:

$$-(xx_2/x_1)\left[\frac{2(1-x)}{(x_{1'}/x_1)}\right]-2x \;=\; -2x\left[x_2(1-x)/x_{1'}+1\right] \;=\; -\frac{2x}{x_1+xx_2}\left[x_2(1-x)+x_1+xx_2\right] \tag{292}$$

$$= -\frac{2x}{x_1+xx_2}\;. \tag{293}$$

The result for ρ_1 is

$$\rho_1 \;=\; m_g^2+m_1^2\frac{x^2x_2^2}{x_1(x_1+xx_2)}+k^{\perp 2}\frac{x_1}{x_1+xx_2}+k_{12}^{\perp 2}\frac{x^2}{x_1(x_1+xx_2)}-2k^\perp k_{12}^\perp\frac{x}{x_1+xx_2}\;. \tag{294}$$

or

$$x_1(x_1+xx_2)(\rho_1-m_g^2) \;=\; m_1^2x^2x_2^2+k^{\perp 2}x_1^2+k_{12}^{\perp 2}x^2-2k^\perp k_{12}^\perp xx_1\;. \tag{295}$$

It looks right because of the simplicity.

$$\rho_1 \;=\; m_g^2+\frac{m_1^2x^2x_2^2+(x_1k^\perp-xk_{12}^\perp)^2}{x_1(x_1+xx_2)}\;. \tag{296}$$

I am sure this is a correct expression. 20250825 00:23 pon Bukowina → 20:36 pon Ekologiczna Now the integrals, $g^2(C_F \text{ or } C_{|2\rangle})/[2(2\pi)^3]\times$

$$\Sigma_{m_2}^\eta \;=\; \frac{1}{p_2^+}\int_0^1 dx\,d^2k^\perp\;\; e^{-2(t/p_2^+)^2\rho_2^2/x^2}\;\frac{4}{x^2}\;, \tag{297}$$

$$K_2^\eta \;=\; \frac{-1}{p_2^+}\int_0^1 dx\,d^2k^\perp\, e^{-(t/p_2^+)^2(\rho_1^2+\rho_2^2)/x^2}\;\frac{\sqrt{x_1}}{\sqrt{(x_1+xx_2)(1-x)}}\;\frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2}\;\frac{4}{x^2}\;. \tag{298}$$

$$\rho_2 \;=\; m_g^2+\frac{m_2^2x^2+k^{\perp 2}}{1-x}\;, \tag{299}$$

$$\rho_1 \;=\; m_g^2+\frac{m_1^2x^2x_2^2+(x_1k^\perp-xk_{12}^\perp)^2}{x_1(x_1+xx_2)}\;. \tag{300}$$

The question is how the sum,

$$S_2^\eta \;=\; \Sigma_{m_2}^\eta+K_2^\eta\;, \tag{301}$$

behaves in the limit $m_g\to 0$.

****

I may have an error in my logic in Eq. (??) etc.

$$\Gamma_{\rm sg} \;=\; \frac{-\delta_{\sigma_1,\sigma_{1'}}\delta_{\sigma_{2'},\sigma_2}}{q^{+2}}4\sqrt{p_1^+p_{1'}^+p_{2'}^+p_2^+}\;, \tag{302}$$

$$\Gamma_{\rm exch} \;=\; \frac{+\delta_{\sigma_1,\sigma_{1'}}\delta_{\sigma_{2'},\sigma_2}}{q^{+2}}4\sqrt{p_1^+p_{1'}^+p_{2'}^+p_2^+}\;\frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2}. \tag{303}$$

Both terms involve the gluon spin singular factor $1/x^2$. One can subtract and add 1,

$$\frac{(\rho_1+\rho_2)^2}{4\rho_1\rho_2}-1+1 \;=\; \frac{(\rho_1-\rho_2)^2}{4\rho_1\rho_2}+1. \tag{304}$$

The term $+1$ on the right-hand side leads to the same singularity that appears in the seagull. The difference between ρ_1 and ρ_2 is

$$\rho_1-\rho_2 \;=\; 2y(\mathcal{M}_{12}^2-\mathcal{M}_{1'2'}^2)\sim y\;. \tag{305}$$

In the limit $y \to 0$ I have

$$\rho_1 = m_g^2 + \frac{4y^2 m_1^2 + 4(xl^\perp - yk^\perp)^2}{x^2 - y^2} \to m_g^2 + 4l^{\perp 2} - 8yl^\perp k^\perp / x \ , \tag{306}$$

$$\rho_2 = m_g^2 + \frac{4y^2 m_2^2 + 4[(1-x)l^\perp + yk^\perp]^2}{(1-x)^2 - y^2} \to m_g^2 + 4l^{\perp 2} + 8yl^\perp k^\perp/(1-x) \ , \tag{307}$$

In the quark-gluon rest frame,

$$p_g^+ = E_g + k^z \ , \ p_q^+ = E_q - k^z \ , \ E = E_g + E_q \ , \tag{308}$$

$$E_g = \sqrt{m_g^2 + k^{\perp 2} + k^{z2}} \ , \ E_q = \sqrt{m^2 + k^{\perp 2} + k^{z2}} \ , \tag{309}$$

$$\int [xk] = \int \frac{dx\, d^2k^\perp}{2(2\pi)^3 x(1-x)} = \int \frac{d^3k}{2(2\pi)^3} \left(\frac{1}{E_g} + \frac{1}{E_q} \right) \ . \tag{310}$$

There is another relation worth knowing and remembering. Namely,

$$q_1^2 - q_2^2 = (p_{1'} - p_1)^2 - (p_2 - p_{2'})^2 \tag{311}$$

$$= (p_{1'} - p_1 - p_2 + p_{2'})(p_{1'} - p_1 + p_2 - p_{2'}) \tag{312}$$

$$= (P_{1'2'} - P_{12})(q_1 + q_2) \ . \tag{313}$$

But $P_{1'2'}$ and P_{12} differ only in their minus components, while $q_1^+ = q_2^+ = zP_{12}^+$. Therefore,

$$q_1^2 - q_2^2 = \frac{1}{2} \frac{\mathcal{M}_{1'2'}^2 - \mathcal{M}_{12}^2}{P_{12}^+} 2zP_{12}^+ \ , \tag{314}$$

which amounts to the universal relation

$$q_1^2 - q_2^2 = z\,(\mathcal{M}_{1'2'}^2 - \mathcal{M}_{12}^2) \ . \tag{315}$$

Generally, I change the definitions of momentum transfers to

$$q_1 = p_{\overline{1}} - p_{\underline{1}} \ , \ q_2 = p_{\overline{2}} - p_{\underline{2}} \ . \tag{316}$$

These momenta match the exchanged gluon momentum except for the minus component.

$$q_1^2 - q_2^2 = (q_1 - q_2)(q_1 + q_2) = (p_{\overline{1}} - p_{\underline{1}} - p_{\overline{2}} + p_{\underline{2}})(q_1 + q_2) \tag{317}$$

$$= (p_{\overline{1}} + p_{\underline{2}} - p_{\underline{1}} - p_{\overline{2}})(q_1 + q_2) \ . \tag{318}$$

****

Substitution $k^\perp = \sqrt{x}\, q^\perp$ yields, with $t_p = t/p_2^+$, the sum in the form
$g^2(C_F \text{ or } C_{|2\rangle})/[2(2\pi)^3]\times$

$$S_2^\eta = \frac{1}{p_2^+} \int_0^1 dx\, d^2q^\perp \ e^{-t_p^2 \sigma_2^2} \ \frac{4}{x} \left[e^{-t_p^2 \sigma_2^2} - e^{-t_p^2 \sigma_1^2} \ \frac{\sqrt{x_1}}{\sqrt{(x_1 + x x_2)(1-x)}} \ \frac{(\sigma_1 + \sigma_2)^2}{4\sigma_1 \sigma_2} \right] \ , \tag{319}$$

$$\sigma_2 = \rho_2/x = \frac{m_g^2}{x} + \frac{m_2^2 x + q^{\perp 2}}{1-x} \ , \tag{320}$$

$$\sigma_1 = \rho_1/x = \frac{m_g^2}{x} + \frac{m_1^2 x x_2^2 + (x_1 q^\perp - \sqrt{x}\, k_{12}^\perp)^2}{x_1(x_1 + x x_2)} \ . \tag{321}$$

Because of the exponential factors, $|q^\perp|$ is limited. The dominant behaviors of σs near $x = 0$ are the same,

$$\sigma_2 = \frac{m_g^2}{x} + q^{\perp 2} \ , \tag{322}$$

$$\sigma_1 = \frac{m_g^2}{x} + q^{\perp 2} \ . \tag{323}$$

Therefore, for small x, the large square bracket vanishes and the small-x logarithmic divergence is actually absent. But there is also a divergence due to a small $q^\perp$ in the denominator. Ide spac 20250826 wto 01:10 Ekologiczna.

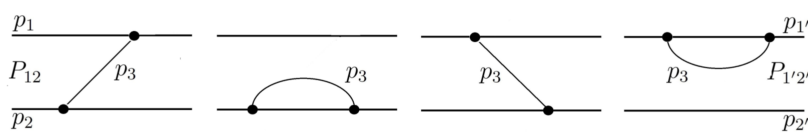
20250901 16:42 pon San Dimas: I need to solve the small denominator problems but now I continue with evaluation of S_2^η for a moment. I recall the option to write

$$\frac{(\sigma_1+\sigma_2)^2}{4\sigma_1\sigma_2} \;=\; \frac{(\sigma_1+\sigma_2)^2}{4\sigma_1\sigma_2} - 1 + 1 \;=\; \frac{(\sigma_1-\sigma_2)^2}{4\sigma_1\sigma_2} + 1 \;. \tag{324}$$

The difference $\sigma_1 - \sigma_2$ vanishes for $x \to 0$. Therefore, the integral with that difference is not diverging when $x \to 0$, which means I could set m_g^2 to zero in this term as far as small x is concerned. The remaining term of 1 contributes to the difference involving the form factors and a factor with square roots that also tends to 1 for $x \to 0$, which implies the logarithmic small-x divergence is canceled.

20250901 17:30 pon San Dimas: To complete the thesis of the paper I need to do 2 things; explain what happens for small denominators, which is the genuine infrared problem, and calculate how the gluon interacts with quarks in the 3-body component.

FIG. 2: Effective gluon exchange and self-interaction terms in the two-body component of a heavy quarkonium with the RGPEP vertex factors $f_{LI} f_{IR}$ represented by the black dots.



IX. INCLUSION OF QUARKONIUM COMPONENTS WITH EFFECTIVE GLUONS

Two-body component is coupled to the three-body states containing an effective gluon in addition to the quark pair,

$$|Q_{s1}\bar{Q}_{s2}G_{s3}\rangle \;=\; b_{s1}^\dagger d_{s2}^\dagger a_{s3}^\dagger|0\rangle \;, \tag{325}$$

and to the states with more effective gluons. States including additional quark-antiquark pairs are assumed so heavy that they do not significantly contribute to the ground and low excited eigenstates. All these components correspond to the basis states $|i\rangle_s$ in Eq. (102), while their coefficients $\psi_i(s)$ are the eigenstate wave functions. Thus, the effective eigenvalue problem differs qualitatively from the equations derived using the Tamm-Dancoff approximation [63–66]. Namely, the Hamiltonian H_s is independent of the regularization parameter $r \to 0$. Instead, the effective interaction terms in H_s include the vertex form factors $f^s_{\bar{p},\bar{q}}$ computed according to Eqs. (108) to (111). As a result, the eigenvalue problem for H_s does not require the so-called sector-dependent counterterms [65]. Nevertheless, the eigenstates have components with different numbers of effective constituents. $|\psi\rangle$ can be written as

$$|\psi\rangle \;=\; |2\rangle + |3\rangle + \ldots \;, \tag{326}$$

where dots indicate the components $|n\rangle$ with $n > 3$. The interaction that couples the component $|3\rangle$ to the component $|2\rangle$, is proportional to the coupling constant. Therefore, the contribution of the component $|3\rangle$ to the dynamics of component $|2\rangle$ is at least of order g^2. The components $|n\rangle$ with $n > 3$ contribute terms of at least 4th order. When the coupling constant g is very small, the effective coupling constant g_s is also very small, since the difference between them is of the third order. Up to the 2nd order, the coupling constants g and g_s are the same.

Self-interactions: the same Eq. (82) for quarks, and Eq. (88) for gluons.

Seagulls: similar to Eq. (61) for quarks but with different color factor, and a new seagull for gluon-quark interaction.

Terms $f - ff$: for qoark-antiquark interaction with different color factor, and new for the gluon-quark interaction

Seagull of quarks: These are new due to the instantaneous quarks

A. 2nd-order seagull interactions in the 3-body space

2025006 21:11 sob San Dimas - Start here tomorrow : In the 3-body space I have the 2nd-order interaction terms . . .

2025007 12:18 nie San Dimas

The initial expression for the quark-gluon seagull term is given in Eq. 52, repeated here

$$H_{JJ} \;=\; \int dx^- d^2x^\perp \;\frac{1}{2}\,(J^{a+}_{\psi f} + J^{a+}_{Af})\frac{-1}{\partial^{+2}}(J^{a+}_{Af} + J^{a+}_{\psi f})\;, \tag{327}$$

The quark-gluon seagull is only

$$H_{sqg} \;=\; \int dx^- d^2x^\perp \; J^{a+}_{Af}\frac{-1}{\partial^{+2}}J^{a+}_{\psi f}\;. \tag{328}$$

The seagull term structure in our notation is illustrated in Fig. 3. Using Eqs. (57) and (58),

$$: J^{a+}_{\psi f} : \;=\; -g\sum_{c_1,c_2}\sum_{\sigma_1,\sigma_2}\chi^\dagger_{c_1}T^a\chi_{c_2}\int[p_1p_2]\,2\sqrt{p_1^+p_2^+}\;\;[:\{\;\;\}:]^r\;, \tag{329}$$

$$\begin{aligned}
[:\{\;\;\}:]^r \;&=\; b_1^\dagger b_2 e^{i(p_1-p_2)x}\left[\theta_{1-2}f^r_{1,2(1-2)} + \theta_{2-1}f^r_{2,1(2-1)}\right]\delta_{\sigma_1,\sigma_2}\\
&-\; d_2^\dagger d_1 e^{i(p_2-p_1)x}\left[\theta_{2-1}f^r_{2,1(2-1)} + \theta_{1-2}f^r_{1,2(1-2)}\right]\delta_{\sigma_1,\sigma_2}\;.
\end{aligned} \tag{330}$$

I dropped the pair creation and annihilation terms from $[:\{\;\;\}:]^r$. The gluon current is given in Eq. (18), or

$$J^{a\mu}_{Af} \;=\; ig[\partial^\mu A_{f\nu}, A^\nu_f]^a\;,\;\; J^{a+}_{Af} \;=\; -ig[\partial^+ A^\perp_f, A^\perp_f]^a \;=\; -igf^{abc}(i\partial^+ A^{b\perp}_f)A^{c\perp}_f\;. \tag{331}$$

Using Eq. (29), knowing that x^+ is set to 0,

$$(i\partial^+ A^{b\perp}_f)A^{c\perp}_f \;=\; \sum_{12}\int[12]\,p_1^+\left[\varepsilon_1^\perp a_{1b}e^{-ip_1x} - \varepsilon_1^{\perp *}a^\dagger_{1b}e^{ip_1x}\right]\left[\varepsilon_2^\perp a_{2c}e^{-ip_2x} + \varepsilon_2^{\perp *}a^\dagger_{2c}e^{ip_2x}\right]\;. \tag{332}$$

For the seagulls in the 3-body space, I only keep terms of the type $a^\dagger a$. The gluon spin is conserved. Its spin conservation is assured by the products

$$\varepsilon_1^{\perp *}\varepsilon_2^\perp \;=\; -\varepsilon_1^*\varepsilon_2 \quad\text{ or }\quad \varepsilon_1^\perp\varepsilon_2^{\perp *} \;=\; -\varepsilon_1\varepsilon_2^*\;. \tag{333}$$

Since components + of ε are 0. Further,

$$(i\partial^+ A^{b\perp}_f)A^{c\perp}_f \;\to\; \sum_\sigma\int[12]\,p_1^+\left[a^\dagger_{2\sigma c}a_{1\sigma b}e^{-ip_1x}e^{ip_2x} - a^\dagger_{1\sigma b}a_{2\sigma c}e^{ip_1x}e^{-ip_2x}\right]\;. \tag{334}$$

The current is normal-ordered and regulated, a la Eqs. (57) and (58). I can simplify notation, keeping 1 for quarks, 2 for antiquarks and 3 for gluons.

$$\begin{aligned}
J^{a+}_{Af} \;&=\; -igf^{abc}(i\partial^+ A^{b\perp}_f)A^{c\perp}_f & (335)\\
&\to\; -igf^{abc}\sum_\sigma\int[33']\,p_3^+\left[a^\dagger_{3'\sigma c}a_{3\sigma b}e^{-ip_3x}e^{ip_{3'}x} - a^\dagger_{3\sigma b}a_{3'\sigma c}e^{ip_3x}e^{-ip_{3'}x}\right]f^r_{\overline{3},\underline{3}(\overline{3}-\underline{3})}\;. & (336)
\end{aligned}$$

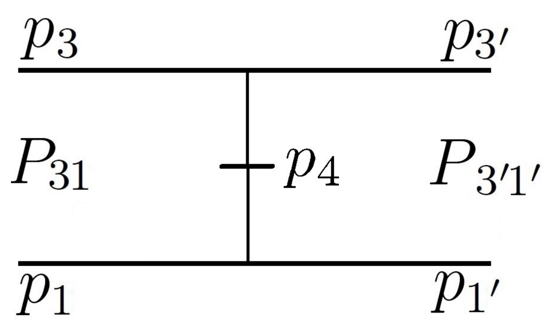
I change $3 \leftrightarrow 3'$ in the first term.

$$J^{a+}_{Af} \;\to\; -igf^{abc}\sum_\sigma\int[33']\left[p^+_{3'}a^\dagger_{3\sigma c}a_{3'\sigma b}e^{-ip_{3'}x}e^{ip_3x} - p_3^+a^\dagger_{3\sigma b}a_{3'\sigma c}e^{ip_3x}e^{-ip_{3'}x}\right]f^r_{\overline{3},\underline{3}(\overline{3}-\underline{3})}\;. \tag{337}$$

I change $b \leftrightarrow c$ in the first term.

$$J^{a+}_{Af} \;\to\; -igf^{abc}\sum_\sigma\int[33']\left[-p^+_{3'}a^\dagger_{3\sigma b}a_{3'\sigma c}e^{-ip_{3'}x}e^{ip_3x} - p_3^+a^\dagger_{3\sigma b}a_{3'\sigma c}e^{ip_3x}e^{-ip_{3'}x}\right]f^r_{\overline{3},\underline{3}(\overline{3}-\underline{3})}\;. \tag{338}$$

FIG. 3: Hamiltonian seagull interaction term of a gluon (3) and a quark (1), with a momentum transfer p_4.



The result is

$$J^{a+}_{Af} \;\to\; igf^{abc}\sum_\sigma \int [33'] \left[p^+_{3'} a^\dagger_{3\sigma b} a_{3'\sigma c} e^{-ip_{3'}x} e^{ip_3 x} + p^+_3 a^\dagger_{3\sigma b} a_{3'\sigma c} e^{ip_3 x} e^{-ip_{3'}x}\right] f^r_{\overline{3},\underline{3}(\overline{3}-\underline{3})} \tag{339}$$

$$= igf^{abc}\sum_\sigma \int [33'] \,(p^+_3 + p^+_{3'})\, a^\dagger_{3\sigma b} a_{3'\sigma c} e^{-i(p_{3'}-p_3)x} f^r_{\overline{3},\underline{3}(\overline{3}-\underline{3})} \;. \tag{340}$$

The seagull term, including both quarks and antiquarks, is

$$\begin{aligned} H_{sqg} \;=\; & \int dx^- d^2x^\perp \; igf^{abc} \int [33'] \,(p^+_3 + p^+_{3'})\; a^\dagger_{3b} a_{3'c}\; e^{-i(p_{3'}-p_3)x}\; \delta_{\sigma_3\sigma_{3'}}\; f^r_{\overline{3},\underline{3}(\overline{3}-\underline{3})} \\ \times\; & \frac{1}{(p^+_{3'} - p^+_3)^2} \\ \times\; & (-g)\; \chi^\dagger_1 T^a \chi_{1'} \int [11']\; 2\sqrt{p^+_1 p^+_{1'}}\; \left[b^\dagger_1 b_{1'} e^{i(p_1 - p_{1'})x} - d^\dagger_{1'} d_1 e^{i(p_{1'}-p_1)x}\right] \\ \times\; & \delta_{\sigma_1\sigma_{1'}}\; f^r_{\overline{1},\underline{1}(\overline{1}-\underline{1})} \;. \end{aligned} \tag{341}$$

Compacted, the gluon-quark seagull is

$$H_{sqg} \;=\; -g^2 \int [33'11']\; \tilde{\delta}_{c.a}\; \delta_{\rm spins}\; f^r_{\overline{1},\underline{1}(\overline{1}-\underline{1})}\; f^r_{\overline{3},\underline{3}(\overline{3}-\underline{3})}\; \frac{(p^+_3 + p^+_{3'})\; 2\sqrt{p^+_1 p^+_{1'}}}{(p^+_{3'} - p^+_3)^2}\; \chi^\dagger_1 T^a \chi_{1'}\, if^{abc}\; a^\dagger_{3b} a_{3'c} b^\dagger_1 b_{1'} \;, \tag{342}$$

and the gluon-antiquark seagull is

$$H_{s\bar{q}g} \;=\; +g^2 \int [33'22']\; \tilde{\delta}_{c.a}\; \delta_{\rm spins}\; f^r_{\overline{2},\underline{2}(\overline{2}-\underline{2})}\; f^r_{\overline{3},\underline{3}(\overline{3}-\underline{3})}\; \frac{(p^+_3 + p^+_{3'})\; 2\sqrt{p^+_2 p^+_{2'}}}{(p^+_{3'} - p^+_3)^2}\; \chi^\dagger_{2'} T^a \chi_2\, if^{abc}\; a^\dagger_{3b} a_{3'c} d^\dagger_2 d_{2'} \;. \tag{343}$$

The wave-function color factor for the globally colorless 3-body states is

$$C^{123}_{\rm 3body} \;=\; \chi^\dagger_1 T^3 \chi_2 \;. \tag{344}$$

The color action of the quark seagull H_{sqg} on the globally colorless 3-body states is described by

$$\chi^\dagger_1 T^s \chi_{1'} if^{s33'}\; \chi^\dagger_{1'} T^{3'} \chi_2 \;=\; if^{s33'} \chi^\dagger_1 T^s T^{3'} \chi_2 \;=\; \frac{i}{2} f^{s33'} \chi^\dagger_1 [T^s, T^{3'}] \chi_2 \;=\; \frac{i}{2} f^{s33'} if^{as3'}\; \chi^\dagger_1 T^a \chi_2 \tag{345}$$

$$=\; -\frac{1}{2}\, f^{s33'} f^{as3'}\; \chi^\dagger_1 T^a \chi_2 \;=\; \frac{1}{2}\, f^{3s3'} f^{as3'}\; \chi^\dagger_1 T^a \chi_2 \;=\; \frac{1}{2}\, N_c\; \chi^\dagger_1 T^3 \chi_2 \;, \tag{346}$$

which means multiplication by $N/2$. The color action of the antiquark seagull $H_{s\bar{q}g}$ on the same globally colorless 3-body states is described by

$$\chi^\dagger_{2'} T^s \chi_2 if^{s33'}\; \chi^\dagger_1 T^{3'} \chi_{2'} \;=\; if^{s33'}\; \chi^\dagger_{2'} T^s \chi_2\; \chi^\dagger_1 T^{3'} \chi_{2'} \;=\; if^{s33'}\; \chi^\dagger_1 T^{3'} T^s \chi_2 \;=\; \frac{i}{2} f^{s33'} if^{3'sa}\; \chi^\dagger_1 T^a \chi_2 \tag{347}$$

$$=\; -\frac{1}{2}\, f^{s33'} f^{3'sa}\; \chi^\dagger_1 T^a \chi_2 \;=\; -\frac{1}{2}\, f^{3's3} f^{3'sa}\; \chi^\dagger_1 T^a \chi_2 \;=\; -\frac{1}{2}\, N_c\; \chi^\dagger_1 T^3 \chi_2 \tag{348}$$

which means multiplication by $-N/2$. Including the sign factors in front of the operators, the net factor is the same in both cases, $-1/x^2$. The sign of the seagull term in the colorless 3-body states is the same as in the colorless 2-body states. The strength is slightly increased from 4/3 to 3/2, by the factor 9/8.

20250908 21:27 pon San Dimas: Start tomorrow from checking what the gluon exchange does.

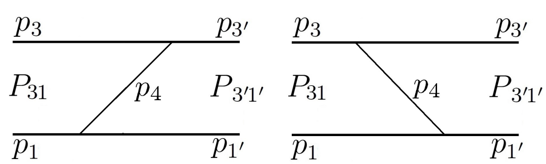
B. 2nd-order quark-gluon interactions through gluon exchange in the 3-body space

The interaction of a gluon with a quark or an antiquark in the 3-body space, proceeding through the exchange of a gluon, results from the double action of the term $-(J^{a\mu}_{\psi f}+J^{a\mu}_{Af})A^a_{f\mu}$ in $\mathcal{H}_{\rm QCD}$ in Eq. (16) and the term $m_g\phi^a\frac{1}{\partial^+}(J^{a+}_{Af}+J^{a+}_{\psi f})$ in Eq. (21), using the currents

$$
\begin{aligned}
J^{a\mu}_{\psi f} &= -g\bar\psi_f T^a\gamma^\mu\psi_f \ , &(349)\\
J^{a\mu}_{Af} &= ig[\partial^\mu A_{f\nu}, A^\nu_f]^a \ . &(350)
\end{aligned}
$$

The exchange term formula in our notation is illustrated in Fig. 4. The formula follows from $\mathcal{H}_f$ and the renormalized

FIG. 4: Hamiltonian interaction term of a gluon (3) with a quark (1) through the exchange of a gluon carrying the momentum transfer p_4.



interaction Hamiltonian terms given in Eq. (112). The gluon exchange in the 3-body space is contained in

$$
\mathcal{H}^{(2)} = f_{LR}\,\mathcal{H}^{(2)}_0 + (f_{LR} - f_{LI}f_{IR})\,\Delta_{LIR}\,\mathcal{H}^{(1)}_0\,\mathcal{H}^{(1)}_0 \ , \tag{351}
$$

where we recognize analogs of the 2-body terms present in Eqs. (125) and (**??**). We identify the the 3-body-space gluon exchange terms between a gluon and a quark in

$$
\begin{aligned}
\mathcal{H}_s &= \mathcal{H}_f + f_{LR}\left[\Sigma_{g\,\rm sing} + \delta m^2_{g\,\rm ln} + \Sigma_{q\,\rm sing} + \delta m^2_{q\,\rm ln} + f_r f_r H_{\rm sggq} + F_r S_{\rm sggq} + F_r S_{\rm exch}\right]\\
&+ (f_{LR} - f_{LI}f_{IR})\,\Delta_{LIR}\,f_r\mathcal{H}^{(1)}_0\,f_r\mathcal{H}^{(1)}_0 \ . &(352)
\end{aligned}
$$

Question: Do the divergences of gluon-quark seagull $f_{LR}f_rf_rH_{\rm sggq}$ and gluon exchange in $f_{LR}\,\Delta_{LIR}\,f_r\mathcal{H}^{(1)}_0\,f_r\mathcal{H}^{(1)}_0$ cancel each other? Gluon-quark seagull operator is given in Eq. (342). The gluon exchange interaction between a gluon and a quark follows from the product $\mathcal{H}^{(1)}_0\,\mathcal{H}^{(1)}_0$, in which $\mathcal{H}^{(1)}_0$ includes the following terms from Eq. (21),

$$
\mathcal{H}^{(1)}_0 = \int_F\left[-(J^{a\mu}_{\psi f}+J^{a\mu}_{Af})A^a_{f\mu} + m_g\phi^a\frac{1}{\partial^+}(J^{a+}_{Af}+J^{a+}_{\psi f})\right] \ . \tag{353}
$$

Integration by parts yields

$$
\mathcal{H}^{(1)}_0 \equiv -\int_F\left(J^{a\mu}_{\psi f}+J^{a\mu}_{Af}\right)\left(A^a_{f\mu}+\frac{m_g\eta_\mu}{\partial^+}\phi^a\right) \ . \tag{354}
$$

I called a file toolong on 20250912 16:00 in San Dimas and start from scratch evaluation of the interaction terms. There are 4 terms to evaluate

$$
\begin{aligned}
\mathcal{H}_{\psi A} &= -\int_F J^{a\mu}_{\psi f}A^a_{f\mu} \ , &(355)\\
\mathcal{H}_{\psi\phi} &= -\int_F J^{a\mu}_{\psi f}\frac{m_g\eta_\mu}{\partial^+}\phi^a \ , &(356)\\
\mathcal{H}_{AA} &= -\int_F J^{a\mu}_{Af}A^a_{f\mu} \ , &(357)\\
\mathcal{H}_{A\phi} &= -\int_F J^{a\mu}_{Af}\frac{m_g\eta_\mu}{\partial^+}\phi^a \ . &(358)
\end{aligned}
$$

Reminder

$$\frac{m_g\eta_\mu}{\partial^+}\phi^a \;=\; \frac{im_g\eta_\mu}{i\partial^+}\phi^a \;=\; \int[4]\,\frac{m_g\eta_\mu}{p_4^+}\left(a_{4a}e^{-ip_4x}+a^\dagger_{4a}e^{ip_4x}\right)\;. \tag{359}$$

Evaluation of $\mathcal{H}_{\psi A}$

$$\mathcal{H}_{\psi A} \;=\; -\int_F J^{a\mu}_{\psi f}A^a_{f\mu}\;, \tag{360}$$

$$\;=\; -g_{\mu\nu}(-g)\int[124]\,\tilde{\delta}_{c.a}\,f\left[u_1\chi_1b_1+v_1\chi_1d_1^\dagger\right]^\dagger\gamma^0\gamma^\mu\left[\varepsilon_4^\nu T^4a_4+\varepsilon_4^{\nu*}T^4a_4^\dagger\right]\left[u_2\chi_2b_2+v_2\chi_2d_2^\dagger\right] \tag{361}$$

In the 3-body space, two terms count: quark or antiquark emitting and absorbing a gluon, while the pair creation or annihilation are ignored as too far off shell.

$$\begin{aligned}\mathcal{H}_{\psi A} \;=\; -\int_F J^{a\mu}_{\psi f}A^a_{f\mu} \;=\; & g\int[11'4]\,\tilde{\delta}_{c.a}\,f\;\left[u_1\chi_1b_1\right]^\dagger\gamma^0\gamma_\nu\left[\varepsilon_4^\nu T^4a_4+\varepsilon_4^{\nu*}T^4a_4^\dagger\right]\left[u_{1'}\chi_{1'}b_{1'}\right]\\ &+\;g\int[22'4]\tilde{\delta}_{c.a}f\;\left[v_{2'}\chi_{2'}d^\dagger_{2'}\right]^\dagger\gamma^0\gamma_\nu\left[\varepsilon_4^\nu T^4a_4+\varepsilon_4^{\nu*}T^4a_4^\dagger\right]\left[v_2\chi_2d_2^\dagger\right]\;.\end{aligned} \tag{362}$$

$$\begin{aligned}\mathcal{H}_{\psi A} \;=\; & g\int[11'4]\,\tilde{\delta}_{c.a}\,f^r_{\overline{1},\underline{1}4}\;\chi_1^\dagger T^4\chi_{1'}\;\bar{u}_1\gamma_\nu u_{1'}\left(\varepsilon_4^\nu\;b_1^\dagger b_{1'}a_4+\varepsilon_4^{\nu*}a_4^\dagger b_1^\dagger b_{1'}\right)\\ -\;& g\int[22'4]\,\tilde{\delta}_{c.a}\,f^r_{\overline{2},\underline{2}4}\;\chi^\dagger_{2'}T^4\chi_2\;\bar{v}_{2'}\gamma_\nu v_2\;\left(\varepsilon_4^\nu\;d_2^\dagger d_{2'}a_4+\varepsilon_4^{\nu*}\;a_4^\dagger d_2^\dagger d_{2'}\right)\;.\end{aligned} \tag{363}$$

Evaluation of $\mathcal{H}_{\psi\phi}$

$$\mathcal{H}_{\psi\phi} \;=\; -\int_F J^{a\mu}_{\psi f}\frac{m_g\eta_\mu}{\partial^+}\phi^a \tag{364}$$

$$\;=\; -g_{\mu\nu}(-g)\int[12\tilde{4}]\,\tilde{\delta}_{c.a}\,f\left[u_1\chi_1b_1+v_1\chi_1d_1^\dagger\right]^\dagger\gamma^0\gamma^\mu\frac{m_g\eta^\nu}{p_{\tilde{4}}^+}T^{\tilde{4}}\left[a_{\tilde{4}}+a^\dagger_{\tilde{4}}\right]\left[u_2\chi_2b_2+v_2\chi_2d_2^\dagger\right] \tag{365}$$

Again, in the 3-body space, two terms count: quark or antiquark emitting and absorbing a gluon, while the pair creation or annihilation are ignored as too far off shell.

$$\begin{aligned}\mathcal{H}_{\psi\phi} \;=\; -\int_F J^{a\mu}_{\psi f}\frac{m_g\eta_\mu}{\partial^+}\phi^a \;=\; & g\int[11'\tilde{4}]\,\tilde{\delta}_{c.a}\,f^r_{\overline{1},\underline{1}\tilde{4}}\;\chi_1^\dagger T^{\tilde{4}}\chi_{1'}\;\bar{u}_1\gamma^+u_{1'}\;\frac{m_g}{p^+_{\tilde{4}}}\left(b_1^\dagger b_{1'}a_{\tilde{4}}+a^\dagger_{\tilde{4}}b_1^\dagger b_{1'}\right)\\ -\;& g\int[22'\tilde{4}]\,\tilde{\delta}_{c.a}\,f^r_{\overline{2},\underline{2}\tilde{4}}\;\chi^\dagger_{2'}T^{\tilde{4}}\chi_2\;\bar{v}_{2'}\gamma^+v_2\;\frac{m_g}{p^+_{\tilde{4}}}\left(d_2^\dagger d_{2'}a_{\tilde{4}}+a^\dagger_{\tilde{4}}d_2^\dagger d_{2'}\right)\;.\end{aligned} \tag{366}$$

Below, I tend to omit regularization factors $f^r_{\overline{x},\underline{xy}}$ till I come to the final expression for the gluon-exchange interaction between the gluon and quark in the 3- body space.

Evaluation of $\mathcal{H}_{AA}$

$$\mathcal{H}_{AA} \;=\; -\int_F J^{a\mu}_{Af}A^a_{f\mu}\;, \tag{367}$$

$$A^a_{f\mu} \;=\; \sum_{3'}\int[3']\left[\varepsilon_{3'\mu}a_{3'a}e^{-ip_{3'}x}+\varepsilon^*_{3'\mu}a^\dagger_{3'a}e^{ip_{3'}x}\right]\;, \tag{368}$$

$$J^{a\mu}_{Af} \;=\; g[i\partial^\mu A_{f\nu},A^\nu_f]^a \;=\; igf^{abc}(i\partial^\mu A^b_{f\nu})A^{c\nu}_f \tag{369}$$

$$\;=\; igf^{abc}\sum_{34}\int[34]\;p_3^\mu\left[\varepsilon_3a_{3b}e^{-ip_3x}-\varepsilon_3^*a^\dagger_{3b}e^{ip_3x}\right]\left[\varepsilon_4a_{4c}e^{-ip_4x}+\varepsilon_4^*a^\dagger_{4c}e^{ip_4x}\right] \tag{370}$$

$$\mathcal{H}_{AA} = -\int_F igf^{abc}\sum_{33'4}\int[33'4]\ \left[\varepsilon_3 a_{3b}e^{-ip_3x} - \varepsilon_3^* a_{3b}^\dagger e^{ip_3x}\right]\left[\varepsilon_4 a_{4c}e^{-ip_4x} + \varepsilon_4^* a_{4c}^\dagger e^{ip_4x}\right]$$
$$\times\ \left[p_3\varepsilon_{3'}a_{3'a}e^{-ip_{3'}x} + p_3\varepsilon_{3'}^* a_{3'a}^\dagger e^{ip_{3'}x}\right] \tag{371}$$
$$= -igf^{abc}\sum_{33'4}\int[33'4]\,\tilde{\delta}_{c.a}\left[\varepsilon_3 a_{3b} - \varepsilon_3^* a_{3b}^\dagger\right]\left[\varepsilon_4 a_{4c} + \varepsilon_4^* a_{4c}^\dagger\right]\left[p_3\varepsilon_{3'}a_{3'a} + p_3\varepsilon_{3'}^* a_{3'a}^\dagger\right]\ . \tag{372}$$

There are 8-2=6 terms to evaluate.

$$112\,[\varepsilon_3 a_{3b}]\,[\varepsilon_4 a_{4c}]\left[p_3\varepsilon_{3'}^* a_{3'a}^\dagger\right] + 121\,[\varepsilon_3 a_{3b}]\left[\varepsilon_4^* a_{4c}^\dagger\right][p_3\varepsilon_{3'}a_{3'a}] + 122\,[\varepsilon_3 a_{3b}]\left[\varepsilon_4^* a_{4c}^\dagger\right]\left[p_3\varepsilon_{3'}^* a_{3'a}^\dagger\right]$$
$$+\ 211\left[-\varepsilon_3^* a_{3b}^\dagger\right][\varepsilon_4 a_{4c}]\,[p_3\varepsilon_{3'}a_{3'a}] + 212\left[-\varepsilon_3^* a_{3b}^\dagger\right][\varepsilon_4 a_{4c}]\left[p_3\varepsilon_{3'}^* a_{3'a}^\dagger\right] + 221\left[-\varepsilon_3^* a_{3b}^\dagger\right]\left[\varepsilon_4^* a_{4c}^\dagger\right][p_3\varepsilon_{3'}a_{3'a}]\ . \tag{373}$$

Clean.

$$\mathcal{H}_{AA} = -igf^{abc}\sum_{33'4}\int[33'4]\,\tilde{\delta}_{c.a}\ X\ , \tag{374}$$
$$X = \varepsilon_3\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}a_{3'a}^\dagger + \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}a_{4c}^\dagger a_{3'a} + \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}^\dagger a_{3'a}^\dagger$$
$$-\ \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}a_{3'a} - \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}^\dagger a_{4c}a_{3'a}^\dagger - \varepsilon_3^*\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}^\dagger a_{3'a}\ . \tag{375}$$

Separate terms with 2 daggers and 1 dagger.

$$X = \varepsilon_3\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}a_{3'a}^\dagger + \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}a_{4c}^\dagger a_{3'a} - \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}a_{3'a}$$
$$+\ \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}^\dagger a_{3'a}^\dagger - \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}^\dagger a_{4c}a_{3'a}^\dagger - \varepsilon_3^*\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}^\dagger a_{3'a}\ . \tag{376}$$

check

$$X = \varepsilon_3\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}a_{3'a}^\dagger - \varepsilon_3^*\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}^\dagger a_{3'a}$$
$$+\ \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}a_{4c}^\dagger a_{3'a} - \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}^\dagger a_{4c}a_{3'a}^\dagger$$
$$+\ \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}^\dagger a_{3'a}^\dagger - \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}a_{3'a}\ . \tag{377}$$

$$-X = \varepsilon_3^*\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}^\dagger a_{3'a} - \varepsilon_3\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}a_{3'a}^\dagger$$
$$+\ \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}^\dagger a_{4c}a_{3'a}^\dagger - \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}a_{4c}^\dagger a_{3'a}$$
$$+\ \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}a_{3'a} - \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}^\dagger a_{3'a}^\dagger\ . \tag{378}$$

Arrange h.c.

$$-X = \varepsilon_3^*\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}^\dagger a_{3'a} - \varepsilon_3\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}a_{3'a}^\dagger$$
$$+\ \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}^\dagger a_{4c}a_{3'a}^\dagger - \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}a_{4c}^\dagger a_{3'a}$$
$$+\ \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}^\dagger a_{3'a}^\dagger - \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}a_{3'a}\ . \tag{379}$$

Rewrite.

$$-X = \varepsilon_3^*\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}^\dagger a_{3'a} + \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}^\dagger a_{4c}a_{3'a}^\dagger + \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}^\dagger a_{3'a}^\dagger$$
$$-\ \varepsilon_3\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_{3b}a_{4c}a_{3'a}^\dagger - \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_{3b}a_{4c}^\dagger a_{3'a} - \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}\ \ a_{3b}^\dagger a_{4c}a_{3'a}\ . \tag{380}$$

Notation.

$$\mathcal{H}_{AA} = g\sum_{33'4}\int[33'4]\,\tilde{\delta}_{c.a}\ if^{c_{3'}c_3c_4}\ Z_{33'4} + h.c.\ , \tag{381}$$
$$Z_{33'4} = \varepsilon_3^*\varepsilon_4^*\ p_3\varepsilon_{3'}\ \ a_3^\dagger a_4^\dagger a_{3'} + \varepsilon_3^*\varepsilon_4\ p_3\varepsilon_{3'}^*\ \ a_3^\dagger a_4 a_{3'}^\dagger + \varepsilon_3\varepsilon_4^*\ p_3\varepsilon_{3'}^*\ \ a_3 a_4^\dagger a_{3'}^\dagger\ . \tag{382}$$

I change notation in the terms second and third to make them look like the first term. I rename $Z_{33'4}$ to $Y_{343'}$ including $if^{c_3c_4c_{3'}}$. In the second term, I change $4 \leftrightarrow 3'$, and in the third term I change $3 \leftrightarrow 3'$. The result is

$$\mathcal{H}_{AA} = g\sum_{33'4}\int[33'4]\,\tilde{\delta}_{c.a}\ Y_{343'} + h.c.\ , \tag{383}$$
$$Y_{343'} = if^{c_3c_4c_{3'}}\ \varepsilon_3^*\varepsilon_4^*\ p_3\varepsilon_{3'}\ a_3^\dagger a_4^\dagger a_{3'} - if^{c_3c_4c_{3'}}\ \varepsilon_3^*\varepsilon_{3'}\ p_3\varepsilon_4^*\ a_3^\dagger a_{3'}a_4^\dagger + if^{c_3c_4c_{3'}}\ \varepsilon_{3'}\varepsilon_4^*\ p_{3'}\varepsilon_3^*\ a_{3'}a_4^\dagger a_3^\dagger\ . \tag{384}$$

Normal ordering.

$$\mathcal{H}_{AA} = g \sum_{343'} \int [343'] \, \tilde{\delta}_{c.a} \; Y_{343'} + h.c. \; , \tag{385}$$

$$Y_{343'} = i f^{c_3 c_4 c_{3'}} \left(\varepsilon_3^* \varepsilon_4^* \; \varepsilon_{3'} p_3 - \varepsilon_3^* \varepsilon_{3'} \; \varepsilon_4^* p_3 + \varepsilon_4^* \varepsilon_{3'} \; \varepsilon_3^* p_{3'} \right) \; a_3^\dagger a_4^\dagger a_{3'} \; . \tag{386}$$

crosscheck of $\mathcal{H}_{AA}$

$$\begin{aligned}
\mathcal{H}_{AA} &= -\int_F igf^{abc} \sum_{343'} \int [343'] \; \left[\varepsilon_3 a_{3b} e^{-ip_3 x} - \varepsilon_3^* a_{3b}^\dagger e^{ip_3 x}\right] \left[\varepsilon_4 a_{4c} e^{-ip_4 x} + \varepsilon_4^* a_{4c}^\dagger e^{ip_4 x}\right] \\
&\times \; \left[p_3 \varepsilon_{3'} a_{3'a} e^{-ip_{3'} x} + p_3 \varepsilon_{3'}^* a_{3'a}^\dagger e^{ip_{3'} x}\right] \qquad (387) \\
&= -igf^{abc} \sum_{343'} \int [343'] \, \tilde{\delta}_{c.a} \left[\varepsilon_3 a_{3b} - \varepsilon_3^* a_{3b}^\dagger\right] \left[\varepsilon_4 a_{4c} + \varepsilon_4^* a_{4c}^\dagger\right] \left[p_3 \varepsilon_{3'} a_{3'a} + p_3 \varepsilon_{3'}^* a_{3'a}^\dagger\right] \; . \qquad (388)
\end{aligned}$$

$$\begin{aligned}
X &= \left[\varepsilon_3 a_{3b} - \varepsilon_3^* a_{3b}^\dagger\right] \left[\varepsilon_4 a_{4c} + \varepsilon_4^* a_{4c}^\dagger\right] \left[p_3 \varepsilon_{3'} a_{3'a} + p_3 \varepsilon_{3'}^* a_{3'a}^\dagger\right] \qquad (389) \\
&= 112 \left[\varepsilon_3 a_{3b}\right] \left[\varepsilon_4 a_{4c}\right] \left[p_3 \varepsilon_{3'}^* a_{3'a}^\dagger\right] + 121 \left[\varepsilon_3 a_{3b}\right] \left[\varepsilon_4^* a_{4c}^\dagger\right] \left[p_3 \varepsilon_{3'} a_{3'a}\right] + 122 \left[\varepsilon_3 a_{3b}\right] \left[\varepsilon_4^* a_{4c}^\dagger\right] \left[p_3 \varepsilon_{3'}^* a_{3'a}^\dagger\right] \\
&- \; 211 \left[\varepsilon_3^* a_{3b}^\dagger\right] \left[\varepsilon_4 a_{4c}\right] \left[p_3 \varepsilon_{3'} a_{3'a}\right] - 212 \left[\varepsilon_3^* a_{3b}^\dagger\right] \left[\varepsilon_4 a_{4c}\right] \left[p_3 \varepsilon_{3'}^* a_{3'a}^\dagger\right] - 221 \left[\varepsilon_3^* a_{3b}^\dagger\right] \left[\varepsilon_4^* a_{4c}^\dagger\right] \left[p_3 \varepsilon_{3'} a_{3'a}\right] \; . \qquad (390)
\end{aligned}$$

Clean check:

$$\begin{aligned}
X &= \left[\varepsilon_3 a_{3b} - \varepsilon_3^* a_{3b}^\dagger\right] \left[\varepsilon_4 a_{4c} + \varepsilon_4^* a_{4c}^\dagger\right] \left[p_3 \varepsilon_{3'} a_{3'a} + p_3 \varepsilon_{3'}^* a_{3'a}^\dagger\right] \qquad (391) \\
&= 112 \;\; \varepsilon_3 a_{3b} \varepsilon_4 a_{4c} p_3 \varepsilon_{3'}^* a_{3'a}^\dagger + 121 \;\; \varepsilon_3 a_{3b} \varepsilon_4^* a_{4c}^\dagger p_3 \varepsilon_{3'} a_{3'a} + 122 \;\; \varepsilon_3 a_{3b} \varepsilon_4^* a_{4c}^\dagger p_3 \varepsilon_{3'}^* a_{3'a}^\dagger \\
&- \; 211 \;\; \varepsilon_3^* a_{3b}^\dagger \varepsilon_4 a_{4c} p_3 \varepsilon_{3'} a_{3'a} - 212 \;\; \varepsilon_3^* a_{3b}^\dagger \varepsilon_4 a_{4c} p_3 \varepsilon_{3'}^* a_{3'a}^\dagger - 221 \;\; \varepsilon_3^* a_{3b}^\dagger \varepsilon_4^* a_{4c}^\dagger p_3 \varepsilon_{3'} a_{3'a} \; . \qquad (392)
\end{aligned}$$

Move of operators and change of sign:

$$\begin{aligned}
-X &= -\left[\varepsilon_3 a_{3b} - \varepsilon_3^* a_{3b}^\dagger\right] \left[\varepsilon_4 a_{4c} + \varepsilon_4^* a_{4c}^\dagger\right] \left[p_3 \varepsilon_{3'} a_{3'a} + p_3 \varepsilon_{3'}^* a_{3'a}^\dagger\right] \qquad (393) \\
&= -112 \;\; \varepsilon_3 \varepsilon_4 p_3 \varepsilon_{3'}^* \;\; a_{3b} a_{4c} a_{3'a}^\dagger - 121 \;\; \varepsilon_3 \varepsilon_4^* p_3 \varepsilon_{3'} \;\; a_{3b} a_{4c}^\dagger a_{3'a} - 122 \;\; \varepsilon_3 \varepsilon_4^* p_3 \varepsilon_{3'}^* a_{3b} a_{4c}^\dagger a_{3'a}^\dagger \\
&+ \; 211 \;\; \varepsilon_3^* \varepsilon_4 p_3 \varepsilon_{3'} \;\; a_{3b}^\dagger a_{4c} a_{3'a} + 212 \;\; \varepsilon_3^* \varepsilon_4 p_3 \varepsilon_{3'}^* \;\; a_{3b}^\dagger a_{4c} a_{3'a}^\dagger + 221 \;\; \varepsilon_3^* \varepsilon_4^* p_3 \varepsilon_{3'} \;\; a_{3b}^\dagger a_{4c}^\dagger a_{3'a} \; . \qquad (394)
\end{aligned}$$

Renaming color scripts $a \to c_{3'}$, $b \to c_3$, $c \to c_4$:

$$\mathcal{H}_{AA} = g \sum_{343'} \int [343'] \, \tilde{\delta}_{c.a} \; f^{c_3 c_4 c_{3'}} (-iX_{343'}) \tag{395}$$

$$\begin{aligned}
-iX_{343'} &= -i \left[\varepsilon_3 a_{3b} - \varepsilon_3^* a_{3b}^\dagger\right] \left[\varepsilon_4 a_{4c} + \varepsilon_4^* a_{4c}^\dagger\right] \left[p_3 \varepsilon_{3'} a_{3'a} + p_3 \varepsilon_{3'}^* a_{3'a}^\dagger\right] \qquad (396) \\
&= -112 \; i \; \varepsilon_3 \varepsilon_4 \; p_3 \varepsilon_{3'}^* \;\; a_3 a_4 a_{3'}^\dagger - 121 \; i \; \varepsilon_3 \varepsilon_4^* \; p_3 \varepsilon_{3'} \;\; a_3 a_4^\dagger a_{3'} - 122 \; i \; \varepsilon_3 \varepsilon_4^* \; p_3 \varepsilon_{3'}^* a_3 a_4^\dagger a_{3'}^\dagger \\
&+ \; 211 \; i \; \varepsilon_3^* \varepsilon_4 \; p_3 \varepsilon_{3'} \;\; a_3^\dagger a_4 a_{3'} + 212 \; i \; \varepsilon_3^* \varepsilon_4 \; p_3 \varepsilon_{3'}^* \;\; a_3^\dagger a_4 a_{3'}^\dagger + 221 \; i \; \varepsilon_3^* \varepsilon_4^* \; p_3 \varepsilon_{3'} \;\; a_3^\dagger a_4^\dagger a_{3'} \; . \qquad (397)
\end{aligned}$$

All terms are always normal-ordered and the changes of order are meant to be under the sign of normal ordering. Collecting h.c.

$$\begin{aligned}
-iX_{343'} &= -i \left[\varepsilon_3 a_{3b} - \varepsilon_3^* a_{3b}^\dagger\right] \left[\varepsilon_4 a_{4c} + \varepsilon_4^* a_{4c}^\dagger\right] \left[p_3 \varepsilon_{3'} a_{3'a} + p_3 \varepsilon_{3'}^* a_{3'a}^\dagger\right] \qquad (398) \\
&= -112 \; i \; \varepsilon_3 \varepsilon_4 \; \varepsilon_{3'}^* p_3 \;\; a_3 a_4 a_{3'}^\dagger - 121 \; i \; \varepsilon_3 \varepsilon_4^* \; \varepsilon_{3'} p_3 \;\; a_3 a_4^\dagger a_{3'} - 122 \; i \; \varepsilon_3 \varepsilon_4^* \; \varepsilon_{3'}^* p_3 \;\; a_3 a_4^\dagger a_{3'}^\dagger \\
&+ \; 221 \; i \; \varepsilon_3^* \varepsilon_4^* \; \varepsilon_{3'} p_3 \;\; a_3^\dagger a_4^\dagger a_{3'} + 212 \; i \; \varepsilon_3^* \varepsilon_4 \; \varepsilon_{3'}^* p_3 \;\; a_3^\dagger a_4 a_{3'}^\dagger + 211 \; i \; \varepsilon_3^* \varepsilon_4 \; \varepsilon_{3'} p_3 \;\; a_3^\dagger a_4 a_{3'} \; . \qquad (399)
\end{aligned}$$

$$-iX_{343'} = 221 \; i \; \varepsilon_3^* \varepsilon_4^* \; \varepsilon_{3'} p_3 \;\; a_3^\dagger a_4^\dagger a_{3'} + 212 \; i \; \varepsilon_3^* \varepsilon_4 \; \varepsilon_{3'}^* p_3 \;\; a_3^\dagger a_4 a_{3'}^\dagger - 122 \; i \; \varepsilon_3 \varepsilon_4^* \; \varepsilon_{3'}^* p_3 \;\; a_3 a_4^\dagger a_{3'}^\dagger + h.c. \; . \tag{400}$$

$$\mathcal{H}_{AA} \;=\; g\sum_{343'}\int [343']\,\tilde{\delta}_{c.a}\;Z_{343'}\;,\;Z_{343'}\;=\;f^{c_3c_4c_{3'}}(-iX_{343'})\;, \tag{401}$$

$$Z_{343'} \;=\; if^{c_3c_4c_{3'}}\;\varepsilon_3^*\varepsilon_4^*\;\varepsilon_{3'}p_3\;a_3^\dagger a_4^\dagger a_{3'} + if^{c_3c_4c_{3'}}\;\varepsilon_3^*\varepsilon_4\;\varepsilon_{3'}^*p_3\;a_3^\dagger a_4 a_{3'}^\dagger - if^{c_3c_4c_{3'}}\;\varepsilon_3\varepsilon_4^*\;\varepsilon_{3'}^*p_3\;a_3 a_4^\dagger a_{3'}^\dagger + h.c.\;. \tag{402}$$

I change notation $4 \leftrightarrow 3'$ in the second term and $3 \leftrightarrow 3'$ in the third one.

$$\begin{aligned} Z_{343'} \;&=\; if^{c_3c_4c_{3'}}\;\varepsilon_3^*\varepsilon_4^*\;\varepsilon_{3'}p_3\;a_3^\dagger a_4^\dagger a_{3'} + if^{c_3c_{3'}c_4}\;\varepsilon_3^*\varepsilon_{3'}\;\varepsilon_4^*p_3\;a_3^\dagger a_{3'} a_4^\dagger - if^{c_{3'}c_4c_3}\;\varepsilon_{3'}\varepsilon_4^*\;\varepsilon_3^*p_{3'}\;a_{3'} a_4^\dagger a_3^\dagger + h.c. && (403)\\ &=\; if^{c_3c_4c_{3'}}\;\varepsilon_3^*\varepsilon_4^*\;\varepsilon_{3'}p_3\;a_3^\dagger a_4^\dagger a_{3'} - if^{c_3c_4c_{3'}}\;\varepsilon_3^*\varepsilon_{3'}\;\varepsilon_4^*p_3\;a_3^\dagger a_{3'} a_4^\dagger + if^{c_3c_4c_{3'}}\;\varepsilon_{3'}\varepsilon_4^*\;\varepsilon_3^*p_{3'}\;a_{3'} a_4^\dagger a_3^\dagger + h.c.\;. && (404)\end{aligned}$$

In the third term, I also change $4 \leftrightarrow 3$ to have ε_4^* contracted with $p_{3'}$. As a result,

$$Z_{343'} \;=\; if^{c_3c_4c_{3'}}\;\varepsilon_3^*\varepsilon_4^*\;\varepsilon_{3'}p_3\;a_3^\dagger a_4^\dagger a_{3'} - if^{c_3c_4c_{3'}}\;\varepsilon_3^*\varepsilon_{3'}\;\varepsilon_4^*p_3\;a_3^\dagger a_{3'} a_4^\dagger - if^{c_3c_4c_{3'}}\;\varepsilon_{3'}\varepsilon_3^*\;\varepsilon_4^*p_{3'}\;a_{3'} a_3^\dagger a_4^\dagger + h.c.\;. \tag{405}$$

Normal ordering, harmless because $\int_F$ reduces terms with normal-order commutators to zero modes, which are eliminated in the regularization procedure, yields:

$$\begin{aligned} Z_{343'} \;&=\; if^{c_3c_4c_{3'}}\;Y_{343'}\;a_3^\dagger a_4^\dagger a_{3'} + h.c.\;, && (406)\\ Y_{343'} \;&=\; \varepsilon_3^*\varepsilon_4^*\;\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\;\varepsilon_4^*p_3 - \varepsilon_{3'}\varepsilon_3^*\;\varepsilon_4^*p_{3'} + h.c.\;. && (407)\end{aligned}$$

Result prior to the check:

$$Y_{343'} \;=\; if^{c_3c_4c_{3'}}\left(\varepsilon_3^*\varepsilon_4^*\;\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\;\varepsilon_4^*p_3 - \varepsilon_3^*\varepsilon_{3'}\;\varepsilon_4^*p_{3'}\right)\;, \tag{408}$$

which is also

$$\begin{aligned} Y_{343'} \;&=\; if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\;\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\;\varepsilon_4^*(p_3+p_{3'})\right] && (409)\\ &=\; if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\;\varepsilon_{3'}p_3 + \delta_{\sigma_3\sigma_{3'}}\;\varepsilon_4^*(p_3+p_{3'})\right]\;, && (410)\end{aligned}$$

Above black text is verified as correct
** new version check for $p_3 + p_{3'}$

$$\varepsilon_4^*p_3 \;=\; \varepsilon_4^*(p_{3'}-p_4) \;=\; \varepsilon_4^*p_{3'}\;. \tag{411}$$

$$Y_{343'} \;=\; if^{c_3c_4c_{3'}}\left(\varepsilon_3^*\varepsilon_4^*\;\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\;\varepsilon_4^*p_3 - \varepsilon_3^*\varepsilon_{3'}\;\varepsilon_4^*p_{3'}\right)\;. \tag{412}$$

Now I use the usual relative momentum variables for components $+$ and $\perp$,

$$\begin{aligned} y_4 \;&=\; x_4/x_{3'}\;,\; y_3 \;=\; x_3/x_{3'}\;,\; k_{43} \;=\; y_3p_4 - y_4p_3\;, && (413)\\ p_4 \;&=\; y_4p_{3'} + k_{43}\;,\; p_3 \;=\; y_3p_{3'} - k_{43}\;. && (414)\end{aligned}$$

In these variables,

$$\begin{aligned} \varepsilon_{3'}p_3 \;&=\; \varepsilon_{3'}(y_3p_{3'} - k_{43}) \;=\; -\varepsilon_{3'}k_{43}\;, && (415)\\ \varepsilon_4^*p_3 \;&=\; \varepsilon_4^*p_{3'} \;=\; \frac{1}{2}\,\varepsilon_4^{*-}\,p_{3'}^+ - \varepsilon_4^{*\perp}p_{3'}^\perp \;=\; \frac{1}{2}\,2\varepsilon_4^{*\perp}p_4^\perp/y_4 - \varepsilon_4^{*\perp}p_{3'}^\perp && (416)\\ &=\; \varepsilon_4^{*\perp}(y_4p_{3'}^\perp + k_{43}^\perp)/y_4 - \varepsilon_4^{*\perp}p_{3'}^\perp \;=\; \varepsilon_4^{*\perp}k_{43}^\perp/y_4 \;=\; -\varepsilon_4^*k_{43}/y_4\;, && (417)\\ {\varepsilon_4}^*p_{3'} \;&=\; \frac{1}{2}\,2{\varepsilon_4}^{*\perp}{p_4}^\perp/y_4 - {\varepsilon_4}^{*\perp}p_{3'}^\perp \;=\; {\varepsilon_4}^{*\perp}(y_4p_{3'}^\perp + k_{43}^\perp)/y_4 - {\varepsilon_4}^{*\perp}p_{3'}^\perp && (418)\\ &=\; +{\varepsilon_4}^{*\perp}k_{43}^\perp)/y_4 \;=\; -{\varepsilon_4}^*k_{43}/y_4\;. && (419)\end{aligned}$$

$$Y_{343'} \;=\; if^{c_3c_4c_{3'}}\left(-\varepsilon_3^*\varepsilon_4^*\;\varepsilon_{3'}k_{43} + \varepsilon_3^*\varepsilon_{3'}\;\varepsilon_4^*k_{43}/y_4 + \varepsilon_3^*\varepsilon_{3'}\;{\varepsilon_4}^*k_{43}/y_4\right)\;a_3^\dagger a_4^\dagger a_{3'}\;. \tag{420}$$

Changing notation $3 \leftrightarrow 4$, one has to remember that $k_{34} = -k_{43}$.
*** old version verified as correct

$$\varepsilon_4^*p_3 \;=\; \varepsilon_4^*(p_{3'}-p_4) \;=\; \varepsilon_4^*p_{3'}\;. \tag{421}$$

$$Y_{343'} = if^{c_3c_4c_{3'}} \left(\varepsilon_3^*\varepsilon_4^* \, \varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'} \, \varepsilon_4^*p_{3'} + \varepsilon_4^*\varepsilon_{3'} \, \varepsilon_3^*p_{3'}\right) \, a_3^\dagger a_4^\dagger a_{3'} \ . \tag{422}$$

Now I use the usual relative momentum variables for components $+$ and $\perp$,

$$y_4 = x_4/x_{3'} \ , \ y_3 = x_3/x_{3'} \ , \ k_{43} = y_3 p_4 - y_4 p_3 \ , \tag{423}$$
$$p_4 = y_4 p_{3'} + k_{43} \ , \ p_3 = y_3 p_{3'} - k_{43} \ . \tag{424}$$

In these variables,

$$\varepsilon_{3'}p_3 = \varepsilon_{3'}(y_3p_{3'} - k_{43}) = -\varepsilon_{3'}k_{43} \ , \tag{425}$$
$$\varepsilon_4^*p_{3'} = \frac{1}{2}\,\varepsilon_4^{*-}p_{3'}^+ - \varepsilon_4^{*\perp}p_{3'}^\perp = \frac{1}{2}\,2\varepsilon_4^{*\perp}p_4^\perp/y_4 - \varepsilon_4^{*\perp}p_{3'}^\perp \tag{426}$$
$$= \varepsilon_4^{*\perp}(y_4p_{3'}^\perp + k_{43}^\perp)/y_4 - \varepsilon_4^{*\perp}p_{3'}^\perp = \varepsilon_4^{*\perp}k_{43}^\perp/y_4 = -\varepsilon_4^*k_{43}/y_4 \ , \tag{427}$$
$$\varepsilon_3^*p_{3'} = \frac{1}{2}\,2\varepsilon_3^{*\perp}p_3^\perp/y_3 - \varepsilon_3^{*\perp}p_{3'}^\perp = \varepsilon_3^{*\perp}(y_3p_{3'}^\perp - k_{43}^\perp)/y_3 - \varepsilon_3^{*\perp}p_{3'}^\perp \tag{428}$$
$$= -\varepsilon_3^{*\perp}k_{43}^\perp)/y_3 = \varepsilon_3^*k_{43}/y_3 \ . \tag{429}$$

$$Y_{343'} = if^{c_3c_4c_{3'}} \left(-\varepsilon_3^*\varepsilon_4^* \, \varepsilon_{3'}k_{43} + \varepsilon_3^*\varepsilon_{3'} \, \varepsilon_4^*k_{43}/y_4 + \varepsilon_4^*\varepsilon_{3'} \, \varepsilon_3^*k_{43}/y_3\right) \, a_3^\dagger a_4^\dagger a_{3'} \ . \tag{430}$$

Changing notation $3 \leftrightarrow 4$, one has to remember that $k_{34} = -k_{43}$.

****

In verified conclusion,

$$\mathcal{H}_{AA} = g\sum_{433'}\int [433'] \, a_4^\dagger a_3^\dagger a_{3'} \, Y_{433'} \, \tilde{\delta}_{43.3'} + h.c. \ , \tag{431}$$
$$Y_{433'} = if^{c_4c_3c_{3'}} \left(\varepsilon_4^*\varepsilon_3^* \, \varepsilon_{3'}k_{43} - \varepsilon_4^*\varepsilon_{3'} \, \varepsilon_3^*k_{43}/y_3 - \varepsilon_3^*\varepsilon_{3'} \, \varepsilon_4^*k_{43}/y_4\right) \ , \tag{432}$$

which matches Eqs. (45) and (46). Also, changing notation $3 \leftrightarrow 4$, I obtain

$$Y_{343'} = if^{c_3c_4c_{3'}} \left(\varepsilon_3^*\varepsilon_4^* \, \varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'} \, \varepsilon_4^*p_{3'} + \varepsilon_4^*\varepsilon_{3'} \, \varepsilon_3^*p_{3'}\right) \, a_3^\dagger a_4^\dagger a_{3'} \ , \tag{433}$$

whose equivalent is

$$Y_{433'} = if^{c_4c_3c_{3'}} \left(\varepsilon_4^*\varepsilon_3^* \, \varepsilon_{3'}p_4 - \varepsilon_4^*\varepsilon_{3'} \, \varepsilon_3^*p_{3'} + \varepsilon_3^*\varepsilon_{3'} \, \varepsilon_4^*p_{3'}\right) \, a_4^\dagger a_3^\dagger a_{3'} \ . \tag{434}$$

Using

$$\varepsilon_4^*p_{3'} = \frac{1}{2}\,\varepsilon_4^{*-}p_{3'}^+ - \varepsilon_4^{*\perp}p_{3'}^\perp = \frac{1}{2}\,2\varepsilon_4^{*\perp}p_4^\perp/y_4 - \varepsilon_4^{*\perp}p_{3'}^\perp \tag{435}$$
$$= \varepsilon_4^{*\perp}(y_4p_{3'}^\perp + k_{43}^\perp)/y_4 - \varepsilon_4^{*\perp}p_{3'}^\perp = \varepsilon_4^{*\perp}k_{43}^\perp/y_4 = -\varepsilon_4^*k_{43}/y_4 \ , \tag{436}$$
$$\varepsilon_3^*p_{3'} = \frac{1}{2}\,2\varepsilon_3^{*\perp}p_3^\perp/y_3 - \varepsilon_3^{*\perp}p_{3'}^\perp = \varepsilon_3^{*\perp}(y_3p_{3'}^\perp - k_{43}^\perp)/y_3 - \varepsilon_3^{*\perp}p_{3'}^\perp \tag{437}$$
$$= -\varepsilon_3^{*\perp}k_{43}^\perp)/y_3 = \varepsilon_3^*k_{43}/y_3 \ , \tag{438}$$

one obtains

$$Y_{433'} = if^{c_4c_3c_{3'}} \left[\varepsilon_4^*\varepsilon_3^* \, \varepsilon_{3'}k_{43} - \varepsilon_4^*\varepsilon_{3'} \, \varepsilon_3^*k_{43}/y_3 + \varepsilon_3^*\varepsilon_{3'} \, (-\varepsilon_4^*k_{43}/y_4)\right] \, a_4^\dagger a_3^\dagger a_{3'} \ , \tag{439}$$

or

$$Y_{433'} = if^{c_4c_3c_{3'}} \left(\varepsilon_4^*\varepsilon_3^* \, \varepsilon_{3'}k_{43} - \varepsilon_4^*\varepsilon_{3'} \, \varepsilon_3^*k_{43}/y_3 - \varepsilon_3^*\varepsilon_{3'} \, \varepsilon_4^*k_{43}/y_4\right) \, a_4^\dagger a_3^\dagger a_{3'} \ . \tag{440}$$

Blue text above is verified.

I check if a change in Eq. (45) that leads to $p_3^+ + p_{3'}^+$ agrees with the above blue and red results.

$$H_{A^3} = g\sum_{123}^{2}\int [123] \, a_1^\dagger a_2^\dagger a_3 \, Y_{123} \, \tilde{\delta}_{12.3} + h.c. \ , \tag{441}$$
$$Y_{123} = if^{c_1c_2c_3} \left[\varepsilon_1^*\varepsilon_2^* \, \varepsilon_3 k_{12} - \varepsilon_1^*\varepsilon_3 \, \varepsilon_2^*k_{12}/x_2 - \varepsilon_2^*\varepsilon_3 \, \varepsilon_1^*k_{12}/x_1\right] \tag{442}$$

$$H_{A^3} \;=\; g\sum_{343'}\int [343']\, a_3^\dagger a_4^\dagger a_{3'}\; Y_{343'}\,\tilde\delta_{34.3'} + h.c.\ , \tag{443}$$

$$Y_{343'} \;=\; if^{c_3c_4c_{3'}}\left[-\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}k_{43} + \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*k_{43}/y_4 + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^*k_{43}/y_3\right]\ . \qquad \text{verified} \tag{444}$$

Switch to momenta p,

$$k_{43} \;=\; (y_3p_4 - y_4p_3)\ , \tag{445}$$

$$Y_{343'} \;=\; if^{c_3c_4c_{3'}}\left[-\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}(y_3p_4-y_4p_3) + \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*(y_3p_4-y_4p_3)/y_4 + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^*(y_3p_4-y_4p_3)/y_3\right] \tag{446}$$

$$= if^{c_3c_4c_{3'}}\left[-\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}(-p_3) + \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*(-y_4p_3)/y_4 + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^*(y_3p_4)/y_3\right] \tag{447}$$

$$= if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*p_3 + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^*p_4\right] \tag{448}$$

$$= if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*p_{3'} + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^*p_4\right]\ , \tag{449}$$

matching Eqs. (433) and (422) (remember $\varepsilon_4^*p_{3'} = \varepsilon_4^*p_3$). In the sums and integrals of all terms, one can rename summed and integrated variables in each term of Y independently. So, one can relabel $3 \leftrightarrow 4$ in the 3rd term. Creation operators retain their form, they only change order which is arbitrary anyway. Factor $if^{c_3c_4c_{3'}}$ changes sign. The result is the same formula for H_{A^3} with

$$H_{A^3} \;=\; g\sum_{343'}\int [343']\, a_3^\dagger a_4^\dagger a_{3'}\; Y_{343'}\,\tilde\delta_{34.3'} + h.c.\ , \tag{450}$$

$$Y_{343'} \;=\; if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*(p_3+p_{3'})\right]\ . \tag{451}$$

When 4 corresponds to the exchanges gluon, the only divergent part of H_{A^3} when $y_4 \to 0$ is

$$H_{A^3}^{\rm div} \;=\; g\sum_{343'}\int [343']\, a_3^\dagger a_4^\dagger a_{3'}\; Y_{343'}^{\rm div}\,\tilde\delta_{34.3'} + h.c.\ , \tag{452}$$

$$Y_{343'}^{\rm div} \;=\; -if^{c_3c_4c_{3'}}\,\varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*(p_3+p_{3'})\ . \tag{453}$$

The divergence comes from ϵ_4^{*-}. Now I can check the whole calculation of the gluon-quark interaction. The full expression for $H_{AA} = H_{A^3}$ can be written in two equivalent forms.

C. Summary of the 3-transverse-gluons term

Equalities mean equivalence when all indices and momenta are fully summed and integrated, respectively.

$$\mathcal{H}_{A^3} \;=\; \mathcal{H}_{AA} \;=\; -\int_F J^{a\mu}_{Af}A^a_{f\mu}\ , \tag{454}$$

$$H_{AA} \;=\; H_{A^3} \;=\; g\sum_{343'}\int [343']\, a_3^\dagger a_4^\dagger a_{3'}\; Y_{343'}\,\tilde\delta_{34.3'} + g\sum_{3'43}\int [3'43]\, a_{3'}^\dagger a_4 a_3\; Y^*_{343'}\,\tilde\delta_{3'.43} \tag{455}$$

$$= g\sum_{343'}\int [343']\, a_3^\dagger a_4^\dagger a_{3'}\; Y_{343'}\,\tilde\delta_{34.3'} + g\sum_{343'}\int [343']\, a_3^\dagger a_4 a_{3'}\; Y^*_{3'43}\,\tilde\delta_{3.43'}\ , \tag{456}$$

$$Y_{343'} \;=\; if^{c_3c_4c_{3'}}\left[-\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}k_{43} + \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*k_{43}/y_4 + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^*k_{43}/y_3\right]\ , \quad k_{43} \;=\; (y_3p_4-y_4p_3)\ , \tag{457}$$

$$Y_{343'} \;=\; if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*p_{3'} + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^*p_4\right]\ , \tag{458}$$

$$Y_{343'} \;\equiv\; if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*(p_{3'}+p_3)\right]\ , \tag{459}$$

$$Y^*_{3'43} \;=\; if^{c_3c_4c_{3'}}\left[\varepsilon_{3'}\varepsilon_4\,\varepsilon_3^*p_{3'} - \varepsilon_{3'}\varepsilon_3^*\,\varepsilon_4p_3 - \varepsilon_{3'}\varepsilon_3^*\,\varepsilon_4p_{3'}\right] \tag{460}$$

$$\equiv if^{c_3c_4c_{3'}}\left[\varepsilon_{3'}\varepsilon_4\,\varepsilon_3^*p_{3'} - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4(p_3+p_{3'})\right]\ . \tag{461}$$

The identities above include the change $3 \leftrightarrow 4$ in the last terms of Y, and from $\epsilon_4p_4 = 0$ after adding p_4 to p_3 in the product with ε_4. One can change notation under the sign of full sum and integral in each term separately.

$$Y^*_{3'43} \;=\; \left\{if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*p_{3'} + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^*p_4\right]\quad 3 \leftrightarrow 3'\right\}^* \tag{462}$$

$$= if^{c_3c_4c_{3'}}\left[\varepsilon_{3'}\varepsilon_4\,\varepsilon_3^*p_{3'} - \varepsilon_{3'}\varepsilon_3^*\,\varepsilon_4p_3 + \varepsilon_4\varepsilon_3^*\,\varepsilon_{3'}p_4\right] \tag{463}$$

Changing notation $4 \leftrightarrow 3'$ under signs of full sums and integrals in the last term alone, one gets

$$Y^*_{3'43} \;=\; if^{c_3c_4c_{3'}}\left[\varepsilon_{3'}\varepsilon_4\,\varepsilon_3^*p_{3'} - \varepsilon_{3'}\varepsilon_3^*\,\varepsilon_4p_3 - \varepsilon_{3'}\varepsilon_3^*\,\varepsilon_4p_{3'}\right] \tag{464}$$

$$= if^{c_3c_4c_{3'}}\left[\varepsilon_{3'}\varepsilon_4\,\varepsilon_3^*p_{3'} - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4(p_3+p_{3'})\right]\ . \tag{465}$$

$$\mathbf{There\,was\,the\,error:\;confusion\,of\,p\,with\,k} \tag{466}$$

$$\begin{aligned} Y_{433'} &= if^{c_4c_3c_{3'}}\left(\varepsilon_4^*\varepsilon_3^*\ \varepsilon_{3'}k_{43} - \varepsilon_4^*\varepsilon_{3'}\ \varepsilon_3^*k_{43}/y_3 - \varepsilon_3^*\varepsilon_{3'}\ \varepsilon_4^*k_{43}/y_4\right) && (467)\\ &= if^{c_4c_3c_{3'}}\left(\varepsilon_4^*\varepsilon_3^*\ \varepsilon_{3'}p_4 - \varepsilon_4^*\varepsilon_{3'}\ \varepsilon_3^*p_{3'} + \varepsilon_3^*\varepsilon_{3'}\ \varepsilon_4^*p_{3'}\right)\ a_4^\dagger a_3^\dagger a_{3'}\ . && (468) \end{aligned}$$

End of the summary of the 3-transverse-gluons term

Evaluation of $\mathcal{H}_{A\phi}$

$$\begin{aligned} \mathcal{H}_{A\phi} &= -\int_F J_{Af}^{a\mu}\frac{m_g\eta_\mu}{\partial^+}\phi^a\ , && (469)\\ J_{Af}^{a\mu} &= \sum_{343'}\int [343']\ igf^{ac_3c_{3'}}p_3^\mu\left[\varepsilon_3a_3e^{-ip_3x} - \varepsilon_3^*a_3^\dagger e^{ip_3x}\right]\left[\varepsilon_{3'}a_{3'}e^{-ip_{3'}x} + \varepsilon_{3'}^*a_{3'}^\dagger e^{ip_{3'}x}\right]\ , && (470)\\ \frac{m_g\eta_\mu}{\partial^+}\phi^a &= \int [4]\frac{m_g\eta_\mu}{p_4^+}\left(\tilde a_{4a}e^{-ip_4x} + \tilde a_{4a}^\dagger e^{ip_4x}\right)\ , && (471)\\ \mathcal{H}_{A\phi} &= -\int_F\sum_{343'}\int [343']\ igf^{c_4c_3c_{3'}}p_3^\mu\left[\varepsilon_3a_3e^{-ip_3x} - \varepsilon_3^*a_3^\dagger e^{ip_3x}\right]\left[\varepsilon_{3'}a_{3'}e^{-ip_{3'}x} + \varepsilon_{3'}^*a_{3'}^\dagger e^{ip_{3'}x}\right]\\ &\times\ \frac{m_g\eta_\mu}{p_4^+}\left(\tilde a_4e^{-ip_4x} + \tilde a_4^\dagger e^{ip_4x}\right) && (472) \end{aligned}$$

Calculation.

$$\mathcal{H}_{A\phi} = -\sum_{343'}\int [343']\,\tilde\delta_{c.a}\ igf^{c_4c_3c_{3'}}\frac{m_gp_3^+}{p_4^+}\ \left[\varepsilon_3a_3 - \varepsilon_3^*a_3^\dagger\right]\left[\varepsilon_{3'}a_{3'} + \varepsilon_{3'}^*a_{3'}^\dagger\right]\left(\tilde a_4 + \tilde a_4^\dagger\right)\ . \tag{473}$$

Again, there are 8 - 2 = 6 terms.

$$\begin{aligned} &\left[\varepsilon_3a_3 - \varepsilon_3^*a_3^\dagger\right]\left[\varepsilon_{3'}a_{3'} + \varepsilon_{3'}^*a_{3'}^\dagger\right]\left(\tilde a_4 + \tilde a_4^\dagger\right) && (474)\\ =&\ 112\left[\varepsilon_3a_3 - \varepsilon_3^*a_3^\dagger\right]\left[\varepsilon_{3'}a_{3'} + \varepsilon_{3'}^*a_{3'}^\dagger\right]\left(\tilde a_4 + \tilde a_4^\dagger\right)\\ +&\ 121\left[\varepsilon_3a_3 - \varepsilon_3^*a_3^\dagger\right]\left[\varepsilon_{3'}a_{3'} + \varepsilon_{3'}^*a_{3'}^\dagger\right]\left(\tilde a_4 + \tilde a_4^\dagger\right)\\ +&\ 122\left[\varepsilon_3a_3 - \varepsilon_3^*a_3^\dagger\right]\left[\varepsilon_{3'}a_{3'} + \varepsilon_{3'}^*a_{3'}^\dagger\right]\left(\tilde a_4 + \tilde a_4^\dagger\right)\\ +&\ 211\left[\varepsilon_3a_3 - \varepsilon_3^*a_3^\dagger\right]\left[\varepsilon_{3'}a_{3'} + \varepsilon_{3'}^*a_{3'}^\dagger\right]\left(\tilde a_4 + \tilde a_4^\dagger\right)\\ +&\ 212\left[\varepsilon_3a_3 - \varepsilon_3^*a_3^\dagger\right]\left[\varepsilon_{3'}a_{3'} + \varepsilon_{3'}^*a_{3'}^\dagger\right]\left(\tilde a_4 + \tilde a_4^\dagger\right)\\ +&\ 221\left[\varepsilon_3a_3 - \varepsilon_3^*a_3^\dagger\right]\left[\varepsilon_{3'}a_{3'} + \varepsilon_{3'}^*a_{3'}^\dagger\right]\left(\tilde a_4 + \tilde a_4^\dagger\right]\ . && (475) \end{aligned}$$

$$\begin{aligned} &112\ \varepsilon_3\varepsilon_{3'}\ a_3a_{3'}\tilde a_4^\dagger + 121\ \varepsilon_3\varepsilon_{3'}^*\ a_3a_{3'}^\dagger\tilde a_4 - 211\ \varepsilon_3^*\varepsilon_{3'}\ a_3^\dagger a_{3'}\tilde a_4\\ -&\ 221\ \varepsilon_3^*\varepsilon_{3'}^*\ a_3^\dagger a_{3'}^\dagger\tilde a_4 - 212\ \varepsilon_3^*\varepsilon_{3'}\ a_3^\dagger a_{3'}\tilde a_4^\dagger + 122\ \varepsilon_3\varepsilon_{3'}^*\ a_3a_{3'}^\dagger\tilde a_4^\dagger && (476) \end{aligned}$$

$$\begin{aligned} X &= \left[\varepsilon_3a_3 - \varepsilon_3^*a_3^\dagger\right]\left[\varepsilon_{3'}a_{3'} + \varepsilon_{3'}^*a_{3'}^\dagger\right]\left(\tilde a_4 + \tilde a_4^\dagger\right) && (477)\\ &= -221\ \varepsilon_3^*\varepsilon_{3'}^*\ a_3^\dagger a_{3'}^\dagger\tilde a_4 - 212\ \varepsilon_3^*\varepsilon_{3'}\ a_3^\dagger a_{3'}\tilde a_4^\dagger + 122\ \varepsilon_3\varepsilon_{3'}^*\ a_3a_{3'}^\dagger\tilde a_4^\dagger - h.c.\ . && (478) \end{aligned}$$

Normal ordering to $\tilde{a}_4^\dagger a_3^\dagger a_{3'}$.

$$\mathcal{H}_{A\phi} \;=\; g m_g \sum_{433'} \int [433']\, \tilde{\delta}_{c.a}\; Z \;,\;\; Z \;=\; f^{c_4 c_3 c_{3'}} \frac{p_3^+}{p_4^+}\; (-iX) \;+\; h.c. \;, \tag{479}$$

$$Z \;=\; f^{c_4 c_3 c_{3'}} \frac{p_3^+}{p_4^+}\; (-iX) \tag{480}$$

$$\;=\; 221 f^{c_4 c_3 c_{3'}} \frac{p_3^+}{p_4^+}\; i\, \varepsilon_3^* \varepsilon_{3'}^*\; a_3^\dagger a_{3'}^\dagger \tilde{a}_4 + 212 f^{c_4 c_3 c_{3'}} \frac{p_3^+}{p_4^+}\; i\, \varepsilon_3^* \varepsilon_{3'}\; \tilde{a}_4^\dagger a_3^\dagger a_{3'} - 122 f^{c_4 c_3 c_{3'}} \frac{p_3^+}{p_4^+}\; i\, \varepsilon_3 \varepsilon_{3'}^*\; \tilde{a}_4^\dagger a_{3'}^\dagger a_3 \;. \tag{481}$$

Here, I am interested in terms that do not change the number of transverse gluons,

$$Z \;\to\; f^{c_4 c_3 c_{3'}} \frac{p_3^+}{p_4^+}\; (-iX) \tag{482}$$

$$\;=\; 212 f^{c_4 c_3 c_{3'}} \frac{p_3^+}{p_4^+}\; i\, \varepsilon_3^* \varepsilon_{3'}\; \tilde{a}_4^\dagger a_3^\dagger a_{3'} - 122 f^{c_4 c_3 c_{3'}} \frac{p_3^+}{p_4^+}\; i\, \varepsilon_3 \varepsilon_{3'}^*\; \tilde{a}_4^\dagger a_{3'}^\dagger a_3 \;. \tag{483}$$

I change notation in the second term, $3 \leftrightarrow 3'$.

$$Z \;\to\; f^{c_4 c_3 c_{3'}} \frac{p_3^+}{p_4^+}\; (-iX) \tag{484}$$

$$\;=\; f^{c_4 c_3 c_{3'}} \frac{p_3^+}{p_4^+}\; i\, \varepsilon_3^* \varepsilon_{3'}\; \tilde{a}_4^\dagger a_3^\dagger a_{3'} - f^{c_4 c_{3'} c_3} \frac{p_{3'}^+}{p_4^+}\; i\, \varepsilon_{3'} \varepsilon_3^*\; \tilde{a}_4^\dagger a_3^\dagger a_{3'} \tag{485}$$

$$\;=\; f^{c_4 c_3 c_{3'}} \frac{p_3^+}{p_4^+}\; i\, \varepsilon_3^* \varepsilon_{3'}\; \tilde{a}_4^\dagger a_3^\dagger a_{3'} + f^{c_4 c_3 c_{3'}} \frac{p_{3'}^+}{p_4^+}\; i\, \varepsilon_3^* \varepsilon_{3'} \varepsilon_3^*\; \tilde{a}_4^\dagger a_3^\dagger a_{3'} \tag{486}$$

$$\;=\; f^{c_4 c_3 c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_4^+}\; i\, \varepsilon_3^* \varepsilon_{3'}\; \tilde{a}_4^\dagger a_3^\dagger a_{3'} \;. \tag{487}$$

The final result reads

$$\mathcal{H}_{A\phi} \;=\; g m_g \sum_{433'} \int [433']\, \tilde{\delta}_{c.a}\; i\, f^{c_4 c_3 c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_4^+}\; \varepsilon_3^* \varepsilon_{3'}\; \tilde{a}_4^\dagger a_3^\dagger a_{3'} \;+\; h.c. \;, \tag{488}$$

D. Summary of terms for gluon-quark interaction in the 3-body space

Integration symbols include sums.

$$\mathcal{H}_0^{(1)} \equiv -\int_F \left(J^{a\mu}_{\psi f} + J^{a\mu}_{Af}\right)\left(A^a_{f\mu} + \frac{m_g\eta_\mu}{\partial^+}\phi^a\right) = \mathcal{H}_{\psi A} + \mathcal{H}_{\psi\phi} + \mathcal{H}_{AA} + \mathcal{H}_{A\phi} \ , \tag{489}$$

$$\begin{aligned}\mathcal{H}_{\psi A} = -\int_F J^{a\mu}_{\psi f} A^a_{f\mu} &= g\int [11'4]\,\tilde{\delta}_{c.a}\, f^r_{\underline{1},\underline{1}4}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\left(\varepsilon_4^\nu\ b_1^\dagger b_{1'} a_4 + \varepsilon_4^{\nu *} a_4^\dagger b_1^\dagger b_{1'}\right) \\ &- g\int [22'4]\,\tilde{\delta}_{c.a}\, f^r_{\underline{2},\underline{2}4}\ \chi_{2'}^\dagger T^4 \chi_2\ \bar{v}_{2'}\gamma_\nu v_2\ \left(\varepsilon_4^\nu\ d_2^\dagger d_{2'} a_4 + \varepsilon_4^{\nu *}\ a_4^\dagger d_2^\dagger d_{2'}\right) \ ,\end{aligned} \tag{490}$$

$$\begin{aligned}\mathcal{H}_{\psi\phi} = -\int_F J^{a\mu}_{\psi f}\frac{m_g\eta_\mu}{\partial^+}\phi^a &= g\int [11'\tilde{4}]\,\tilde{\delta}_{c.a}\, f^r_{\underline{1},\underline{1}\tilde{4}}\ \chi_1^\dagger T^{\tilde{4}} \chi_{1'}\ \bar{u}_1\gamma^+ u_{1'}\ \frac{m_g}{p_{\tilde{4}}^+}\left(b_1^\dagger b_{1'} a_{\tilde{4}} + a_{\tilde{4}}^\dagger b_1^\dagger b_{1'}\right) \\ &- g\int [22'\tilde{4}]\,\tilde{\delta}_{c.a}\, f^r_{\underline{2},\underline{2}\tilde{4}}\ \chi_{2'}^\dagger T^{\tilde{4}} \chi_2\ \bar{v}_{2'}\gamma^+ v_2\ \frac{m_g}{p_{\tilde{4}}^+}\left(d_2^\dagger d_{2'} a_{\tilde{4}} + a_{\tilde{4}}^\dagger d_2^\dagger d_{2'}\right) \ ,\end{aligned} \tag{491}$$

$$\mathcal{H}_{A^3} = \mathcal{H}_{AA} = -\int_F J^{a\mu}_{Af} A^a_{f\mu} \ , \tag{492}$$

$$H_{AA} = H_{A^3} = g\sum_{343'}\int [343']\, a_3^\dagger a_4^\dagger a_{3'}\ Y_{343'}\,\tilde{\delta}_{34.3'} + g\sum_{3'43}\int [3'43]\, a_{3'}^\dagger a_4 a_3\ Y^*_{343'}\,\tilde{\delta}_{3'.43} \tag{493}$$

$$= g\sum_{343'}\int [343']\, a_3^\dagger a_4^\dagger a_{3'}\ Y_{343'}\,\tilde{\delta}_{34.3'} + g\sum_{343'}\int [343']\, a_3^\dagger a_4 a_{3'}\ Y^*_{3'43}\,\tilde{\delta}_{3.43'} \ , \tag{494}$$

$$Y_{343'} = if^{c_3c_4c_{3'}}\left[-\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}k_{43} + \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^* k_{43}/y_4 + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^* k_{43}/y_3\right] \ , \ k_{43} = (y_3p_4 - y_4p_3) \ , \tag{495}$$

$$Y_{343'} = if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^* p_{3'} + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^* p_4\right] \ , \tag{496}$$

$$Y_{343'} \equiv if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^* (p_{3'} + p_3)\right] \ , \tag{497}$$

$$Y^*_{3'43} = if^{c_3c_4c_{3'}}\left[\varepsilon_{3'}\varepsilon_4\,\varepsilon_3^* p_{3'} - \varepsilon_{3'}\varepsilon_3^*\,\varepsilon_4 p_3 - \varepsilon_{3'}\varepsilon_3^*\,\varepsilon_4 p_{3'}\right] \tag{498}$$

$$\equiv if^{c_3c_4c_{3'}}\left[\varepsilon_{3'}\varepsilon_4\,\varepsilon_3^* p_{3'} - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4 (p_3 + p_{3'})\right] \ , \tag{499}$$

$$\mathcal{H}_{A\phi} = -\int_F J^{a\mu}_{Af}\frac{m_g\eta_\mu}{\partial^+}\phi^a = gm_g\sum_{433'}\int [433']\,\tilde{\delta}_{c.a}\ if^{c_4c_3c_{3'}}\frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'}\ \tilde{a}_4^\dagger a_3^\dagger a_{3'}\ +\ h.c.\,,\ \text{cons.}\ \#\perp . \tag{500}$$

For gluon-quark interaction in the 3-body space, without gluon-antiquark interaction,

$$\mathcal{H}_0^{(1)} = \mathcal{H}_{\psi A} + \mathcal{H}_{\psi\phi} + \mathcal{H}_{AA} + \mathcal{H}_{A\phi} \ , \tag{501}$$

$$\mathcal{H}_{\psi A} = g\int [11'4]\,\tilde{\delta}_{c.a}\, f^r_{\underline{1},\underline{1}4}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\left(\varepsilon_4^\nu\ b_1^\dagger b_{1'} a_4 + \varepsilon_4^{\nu *} a_4^\dagger b_1^\dagger b_{1'}\right) \ , \tag{502}$$

$$\mathcal{H}_{\psi\phi} = g\int [11'\tilde{4}]\,\tilde{\delta}_{c.a}\, f^r_{\underline{1},\underline{1}\tilde{4}}\ \chi_1^\dagger T^{\tilde{4}} \chi_{1'}\ \bar{u}_1\gamma^+ u_{1'}\ \frac{m_g}{p_{\tilde{4}}^+}\left(b_1^\dagger b_{1'} a_{\tilde{4}} + a_{\tilde{4}}^\dagger b_1^\dagger b_{1'}\right) \ , \tag{503}$$

$$H_{AA} = g\sum_{343'}\int [343']\, a_3^\dagger a_4^\dagger a_{3'}\ Y_{343'}\,\tilde{\delta}_{34.3'} + g\sum_{343'}\int [343']\, a_3^\dagger a_4 a_{3'}\ Y^*_{3'43}\,\tilde{\delta}_{3.43'} \ , \tag{504}$$

$$Y_{343'} = if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^* p_{3'} + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^* p_4\right] \ , \tag{505}$$

$$\equiv if^{c_3c_4c_{3'}}\left[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}p_3 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^* (p_{3'} + p_3)\right] \ , \tag{506}$$

$$Y^*_{3'43} = if^{c_3c_4c_{3'}}\left[\varepsilon_{3'}\varepsilon_4\,\varepsilon_3^* p_{3'} - \varepsilon_{3'}\varepsilon_3^*\,\varepsilon_4 p_3 - \varepsilon_{3'}\varepsilon_3^*\,\varepsilon_4 p_{3'}\right] \tag{507}$$

$$\equiv if^{c_3c_4c_{3'}}\left[\varepsilon_{3'}\varepsilon_4\,\varepsilon_3^* p_{3'} - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4 (p_3 + p_{3'})\right] \ , \tag{508}$$

$$\mathcal{H}_{A\phi} = gm_g\int [433']\,\tilde{\delta}_{c.a}\ if^{c_4c_3c_{3'}}\frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'}\ \tilde{a}_4^\dagger a_3^\dagger a_{3'}\ +\ h.c.\ . \tag{509}$$

E. Exchange with $\theta_{1-1'}$ using file "FirstLine20250927"

In conclusion from "FirstLine20250927":

$$
\begin{aligned}
L1 &= g^2 \sum_4 \int [11'33']\, \theta_{1-1'} \frac{\tilde{\delta}_{13.1'3'}}{p_4} \chi_1^\dagger T^4 \chi_{1'}\, \bar{u}_1 \gamma_\nu u_{1'}\; i f^{c_3 c_4 c_{3'}}\; Y_{131'3'}\; b_1^\dagger b_{1'}\, a_3^\dagger a_{3'}\ , & (510)\\
Y_{131'3'} &= 2\varepsilon_4^\nu \left[\varepsilon_3^* \varepsilon_4^*\, \varepsilon_{3'} p_3 + \varepsilon_4^* \varepsilon_{3'}\, \varepsilon_3^* p_4 - \varepsilon_3^* \varepsilon_{3'}\, \varepsilon_4^* p_{3'}\right]\ . & (511)
\end{aligned}
$$

$L1$ is the expression obtained for

$$
\begin{aligned}
L1 &= g^2 \int [11'4]\, \tilde{\delta}_{1.1'4}\; \chi_1^\dagger T^4 \chi_{1'}\; \bar{u}_1 \gamma_\nu u_{1'}\; \varepsilon_4^\nu\; b_1^\dagger b_{1'} a_4 \quad \int [34'3']\, a_3^\dagger a_{4'}^\dagger a_{3'}\; Y_{34'3'}\, \tilde{\delta}_{34'.3'} & (512)\\
&= g^2 \int [11'4]\, \tilde{\delta}_{1.1'4} \theta_{1-1'}\; \chi_1^\dagger T^4 \chi_{1'}\; \bar{u}_1 \gamma_\nu u_{1'}\; \varepsilon_4^\nu \\
&\times \int [34'3']\, \tilde{\delta}_{34'.3'} \theta_{3'-3}\; i f^{c_3 c_{4'} c_{3'}} \left[\varepsilon_3^* \varepsilon_{4'}^*\, \varepsilon_{3'} p_3 - \varepsilon_3^* \varepsilon_{3'}\, \varepsilon_{4'}^* p_{3'} + \varepsilon_{4'}^* \varepsilon_{3'}\, \varepsilon_3^* p_{4'}\right]\; b_1^\dagger b_{1'} a_4\; a_3^\dagger a_{4'}^\dagger a_{3'}\ . & (513)
\end{aligned}
$$

Exchange of ϕ with $\theta_{1-1'}$ from Eq. (566)

$$
L3 = g^2 \int [11'4]\, \tilde{\delta}_{1.1'4}\; \chi_1^\dagger T^4 \chi_{1'}\; \bar{u}_1 \gamma^+ u_{1'}\; \frac{m_g}{p_4^+}\; b_1^\dagger b_{1'} \tilde{a}_4\;\; m_g \int [4'33']\, \tilde{\delta}_{4'3.3'}\; i f^{c_{4'} c_3 c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_{4'}^+}\; \varepsilon_3^* \varepsilon_{3'}\; \tilde{a}_{4'}^\dagger a_3^\dagger a_{3'}\ . \quad (514)
$$

Commuting and integration over 4.

$$
\begin{aligned}
L3 &= g^2 \int [11'433']\, \tilde{\delta}_{1.1'4} \theta_{1-1'}\; \chi_1^\dagger T^4 \chi_{1'}\; \bar{u}_1 \gamma_\nu u_{1'}\; \frac{m_g \eta^\nu}{p_4^+}\; b_1^\dagger b_{1'}\;\; m_g\, \tilde{\delta}_{43.3'}\; i f^{c_4 c_3 c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_4^+}\; \varepsilon_3^* \varepsilon_{3'}\; a_3^\dagger a_{3'} & (515)\\
&= g^2 \int [11'33']\, \theta_{1-1'} \frac{\tilde{\delta}_{13.1'3'}}{p_4^+}\; \chi_1^\dagger T^4 \chi_{1'}\; \bar{u}_1 \gamma_\nu u_{1'}\; \frac{m_g^2 \eta^\nu}{p_4^{+2}}\;\; i f^{c_4 c_3 c_{3'}} (p_3^+ + p_{3'}^+)\; \varepsilon_3^* \varepsilon_{3'}\; b_1^\dagger b_{1'}\, a_3^\dagger a_{3'}\ . & (516)
\end{aligned}
$$

Exchange of A and ϕ with $\theta_{1-1'}$ together with the seagull

The exchanges of A and ϕ work with Δ_{LIR} of Eq. (607),

$$
\frac{1}{p_4^+} \Delta_{LIR} = \frac{1}{2p_4^+} \left(\frac{1}{p_3^- + p_1^- - P_{\rm three}^-} + \frac{1}{p_{3'}^- + p_{1'}^- - P_{\rm three}^-} \right) = -\,\frac{1}{2} \left(\frac{1}{\rho_1} + \frac{1}{\rho_3} \right)\ . \quad (517)
$$

The exchanged gluon p_4^- in $P_{\rm three}^-$,

$$
p_4^- = (p_4^{\perp 2} + m_g^2)/p_4^+\ , \quad (518)
$$

is the largest denominator term when p_4^+ tends to 0. This implies

$$
\frac{1}{p_4^+} \Delta_{LIR} \to \frac{-1}{p_4^{\perp 2} + m_g^2}\ . \quad (519)
$$

From Eqs. (520), (516), and (342), Start here 20250928 20:29 nie San Dimas

$$
\begin{aligned}
L1 &= g^2 \sum_4 \int [11'33']\, \theta_{1-1'} \frac{\tilde{\delta}_{13.1'3'}}{p_4} \chi_1^\dagger T^4 \chi_{1'}\, \bar{u}_1 \gamma_\nu u_{1'}\; i f^{c_3 c_4 c_{3'}}\; Y_{131'3'}\; b_1^\dagger b_{1'}\, a_3^\dagger a_{3'}\ , & (520)\\
Y_{131'3'} &= 2\varepsilon_4^\nu \left[\varepsilon_3^* \varepsilon_4^*\, \varepsilon_{3'} p_3 + \varepsilon_4^* \varepsilon_{3'}\, \varepsilon_3^* p_4 - \varepsilon_3^* \varepsilon_{3'}\, \varepsilon_4^* p_{3'}\right]\ . & (521)
\end{aligned}
$$

$$
L3 = g^2 \int [11'33']\, \theta_{1-1'} \frac{\tilde{\delta}_{13.1'3'}}{p_4^+}\; \chi_1^\dagger T^4 \chi_{1'}\; \bar{u}_1 \gamma_\nu u_{1'}\; \frac{m_g^2 \eta^\nu}{p_4^{+2}}\;\; i f^{c_4 c_3 c_{3'}} (p_3^+ + p_{3'}^+)\; \varepsilon_3^* \varepsilon_{3'}\; b_1^\dagger b_{1'}\, a_3^\dagger a_{3'}\ , \quad (522)
$$

$$
H_{sqg} = -g^2 \int [33'11']\; \tilde{\delta}_{c.a}\; \delta_{\rm spins}\; \frac{(p_3^+ + p_{3'}^+)\; \bar{u}_1 \gamma_\nu u_{1'} \eta^\nu}{p_4^{+2}}\; \chi_1^\dagger T^{c_4} \chi_{1'}\, i f^{c_3 c_{3'} c_4}\; a_3^\dagger b_1^\dagger b_{1'} a_{3'}\ . \quad (523)
$$

Regularization factors $f^{r}_{\overline{1},\underline{1}(\overline{1}-\underline{1})}$ $f^{r}_{\overline{3},\underline{3}(\overline{3}-\underline{3})}$ are omitted. Only the part with $\theta_{1-1'}$ is taken from the seagull. Collection of common factors yields

$$CF \;=\; g^2 \sum_4 \int [11'33']\, \theta_{1-1'}\, \frac{\tilde{\delta}_{13.1'3'}}{p_4}\, \chi_1^\dagger T^4 \chi_{1'}\, \bar{u}_1\gamma_\nu u_{1'}\; if^{c_3c_4c_{3'}}\; a_3^\dagger b_1^\dagger b_{1'} a_{3'} \;. \tag{524}$$

For the 3-body Hamiltonian, Eqs. (112) or (351) imply

$$\mathcal{H}^{(2)} \;=\; f_{LR}\,\mathcal{H}_0^{(2)} + (f_{LR} - f_{LI}f_{IR})\,\Delta_{LIR}\,\mathcal{H}_0^{(1)}\,\mathcal{H}_0^{(1)} \;. \tag{525}$$

Only the terms with f_{LR} involve the gluon severe small-x divergences. These divergences appear in the same way with $\theta_{1-1'}$ and with $\theta_{1'-1}$. For $\theta_{1-1'}$, I have the formula

$$\mathcal{H}^{(2)} \;=\; f_{LR} CF \left[\frac{1}{p_4^+}\Delta_{LIR}(L_1+L_3) + L_s\right] \;. \tag{526}$$

where L_1, L_3, and L_s denote the remaining factors of the lines 1, 2, and the seagull, respectively.

$$L_1 \;=\; 2\sum_4 \varepsilon_4^\nu\,[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'}p_3 + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^*p_4 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^*p_{3'}] \;, \tag{527}$$

$$L_3 \;=\; \frac{m_g^2\eta^\nu}{p_4^{+2}}\;(-1)_{\rm color}(p_3^+ + p_{3'}^+)\;\varepsilon_3^*\varepsilon_{3'} \;, \tag{528}$$

$$L_s \;=\; -[\delta_{\rm spins} = -\varepsilon_3^*\varepsilon_{3'}]\;\frac{p_3^+ + p_{3'}^+}{p_4^{+2}}\;\eta^\nu\;(-1)_{\rm color} \;. \tag{529}$$

I need to see if the gluon severe small-x singularity $1/p_4^{+2}$ cancels out. The only part of L_1 that counts is the third one,

$$L_1^{\rm div} \;=\; -2\varepsilon_3^*\varepsilon_{3'}\sum_4 \varepsilon_4^\nu\,\varepsilon_4^*p_{3'} \tag{530}$$

$$\equiv\; -\varepsilon_3^*\varepsilon_{3'}\sum_4 \varepsilon_4^\nu\,\varepsilon_4^{*\mu}(p_{3'}+p_3)_\mu \;. \tag{531}$$

The equivalence is a consequence of $\varepsilon_4^*p_{3'} = \varepsilon_4^*p_3$. The divergence originates solely from the minus component of ε_4. I write

$$\varepsilon_4 \;=\; \frac{1}{2}\,\eta\varepsilon_4^- + \varepsilon_4^\perp \;. \tag{532}$$

The transverse part does not involve small-p_4^+. The singularity is contained in $\varepsilon_4^- = 2\varepsilon_4^\perp p_4^\perp/p_4^+$. Therefore, the singular part of L_1 has the form

$$L_1^{\rm div} \;=\; -\varepsilon_3^*\varepsilon_{3'}\sum_4 (p_4^\perp\varepsilon_4^\perp/p_4^+)(\varepsilon_4^{*\perp}p_4^\perp/p_4^+)\eta^\nu\eta^\mu(p_{3'}+p_3)_\mu \tag{533}$$

$$=\; -\varepsilon_3^*\varepsilon_{3'}\;\frac{p_4^{\perp 2}}{p_4^{+2}}\,\eta^\nu\,(p_{3'}^+ + p_3^+) \;. \tag{534}$$

All terms with singularity $1/p_4^{+2}$ thus are

$$\mathcal{H}^{(2)}_{\rm div} \;=\; f_{LR} CF \left\{\frac{-1}{p_4^{\perp 2} + m_g^2}\left[\left(p_4^{\perp 2}\right)_{L_1} + \left(m_g^2\right)_{L_3}\right] + (1)_{L_s}\right\}(-\varepsilon_3^*\varepsilon_{3'})\,\eta^\nu\,\frac{p_{3'}^+ + p_3^+}{p_4^{+2}} \;. \tag{535}$$

The curly bracket amounts to 0, which ends my struggle with massive gluons. **20250929 00:12 pon San Dimas**

X. GLUON-QUARK INTERACTION DEPENDENCE ON $m_g \to 0$ IN 3-BODY SPACE 20251002 10:12

The Hamiltonian terms to analyze are

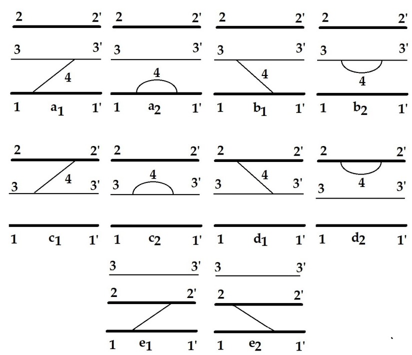
$${}_s\langle 123|H_s|1'2'3'\rangle_s \;=\; \langle 123|\left\{\mathcal{H}_f + f_{LR}\,\mathcal{H}_r - f_{LI}f_{IR}\,\Delta_{LIR}\,\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}\right\}|1'2'3'\rangle \;, \tag{536}$$

$$\mathcal{H}_r \;=\; \mathcal{H}_0^{(2)} + \Delta_{LIR}\;\mathcal{H}_0^{(1)}\,\mathcal{H}_0^{(1)} \;. \tag{537}$$

The terms that matter come only from $-f_{LI} f_{IR}\,\Delta_{LIR}\,\mathcal{H}_0^{(1)}\mathcal{H}_0^{(1)}$. They are finite for finite m_g. I consider gluon-quark interaction and quark and gluon self-interactions. The self interactions are the same as for free particles. The interactions include the RGPEP factors f^t.

I consider two pairs of interaction terms. One pair is the interaction of quark 1 with gluon 3 involving $\theta_{1-1'}$ and quark 1 self-interaction. The other pair is the interaction of quark 1 with gluon 3 involving $\theta_{1'-1}$ and the gluon self-interaction. Color factors for these interaction terms are found in Eqs. (538) to (542).

FIG. 5: Interactions that depend logarithmically on m_g in the 3-body space. In this figure, all vertices include factors f^t making all expressions finite but dependent on s, quark masses, m_1 and m_2, and the gluon mass m_g. The question is what expressions one obtains in the limit $m_g \to 0$.



A. Gluon-quark interaction in the 3-body space, new = 20251011 12:06 sob Ekologiczna

Color factors follow from the wave function

$$C^{123}_{\rm 3body} \;=\; \chi_1^\dagger T^3 \chi_2 \ . \tag{538}$$

- **-3/2** The quark-gluon seagull H_{sqg}, Eq. (342),

$$\chi_1^\dagger T^4 \chi_{1'} i f^{343'}\ \chi_{1'}^\dagger T^{3'}\chi_2 \;=\; i f^{343'}\chi_1^\dagger T^4 T^{3'}\chi_2 \;=\; \frac{i}{2} f^{343'}\chi_1^\dagger [T^4, T^{3'}]\chi_2 \;=\; \frac{i}{2} f^{343'} i f^{43'a}\ \chi_1^\dagger T^a \chi_2 \tag{539}$$

$$= -\frac{1}{2} f^{343'} f^{43'a}\ \chi_1^\dagger T^a \chi_2 \;=\; -\frac{1}{2} N_c\ \chi_1^\dagger T^3 \chi_2 \;=\; -\frac{3}{2}\ \chi_1^\dagger T^3 \chi_2 \;=\; -\frac{3}{2}\ C^{123}_{\rm 3body} \ . \tag{540}$$

- **-3/2** The antiquark-gluon seagull $H_{s\bar q g}$, Eq. (343)

$$\chi_{2'}^\dagger T^4 \chi_2 (-i f^{343'})\ \chi_1^\dagger T^{3'}\chi_{2'} \;=\; -i f^{343'}\ \chi_{2'}^\dagger T^4 \chi_2\ \chi_1^\dagger T^{3'}\chi_{2'} \;=\; -i f^{343'}\ \chi_1^\dagger T^{3'} T^4 \chi_2 \;=\; -\frac{i}{2} f^{343'} i f^{3'4a}\ \chi_1^\dagger T^a \chi_2 \tag{541}$$

$$= \frac{1}{2} f^{343'} f^{3'4a}\ \chi_1^\dagger T^4 \chi_2 \;=\; -\frac{1}{2} f^{33'4} f^{3'4a}\ \chi_1^\dagger T^a \chi_2 \;=\; -\frac{N_c}{2}\ \chi_1^\dagger T^3 \chi_2 \;=\; -\frac{3}{2}\ C^{123}_{\rm 3body} \ . \tag{542}$$

Multiplication by $-N/2$. The strength is negative and slightly increased from 4/3 to 3/2, by the factor 9/8.

- **-3/2** Quark-gluon exchange of a gluon, from Eq. (520),

$$L1\ C^{123}_{\rm 3body} \;=\; g^2 \sum_4 \int [11'33']\,\theta_{1-1'}\,\frac{\tilde\delta_{13.1'3'}}{p_4}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar u_1 \gamma_\nu u_{1'}\ i f^{c_3 c_4 c_{3'}}\ Y_{131'3'}\ b_1^\dagger b_{1'}\ a_3^\dagger a_{3'} \ , \tag{543}$$

$$Y_{131'3'} \;=\; 2\varepsilon_4^\nu \left[\varepsilon_3^*\varepsilon_4^*\,\varepsilon_{3'} p_3 + \varepsilon_4^*\varepsilon_{3'}\,\varepsilon_3^* p_4 - \varepsilon_3^*\varepsilon_{3'}\,\varepsilon_4^* p_{3'}\right] \ , \tag{544}$$

$$\chi_1^\dagger T^4 \chi_{1'}\ i f^{343'}\chi_{1'}^\dagger T^{3'}\chi_2 \;=\; -\frac{3}{2}\ C^{123}_{\rm 3body} \ . \tag{545}$$

- **-3/2** Antiquark-gluon exchange of a gluon, *a la* Eq. (520), The same is then true for the gluon exchange between an antiquark and a gluon.
- **4/3** Quark self-interaction is the same 4/3 for the quark and antiquark self-interactions.
- **4/3** Antiquark self-interaction.
- **3** for the gluon self-interaction is $N = 3$, see App. A, especially its part A 5, and Eq. (A105). I remind myself that in computing the term with f_{LR}, after inclusion of the 2nd-order simultaneous counterterm to severe small-x and quadratic ultraviolet divergences, the remaining divergence is only logarithmic, while the remaining gluon self-interaction result is proportional to the gluon mass squared, m_g^2. In the limit $m_g \to 0$ the remaining, logarithmic gluon self-interaction multiplied by m_g^2 tends to 0.

  In the effective gluon self-interaction with factors f^t instead of f^r, there are no counterterms to include and the entire gluon self-interaction needs to be accounted for. I need to compare factors in the gluon-exchange between quark and gluon and the gluon self-interaction factors without any counterterms. The interaction is finite for $m_g > 0$.
- **-1/6** Gluon exchange between the quark and antiquark.

$$\chi_1^\dagger T^4 \chi_{1'} \; \chi_{1'}^\dagger T^{3'} \chi_{2'} \; \chi_{2'}^\dagger T^4 \chi_2 \;=\; \chi_1^\dagger T^4 \, T^{3'} \, T^4 \chi_2 \;=\; \chi_1^\dagger \left([T^4, T^{3'}] \, T^4 + T^{3'} T^4 T^4 \right) \chi_2 \tag{546}$$

$$= \chi_1^\dagger \left(i f^{43'a} T^a T^4 + T^{3'} T^4 T^4 \right) \chi_2 \tag{547}$$

$$= \chi_1^\dagger \left(\frac{1}{2} \, i f^{43'a} [T^a, T^4] + T^{3'} (N^2 - 1)/(2N) \right) \chi_2 \tag{548}$$

$$= \chi_1^\dagger \left(\frac{1}{2} \, i f^{43'a} i f^{a4b} T^b + T^{3'} (N^2 - 1)/(2N) \right) \chi_2 \tag{549}$$

$$= \chi_1^\dagger \left(-\frac{1}{2} \, N \delta^{3'b} T^b + T^{3'} (N^2 - 1)/(2N) \right) \chi_2 \tag{550}$$

$$= \left(-\frac{1}{2} \, N + (N^2 - 1)/(2N) \right) \; \chi_1^\dagger T^{3'} \chi_2 \;=\; -\frac{1}{2N} \; \chi_1^\dagger T^{3'} \chi_2 \tag{551}$$

$$= -\frac{1}{6} \; \chi_1^\dagger T^{3'} \chi_2 \;=\; -\frac{1}{6} \; C^{123}_{\rm 3body} \; . \tag{552}$$

Summary of color factors in the colorless 3-body space

$$C_{Q\bar{Q}{\rm exch}} \;=\; C_{Q\bar{Q}{\rm seag}} \;=\; -1/6 \; , \; C_{QG{\rm exch}} \;=\; C_{QG{\rm seag}} \;=\; -3/2 \; , \tag{553}$$

$$C_{\Sigma_Q} \;=\; C_{\Sigma_{\bar{Q}}} \;=\; 4/3 \; , \; C_{\Sigma_G} = 3 \tag{554}$$

Vertex change from quark-gluon to quark or gluon self-interaction.

$$\chi_1^\dagger T^4 \chi_{1'} i f^{343'} \; \chi_{1'}^\dagger T^{3'} \chi_2 \;\to \tag{555}$$

B. Gluon-quark interaction in the 3-body space, old

The seagull of Eq. (342) contributes

$$H_{sqg} \;=\; -g^2 \int [33'11'] \; \tilde{\delta}_{c.a} \; \delta_{\rm spins} \; f^r_{\bar{1},\underline{1}(\bar{1}-\underline{1})} \; f^r_{\bar{3},\underline{3}(\bar{3}-\underline{3})} \; \frac{(p_3^+ + p_{3'}^+) \, 2\sqrt{p_1^+ p_{1'}^+}}{(p_{3'}^+ - p_3^+)^2} \; \chi_1^\dagger T^{c_4} \chi_{1'} \, i f^{c_3 c_{3'} c_4} \; a_3^\dagger b_1^\dagger b_{1'} a_{3'} \; , \tag{556}$$

The 2nd-order gluon-quark interaction contained in the gluon-exchange product

$$\mathcal{H}_0^{(1)} \mathcal{H}_0^{(1)} \;=\; (\mathcal{H}_{\psi A} + \mathcal{H}_{\psi\phi} + \mathcal{H}_{AA} + \mathcal{H}_{A\phi}) \, (\mathcal{H}_{\psi A} + \mathcal{H}_{\psi\phi} + \mathcal{H}_{AA} + \mathcal{H}_{A\phi}) \; . \tag{557}$$

The terms that count must be products that involve ψ only in one $\mathcal{H}_0^{(1)}$. This condition is satisfied only by the terms

$$(\mathcal{H}_{\psi A} + \mathcal{H}_{\psi\phi}) \, (\mathcal{H}_{AA} + \mathcal{H}_{A\phi}) + (\mathcal{H}_{AA} + \mathcal{H}_{A\phi}) \, (\mathcal{H}_{\psi A} + \mathcal{H}_{\psi\phi}) \; . \tag{558}$$

If there is ϕ in one factor, there must be ϕ in the other. This condition allows only for

$$\mathcal{H}_{\psi A}\mathcal{H}_{AA} + \mathcal{H}_{AA}\mathcal{H}_{\psi A} + \mathcal{H}_{\psi\phi}\mathcal{H}_{A\phi} + \mathcal{H}_{A\phi}\mathcal{H}_{\psi\phi} \ . \tag{559}$$

Each of the Hamiltonian terms is a sum of two terms, in the forms

$$\begin{aligned}
\mathcal{H}_{\psi A} &= (\mathcal{H}_{\psi a} + \mathcal{H}_{\psi a^\dagger}) \ , &(560)\\
\mathcal{H}_{\psi\phi} &= (\mathcal{H}_{\psi\tilde a} + \mathcal{H}_{\psi\tilde a^\dagger}) \ , &(561)\\
\mathcal{H}_{AA} &= (\mathcal{H}_{Aa^\dagger a^\dagger a} + \mathcal{H}_{Aa^\dagger aa}) \ , &(562)\\
\mathcal{H}_{A\phi} &= (\mathcal{H}_{A\tilde a} + \mathcal{H}_{A\tilde a^\dagger}) \ , &(563)
\end{aligned}$$

where in each case the 2nd term is the h.c. of the first. The terms that count are

$$\begin{aligned}
&(\mathcal{H}_{\psi a} + \mathcal{H}_{\psi a^\dagger})(\mathcal{H}_{Aa^\dagger a^\dagger a} + \mathcal{H}_{Aa^\dagger aa}) + (\mathcal{H}_{Aa^\dagger a^\dagger a} + \mathcal{H}_{Aa^\dagger aa})(\mathcal{H}_{\psi a} + \mathcal{H}_{\psi a^\dagger})\\
+\ &(\mathcal{H}_{\psi\tilde a} + \mathcal{H}_{\psi\tilde a^\dagger})(\mathcal{H}_{A\tilde a} + \mathcal{H}_{A\tilde a^\dagger}) + (\mathcal{H}_{A\tilde a} + \mathcal{H}_{A\tilde a^\dagger})(\mathcal{H}_{\psi\tilde a} + \mathcal{H}_{\psi\tilde a^\dagger}) \ . \qquad (564)
\end{aligned}$$

The gluon exchange interaction results from contraction of a and $a^\dagger$ that leaves behind $a^\dagger a$, just 4 terms.

$$\mathcal{H}_{\psi a}\mathcal{H}_{Aa^\dagger a^\dagger a} + \mathcal{H}_{Aa^\dagger aa}\mathcal{H}_{\psi a^\dagger} + \mathcal{H}_{\psi\tilde a}\mathcal{H}_{A\tilde a^\dagger} + \mathcal{H}_{A\tilde a}\mathcal{H}_{\psi\tilde a^\dagger} \ . \tag{565}$$

I omit the regularization factors f^r. I need to write all the contributing terms explicitly, as g^2 times

$$\begin{aligned}
&\int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \bar u_1\gamma_\nu u_{1'}\ \varepsilon_4^\nu\ b_1^\dagger b_{1'}a_4\quad \int[34'3']\,a_3^\dagger a_{4'}^\dagger a_{3'}\ Y_{34'3'}\,\tilde\delta_{34'.3'}\\
+\ &\int[343']\,a_3^\dagger a_4 a_{3'}\ Y^*_{3'43}\,\tilde\delta_{3.43'}\quad \int[11'4']\,\tilde\delta_{4'1.1'}\ \chi_1^\dagger T^{4'}\chi_{1'}\ \bar u_1\gamma_\nu u_{1'}\ \varepsilon_{4'}^{\nu*}a_{4'}^\dagger b_1^\dagger b_{1'}\\
+\ &\int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \bar u_1\gamma^+u_{1'}\ \frac{m_g}{p_4^+}\ b_1^\dagger b_{1'}\tilde a_4\ \ m_g\int[4'33']\,\tilde\delta_{4'3.3'}\ if^{c_{4'}c_3c_{3'}}\frac{p_3^+ + p_{3'}^+}{p_{4'}^+}\ \varepsilon_3^*\varepsilon_{3'}\ \tilde a_{4'}^\dagger a_3^\dagger a_{3'}\\
+\ &m_g\int[433']\,\tilde\delta_{3'.43}\ \left[if^{c_4c_3c_{3'}}\frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'}\ \tilde a_4^\dagger a_3^\dagger a_{3'}\right]^\dagger\ \int[11'4']\,\tilde\delta_{4'1.1'}\ \chi_1^\dagger T^{4'}\chi_{1'}\ \bar u_1\gamma^+u_{1'}\ \frac{m_g}{p_{4'}^+}\ \tilde a_{4'}^\dagger b_1^\dagger b_{1'} \ . \quad (566)
\end{aligned}$$

Start here The interaction between the quark (1) and gluon (3) via gluon exchange (4) is given by the formula g^2 times

$$\begin{aligned}
\mathcal{H}_{13\,\mathrm{exch}} =\ &\int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \bar u_1\gamma_\nu u_{1'}\ \varepsilon_4^\nu\ b_1^\dagger b_{1'}\quad \int[4'33']\,\tilde\delta_{4'3.3'}\ Y_{4'33'}\ a_4a_{4'}^\dagger a_3^\dagger a_{3'}\int[34'3']\ Y_{34'3'}\ \tilde\delta_{34'.3'}\ a_4a_3^\dagger a_{4'}^\dagger a_{3'}\\
+\ &\int[433']\,\tilde\delta_{3'.43}\ Y^*_{433'}\ a_{3'}^\dagger\ \int[343']\,a_{3'}^\dagger\ Y^*_{343'}\,\tilde\delta_{3'.43}\quad \int[11'4']\,\tilde\delta_{4'1.1'}\ \chi_1^\dagger T^{4'}\chi_{1'}\ \bar u_1\gamma_\nu u_{1'}\ \varepsilon_{4'}^{\nu*}\ a_3a_4a_{4'}^\dagger b_1^\dagger b_{1'}\\
+\ &\int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \bar u_1\gamma^+u_{1'}\ \frac{m_g}{p_4^+}\ b_1^\dagger b_{1'}\ \ m_g\int[4'33']\,\tilde\delta_{4'3.3'}\ if^{c_{4'}c_3c_{3'}}\frac{p_3^+ + p_{3'}^+}{p_{4'}^+}\ \varepsilon_3^*\varepsilon_{3'}\ \tilde a_4\tilde a_{4'}^\dagger a_3^\dagger a_{3'}\\
+\ &m_g\int[433']\,\tilde\delta_{3'.43}\ if^{c_{3'}c_3c_4}\frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_{3'}^*\varepsilon_3\ a_{3'}^\dagger\quad \int[11'4']\,\tilde\delta_{4'1.1'}\ \chi_1^\dagger T^{4'}\chi_{1'}\ \bar u_1\gamma^+u_{1'}\ \frac{m_g}{p_{4'}^+}\ a_3\tilde a_4\tilde a_{4'}^\dagger b_1^\dagger b_{1'} \ . \quad (567)
\end{aligned}$$

I contract as with $a^\dagger$s in $g^2\ b_1^\dagger b_{1'}$ times

$$\begin{aligned}
\mathcal{H}_{13\,\mathrm{exch}} =\ &\int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \bar u_1\gamma_\nu u_{1'}\ \varepsilon_4^\nu\quad \int[34'3']\ Y_{34'3'}\ \tilde\delta_{34'.3'}\ \left(\delta_{44'}a_3^\dagger + a_{4'}^\dagger\delta_{43}\right)a_{3'}\\
+\ &\int[343']\ Y^*_{343'}\ \tilde\delta_{3'.43}\quad \int[11'4']\,\tilde\delta_{4'1.1'}\ \chi_1^\dagger T^{4'}\chi_{1'}\ \bar u_1\gamma_\nu u_{1'}\ \varepsilon_{4'}^{\nu*}\ a_{3'}^\dagger\left(a_3\delta_{44'} + a_4\delta_{34'}\right)\\
+\ &\int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \bar u_1\gamma^+u_{1'}\ \frac{m_g^2}{p_4^+}\quad \int[4'33']\,\tilde\delta_{4'3.3'}\ if^{c_{4'}c_3c_{3'}}\frac{p_3^+ + p_{3'}^+}{p_{4'}^+}\ \varepsilon_3^*\varepsilon_{3'}\ \delta_{44'}a_3^\dagger a_{3'}\\
+\ &\int[433']\,\tilde\delta_{3'.43}\ if^{c_{3'}c_3c_4}\frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_{3'}^*\varepsilon_3\quad \int[11'4']\,\tilde\delta_{4'1.1'}\ \chi_1^\dagger T^{4'}\chi_{1'}\ \bar u_1\gamma^+u_{1'}\ \frac{m_g^2}{p_{4'}^+}\ a_{3'}^\dagger a_3\delta_{44'} \ . \quad (568)
\end{aligned}$$

After expansion of the brackets, I have $g^2\ b_1^\dagger b_{1'}$ times

$$\begin{aligned}
old\ \ \mathcal{H}_{13\,\mathrm{exch}} \ =\ & \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\ \varepsilon_4^\nu \ \ \int [4'33']\, \tilde{\delta}_{4'3.3'}\ Y_{4'33'}\ \delta_{44'} a_3^\dagger a_{3'} \\
+\ & \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\ \varepsilon_4^\nu \ \ \int [4'33']\, \tilde{\delta}_{4'3.3'}\ Y_{4'33'}\ a_{4'}^\dagger \delta_{43} a_{3'} \\
+\ & \int [433']\, \tilde{\delta}_{3'.43}\ Y_{433'}^* \ \ \int [11'4']\, \tilde{\delta}_{4'1.1'}\ \chi_1^\dagger T^{4'} \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\ \varepsilon_{4'}^{\nu *}\ a_{3'}^\dagger a_3 \delta_{44'} \\
+\ & \int [433']\, \tilde{\delta}_{3'.43}\ Y_{433'}^* \ \ \int [11'4']\, \tilde{\delta}_{4'1.1'}\ \chi_1^\dagger T^{4'} \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\ \varepsilon_{4'}^{\nu *}\ a_{3'}^\dagger a_4 \delta_{34'} \\
+\ & \int [11'4]\, \tilde{\delta}_{1.1'4'}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1\gamma^+ u_{1'}\ \frac{m_g^2}{p_4^+} \ \ \int [4'33']\, \tilde{\delta}_{4'3.3'}\ if^{c_{4'}c_3c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_{4'}^+}\ \varepsilon_3^* \varepsilon_{3'}\ \delta_{44'} a_3^\dagger a_{3'} \\
+\ & \int [433']\, \tilde{\delta}_{3'.43}\ if^{c_{3'}c_3c_4} \frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_{3'}^* \varepsilon_3 \ \ \int [11'4']\, \tilde{\delta}_{4'1.1'}\ \chi_1^\dagger T^{4'} \chi_{1'}\ \bar{u}_1\gamma^+ u_{1'}\ \frac{m_g^2}{p_{4'}^+}\ a_{3'}^\dagger a_3 \delta_{44'}\ .
\end{aligned} \tag{569}$$

$$\begin{aligned}
new\ \ \mathcal{H}_{13\,\mathrm{exch}} \ =\ & \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\ \varepsilon_4^\nu \ \ \int [34'3']\ Y_{34'3'}\ \tilde{\delta}_{34'.3'}\ \delta_{44'} a_3^\dagger a_{3'} \\
+\ & \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\ \varepsilon_4^\nu \ \ \int [34'3']\ Y_{34'3'}\ \tilde{\delta}_{34'.3'}\ a_{4'}^\dagger \delta_{43} a_{3'} \\
+\ & \int [343']\ Y_{343'}^*\ \tilde{\delta}_{3'.43} \ \ \int [11'4']\, \tilde{\delta}_{4'1.1'}\ \chi_1^\dagger T^{4'} \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\ \varepsilon_{4'}^{\nu *}\ a_{3'}^\dagger a_3 \delta_{44'} \\
+\ & \int [343']\ Y_{343'}^*\ \tilde{\delta}_{3'.43} \ \ \int [11'4']\, \tilde{\delta}_{4'1.1'}\ \chi_1^\dagger T^{4'} \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\ \varepsilon_{4'}^{\nu *}\ a_{3'}^\dagger a_4 \delta_{34'} \\
+\ & \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1\gamma^+ u_{1'}\ \frac{m_g^2}{p_4^+} \ \ \int [4'33']\, \tilde{\delta}_{4'3.3'}\ if^{c_{4'}c_3c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_{4'}^+}\ \varepsilon_3^* \varepsilon_{3'}\ \delta_{44'} a_3^\dagger a_{3'} \\
+\ & \int [433']\, \tilde{\delta}_{3'.43}\ if^{c_{3'}c_3c_4} \frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_{3'}^* \varepsilon_3 \ \ \int [11'4']\, \tilde{\delta}_{4'1.1'}\ \chi_1^\dagger T^{4'} \chi_{1'}\ \bar{u}_1\gamma^+ u_{1'}\ \frac{m_g^2}{p_{4'}^+}\ a_{3'}^\dagger a_3 \delta_{44'}\ .
\end{aligned} \tag{570}$$

Now I can carry out the integration resulting from the contractions, leading to $g^2\ b_1^\dagger b_{1'}\ \bar{u}_1\gamma_\nu u_{1'}$ combined with

$$\begin{aligned}
old\ \ \mathcal{H}_{13\,\mathrm{exch}} \ =\ & \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \varepsilon_4^\nu \ \ \int [33']\, \tilde{\delta}_{43.3'}\ Y_{433'}\ a_3^\dagger a_{3'} \\
+\ & \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \varepsilon_4^\nu \ \ \int [4'3']\, \tilde{\delta}_{4'4.3'}\ Y_{4'43'}\ a_{4'}^\dagger a_{3'} \qquad 4' \to 3 \\
+\ & \int [433']\, \tilde{\delta}_{3'.43}\ Y_{433'}^* \ \ \int [11']\, \tilde{\delta}_{41.1'}\ \chi_1^\dagger T^4 \chi_{1'}\ \varepsilon_4^{\nu *}\ a_{3'}^\dagger a_3 \qquad 3' \leftrightarrow 3 \\
+\ & \int [433']\, \tilde{\delta}_{3'.43}\ Y_{433'}^* \ \ \int [11']\, \tilde{\delta}_{31.1'}\ \chi_1^\dagger T^3 \chi_{1'}\ \varepsilon_3^{\nu *}\ a_{3'}^\dagger a_4 \qquad 3 \leftrightarrow 4\ \text{ and then }\ 3' \leftrightarrow 3 \\
+\ & \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \frac{m_g^2 \eta^\nu}{p_4^+} \ \ \int [33']\, \tilde{\delta}_{43.3'}\ if^{c_4c_3c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^* \varepsilon_{3'}\ a_3^\dagger a_{3'} \\
+\ & \int [433']\, \tilde{\delta}_{3'.43}\ if^{c_{3'}c_3c_4} \frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_{3'}^* \varepsilon_3 \ \ \int [11']\, \tilde{\delta}_{41.1'}\ \chi_1^\dagger T^4 \chi_{1'}\ \frac{m_g^2 \eta^\nu}{p_4^+}\ a_{3'}^\dagger a_3 \qquad 3' \leftrightarrow 3\ .
\end{aligned} \tag{571}$$

$$
\begin{aligned}
\textit{new}\ \ \mathcal{H}_{13\,\mathrm{exch}} &= \int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \bar u_1\gamma_\nu u_{1'}\ \varepsilon_4^\nu\ \int[33']\ Y_{343'}\,\tilde\delta_{34.3'}\ a_3^\dagger a_{3'} \\
&+ \int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \bar u_1\gamma_\nu u_{1'}\ \varepsilon_4^\nu\ \int[4'3']\ Y_{44'3'}\,\tilde\delta_{44'.3'}\ a_{4'}^\dagger a_{3'} \\
&+ \int[343']\ Y^*_{343'}\,\tilde\delta_{3'.43}\ \int[11']\,\tilde\delta_{41.1'}\ \chi_1^\dagger T^4\chi_{1'}\ \bar u_1\gamma_\nu u_{1'}\ \varepsilon_4^{\nu *}\ a_{3'}^\dagger a_3 \\
&+ \int[343']\ Y^*_{343'}\,\tilde\delta_{3'.43}\ \int[11']\,\tilde\delta_{31.1'}\ \chi_1^\dagger T^3\chi_{1'}\ \bar u_1\gamma_\nu u_{1'}\ \varepsilon_3^{\nu *}\ a_{3'}^\dagger a_4 \\
&+ \int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \bar u_1\gamma^+ u_{1'}\ \frac{m_g^2}{p_4^+}\ \int[4'33']\,\tilde\delta_{4'3.3'}\ if^{c_{4'}c_3c_{3'}}\frac{p_3^+ + p_{3'}^+}{p_{4'}^+}\ \varepsilon_3^*\varepsilon_{3'}\ \delta_{44'}a_3^\dagger a_{3'} \\
&+ \int[433']\,\tilde\delta_{3'.43}\ if^{c_{3'}c_3c_4}\frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_{3'}^*\varepsilon_3\ \int[11'4']\,\tilde\delta_{4'1.1'}\ \chi_1^\dagger T^{4'}\chi_{1'}\ \bar u_1\gamma^+ u_{1'}\ \frac{m_g^2}{p_{4'}^+}\ a_{3'}^\dagger a_3\delta_{44'}\ . \quad (572)
\end{aligned}
$$

I am now (20250918 10:08 czw San Dimas) making these changes in $g^2\ b_1^\dagger b_{1'}\ \bar u_1\gamma_\nu u_{1'}$ combined with

$$
\begin{aligned}
\textit{old}\ \ \mathcal{H}_{13\,\mathrm{exch}} &= \int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \varepsilon_4^\nu\ \int[33']\,\tilde\delta_{43.3'}\ Y_{433'}\ a_3^\dagger a_{3'} \\
&+ \int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \varepsilon_4^\nu\ \int[33']\,\tilde\delta_{34.3'}\ Y_{343'}\ a_3^\dagger a_{3'} \qquad 4'\to 3 \\
&+ \int[43'3]\,\tilde\delta_{3.43'}\ Y^*_{43'3}\ \int[11']\,\tilde\delta_{41.1'}\ \chi_1^\dagger T^4\chi_{1'}\ \varepsilon_4^{\nu *}\ a_3^\dagger a_{3'} \qquad 3'\leftrightarrow 3 \\
&+ \int[3'43]\,\tilde\delta_{3.3'4}\ Y^*_{3'43}\ \int[11']\,\tilde\delta_{41.1'}\ \chi_1^\dagger T^4\chi_{1'}\ \varepsilon_4^{\nu *}\ a_3^\dagger a_{3'} \qquad 3\leftrightarrow 4\ \text{ and then }\ 3'\leftrightarrow 3 \\
&+ \int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \frac{m_g^2\eta^\nu}{p_4^+}\ \int[33']\,\tilde\delta_{43.3'}\ if^{c_4c_3c_{3'}}\frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'}\ a_3^\dagger a_{3'} \\
&+ \int[43'3]\,\tilde\delta_{3.43'}\ if^{c_3c_{3'}c_4}\frac{p_{3'}^+ + p_3^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'}\ \int[11']\,\tilde\delta_{41.1'}\ \chi_1^\dagger T^4\chi_{1'}\ \frac{m_g^2\eta^\nu}{p_4^+}\ a_3^\dagger a_{3'} \qquad 3'\leftrightarrow 3\ . \quad (573)
\end{aligned}
$$

And in the checking calculation in green on 20250924 21:58 sro San Dimas, I make analogous notation changes in $g^2\ b_1^\dagger b_{1'}\ \bar u_1\gamma_\nu u_{1'}$ combined with

$$
\begin{aligned}
\textit{new}\ \ \mathcal{H}_{13\,\mathrm{exch}} &= \int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \varepsilon_4^\nu\ \int[33']\ Y_{343'}\,\tilde\delta_{34.3'}\ a_3^\dagger a_{3'} \\
&+ \int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \varepsilon_4^\nu\ \int[33']\ Y_{433'}\,\tilde\delta_{43.3'}\ a_3^\dagger a_{3'} \qquad \text{from }\ 4'\to 3 \\
&+ \int[3'43]\ Y^*_{3'43}\,\tilde\delta_{3.43'}\ \int[11']\,\tilde\delta_{41.1'}\ \chi_1^\dagger T^4\chi_{1'}\ \varepsilon_4^{\nu *}\ a_3^\dagger a_{3'} \qquad \text{from }\ 3'\leftrightarrow 3 \\
&+ \int[43'3]\ Y^*_{43'3}\,\tilde\delta_{3.3'4}\ \int[11']\,\tilde\delta_{41.1'}\ \chi_1^\dagger T^4\chi_{1'}\ \varepsilon_4^{\nu *}\ a_3^\dagger a_{3'} \quad \text{from }\ 3\leftrightarrow 4\ \text{ and then }\ 3'\leftrightarrow 3 \\
&+ \int[11'4]\,\tilde\delta_{1.1'4}\ \chi_1^\dagger T^4\chi_{1'}\ \frac{m_g^2}{p_4^+}\ \int[33']\,\tilde\delta_{43.3'}\ if^{c_4c_3c_{3'}}\frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'}\ a_3^\dagger a_{3'} \\
&+ \int[43'3]\,\tilde\delta_{3.43'}\ if^{c_3c_{3'}c_4}\frac{p_{3'}^+ + p_3^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'}\ \int[11']\,\tilde\delta_{41.1'}\ \chi_1^\dagger T^4\chi_{1'}\ \frac{m_g^2}{p_4^+}\ a_3^\dagger a_{3'}\ . \quad \text{from }\ 3'\leftrightarrow 3 \quad (574)
\end{aligned}
$$

The result is $g^2\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1 \gamma_\nu u_{1'}\ b_1^\dagger b_{1'}\ a_3^\dagger a_{3'}$ combined with

$$\begin{aligned}
old\ \ \mathcal{H}_{13\,\text{exch}} \ = & \ \int [11'4]\, \tilde{\delta}_{1.1'4}\ \varepsilon_4^\nu \quad \int [33']\, \tilde{\delta}_{43.3'}\ Y_{433'} \\
+ & \ \int [11'4]\, \tilde{\delta}_{1.1'4}\ \varepsilon_4^\nu \quad \int [33']\, \tilde{\delta}_{34.3'}\ Y_{343'} \\
+ & \ \int [43'3]\, \tilde{\delta}_{3.43'}\ Y^*_{43'3} \quad \int [11']\, \tilde{\delta}_{41.1'}\ \varepsilon_4^{\nu *} \\
+ & \ \int [3'43]\, \tilde{\delta}_{3.3'4}\ Y^*_{3'43} \quad \int [11']\, \tilde{\delta}_{41.1'}\ \varepsilon_4^{\nu *} \\
+ & \ \int [11'4]\, \tilde{\delta}_{1.1'4}\ \frac{m_g^2 \eta^\nu}{p_4^+} \quad \int [33']\, \tilde{\delta}_{43.3'}\ i f^{c_4 c_3 c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^* \varepsilon_{3'} \\
+ & \ \int [43'3]\, \tilde{\delta}_{3.43'}\ i f^{c_3 c_{3'} c_4} \frac{p_{3'}^+ + p_3^+}{p_4^+}\ \varepsilon_3^* \varepsilon_{3'} \quad \int [11']\, \tilde{\delta}_{41.1'}\ \frac{m_g^2 \eta^\nu}{p_4^+}\ .
\end{aligned} \tag{575}$$

The new result will be $g^2\ \bar{u}_1 \gamma_\nu u_{1'}\ b_1^\dagger b_{1'}$ combined with

$$\begin{aligned}
new\ \ \mathcal{H}_{13\,\text{exch}} \ = & \ \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \varepsilon_4^\nu \quad \int [33']\ Y_{343'}\, \tilde{\delta}_{34.3'}\ a_3^\dagger a_{3'} \\
+ & \ \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \varepsilon_4^\nu \quad \int [4'3']\ Y_{44'3'}\, \tilde{\delta}_{44'.3'}\ a_{4'}^\dagger a_{3'} \qquad 4' \to 3 \\
+ & \ \int [343']\ Y^*_{343'}\, \tilde{\delta}_{3'.43} \quad \int [11']\, \tilde{\delta}_{41.1'}\ \chi_1^\dagger T^4 \chi_{1'}\ \varepsilon_4^{\nu *}\ a_{3'}^\dagger a_3 \qquad 3' \leftrightarrow 3 \\
+ & \ \int [343']\ Y^*_{343'}\, \tilde{\delta}_{3'.43} \quad \int [11']\, \tilde{\delta}_{31.1'}\ \chi_1^\dagger T^3 \chi_{1'}\ \varepsilon_3^{\nu *}\ a_{3'}^\dagger a_4 \qquad 3 \leftrightarrow 4\ \text{ and then }\ 3' \leftrightarrow 3 \\
+ & \ \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \frac{m_g^2}{p_4^+} \quad \int [33']\, \tilde{\delta}_{43.3'}\ i f^{c_4 c_3 c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^* \varepsilon_{3'}\ a_3^\dagger a_{3'} \\
+ & \ \int [433']\, \tilde{\delta}_{3'.43}\ i f^{c_{3'} c_3 c_4} \frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_{3'}^* \varepsilon_3 \quad \int [11']\, \tilde{\delta}_{41.1'}\ \chi_1^\dagger T^4 \chi_{1'}\ \frac{m_g^2}{p_4^+}\ a_{3'}^\dagger a_3\ . \quad 3' \leftrightarrow 3
\end{aligned} \tag{576}$$

$$\begin{aligned}
new\ \ \mathcal{H}_{13\,\text{exch}} \ = & \ \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \varepsilon_4^\nu \quad \int [33']\ Y_{343'}\, \tilde{\delta}_{34.3'}\ a_3^\dagger a_{3'} \\
+ & \ \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \varepsilon_4^\nu \quad \int [33']\ Y_{433'}\, \tilde{\delta}_{43.3'}\ a_3^\dagger a_{3'} \qquad \text{from }\ 4' \to 3 \\
+ & \ \int [3'43]\ Y^*_{3'43}\, \tilde{\delta}_{3.43'} \quad \int [11']\, \tilde{\delta}_{41.1'}\ \chi_1^\dagger T^4 \chi_{1'}\ \varepsilon_4^{\nu *}\ a_3^\dagger a_{3'} \qquad \text{from }\ 3' \leftrightarrow 3 \\
+ & \ \int [43'3]\ Y^*_{43'3}\, \tilde{\delta}_{3.3'4} \quad \int [11']\, \tilde{\delta}_{41.1'}\ \chi_1^\dagger T^4 \chi_{1'}\ \varepsilon_4^{\nu *}\ a_3^\dagger a_{3'} \quad \text{from }\ 3 \leftrightarrow 4\ \text{ and then }\ 3' \leftrightarrow 3 \\
+ & \ \int [11'4]\, \tilde{\delta}_{1.1'4}\ \chi_1^\dagger T^4 \chi_{1'}\ \frac{m_g^2}{p_4^+} \quad \int [33']\, \tilde{\delta}_{43.3'}\ i f^{c_4 c_3 c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^* \varepsilon_{3'}\ a_3^\dagger a_{3'} \\
+ & \ \int [43'3]\, \tilde{\delta}_{3.43'}\ i f^{c_3 c_{3'} c_4} \frac{p_{3'}^+ + p_3^+}{p_4^+}\ \varepsilon_3^* \varepsilon_{3'} \quad \int [11']\, \tilde{\delta}_{41.1'}\ \chi_1^\dagger T^4 \chi_{1'}\ \frac{m_g^2}{p_4^+}\ a_3^\dagger a_{3'}\ . \quad \text{from }\ 3' \leftrightarrow 3
\end{aligned} \tag{577}$$

The new result is $g^2\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\ b_1^\dagger b_{1'}\ a_3^\dagger a_{3'}$ combined with

$$\begin{aligned}
new\ \ \mathcal{H}_{13\,\mathrm{exch}} \ =\ & \int [11'4]\,\tilde\delta_{1.1'4}\ \varepsilon_4^\nu\ \ \int [33']\ Y_{343'}\,\tilde\delta_{34.3'} \\
+\ & \int [11'4]\,\tilde\delta_{1.1'4}\ \varepsilon_4^\nu\ \ \int [33']\ Y_{433'}\,\tilde\delta_{43.3'} \qquad \text{from}\ \ 4'\to 3 \\
+\ & \int [3'43]\ Y^*_{3'43}\,\tilde\delta_{3.43'}\ \ \int [11']\,\tilde\delta_{41.1'}\ \varepsilon_4^{\nu *} \qquad \text{from}\ \ 3'\leftrightarrow 3 \\
+\ & \int [43'3]\ Y^*_{43'3}\,\tilde\delta_{3.3'4}\ \ \int [11']\,\tilde\delta_{41.1'}\ \varepsilon_4^{\nu *} \quad \text{from}\ \ 3\leftrightarrow 4\ \ \text{and then}\ \ 3'\leftrightarrow 3 \\
+\ & \int [11'4]\,\tilde\delta_{1.1'4}\ \frac{m_g^2}{p_4^+}\ \ \int [33']\,\tilde\delta_{43.3'}\ i f^{c_4 c_3 c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'} \\
+\ & \int [43'3]\,\tilde\delta_{3.43'}\ i f^{c_3 c_{3'} c_4} \frac{p_{3'}^+ + p_3^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'}\ \ \int [11']\,\tilde\delta_{41.1'}\ \frac{m_g^2}{p_4^+}\ . \quad \text{from}\ \ 3'\leftrightarrow 3
\end{aligned} \tag{578}$$

I simplify to $g^2\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\ b_1^\dagger b_{1'}\ a_3^\dagger a_{3'}$ combined with

$$\begin{aligned}
old\ \ \mathcal{H}_{13\,\mathrm{exch}} \ =\ & \int [11'433']\,\tilde\delta_{1.1'4}\tilde\delta_{43.3'}\ \varepsilon_4^\nu\ \ (Y_{433'} + Y_{343'}) \\
+\ & \int [11'433']\,\tilde\delta_{3.43'}\tilde\delta_{41.1'}\ \ (Y^*_{43'3} + Y^*_{3'43})\ \ \varepsilon_4^{\nu *} \\
+\ & \int [11'433']\,\tilde\delta_{1.1'4}\tilde\delta_{43.3'}\ \frac{m_g^2\eta^\nu}{p_4^+}\ \ i f^{c_4 c_3 c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'} \\
+\ & \int [11'433']\,\tilde\delta_{3.43'}\tilde\delta_{41.1'}\ i f^{c_3 c_{3'} c_4} \frac{p_{3'}^+ + p_3^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'}\ \ \frac{m_g^2\eta^\nu}{p_4^+}\ .
\end{aligned} \tag{579}$$

I simplify to $g^2\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1\gamma_\nu u_{1'}\ b_1^\dagger b_{1'}\ a_3^\dagger a_{3'}$ combined with

$$\begin{aligned}
new\ \ \mathcal{H}_{13\,\mathrm{exch}} \ =\ & \int [11'433']\,\tilde\delta_{1.1'4}\tilde\delta_{34.3'}\ \ \ \varepsilon_4^\nu Y_{343'} \\
+\ & \int [11'433']\,\tilde\delta_{1.1'4}\tilde\delta_{43.3'}\ \ \varepsilon_4^\nu Y_{433'} \\
+\ & \int [11'433']\,\tilde\delta_{41.1'}\tilde\delta_{3.43'}\ \ \ \varepsilon_4^{\nu *} Y^*_{3'43} \\
+\ & \int [11'433']\,\tilde\delta_{41.1'}\tilde\delta_{3.3'4}\ \ \ \varepsilon_4^{\nu *} Y^*_{43'3} \\
+\ & \int [11'433']\,\tilde\delta_{1.1'4}\tilde\delta_{43.3'}\ \ \frac{m_g^2}{p_4^+}\ \ i f^{c_4 c_3 c_{3'}} \frac{p_3^+ + p_{3'}^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'} \\
+\ & \int [11'433']\,\tilde\delta_{41.1'}\tilde\delta_{3.43'}\ \ i f^{c_3 c_{3'} c_4} \frac{p_{3'}^+ + p_3^+}{p_4^+}\ \varepsilon_3^*\varepsilon_{3'}\ \ \frac{m_g^2}{p_4^+}\ .
\end{aligned} \tag{580}$$

There are two integrals over 4.

$$\int [4]\,\tilde\delta_{1.1'4}\tilde\delta_{43.3'} \ =\ \int [4]\,2(2\pi)^3\delta_{1-1'-4}2(2\pi)^3\delta_{4+3-3'} \ =\ \tilde\delta_{13.1'3'}\theta_{1-1'}/p_4^+ \qquad \text{and} \qquad p_4 \ =\ p_1 - p_{1'}\ , \tag{581}$$

$$\int [4]\,\tilde\delta_{3.43'}\tilde\delta_{41.1'} \ =\ \int [4]\,2(2\pi)^3\delta_{3-4-3'}2(2\pi)^3\delta_{4+1-1'} \ =\ \tilde\delta_{13.1'3'}\theta_{1'-1}/p_4^+ \qquad \text{and} \qquad p_4 \ =\ p_{1'} - p_1\ . \tag{582}$$

The result for the exchange interaction is

$$\begin{aligned}
\mathcal{H}_{13\,\mathrm{exch}} &= g^2 \sum_{\mathrm{tr}} \int [11'33']\, \tilde{\delta}_{13.1'3'}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1 \gamma_\nu u_{1'}\ \left(\mathcal{J}^\nu_{\mathrm{tr}}/p_4^+\right)\ a_3^\dagger b_1^\dagger b_{1'} a_{3'}\ , \quad p_4 = p_{\bar{1}} - p_{\underline{1}} \\
&+ g^2 \int [11'33']\, \tilde{\delta}_{13.1'3'}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1 \gamma_\nu u_{1'}\ \left(\mathcal{J}^\nu_{\eta}/p_4^+\right)\ a_3^\dagger b_1^\dagger b_{1'} a_{3'}\ , \quad p_4 = p_{\bar{1}} - p_{\underline{1}}\ , \qquad (583) \\
\mathcal{J}^\nu_{\mathrm{tr}} &= \theta_{1-1'}\, \varepsilon_4^\nu \left(Y_{433'} + Y_{343'}\right) + \theta_{1'-1}\, \varepsilon_4^{\nu *} \left(Y^*_{43'3} + Y^*_{3'43}\right)\ , \qquad (584) \\
\mathcal{J}^\nu_{\eta} &= \frac{m_g^2 \eta^\nu}{p_4^{+2}}\ i f^{c_4 c_3 c_{3'}} (p_3^+ + p_{3'}^+)\ \varepsilon_3^* \varepsilon_{3'}\ . \qquad (585)
\end{aligned}$$

$$Y_{433'} + Y_{343'} = 2Y_{433'}\ ,\ Y^*_{43'3} + Y^*_{3'43} = 2Y^*_{43'3}\ . \qquad (586)$$

$$\begin{aligned}
Y_{433'} &= i f^{c_4 c_3 c_{3'}} \left(\varepsilon_4^* \varepsilon_3^*\ \varepsilon_{3'} p_4 - \varepsilon_4^* \varepsilon_{3'}\ \varepsilon_3^* p_{3'} + \varepsilon_3^* \varepsilon_{3'}\ \varepsilon_4^* p_{3'}\right), \quad \text{to multiply by} \quad a_4^\dagger a_3^\dagger a_{3'}\ , \qquad (587) \\
&= i f^{c_4 c_3 c_{3'}} \left(\varepsilon_4^* \varepsilon_3^*\ \varepsilon_{3'} k_{43} - \varepsilon_4^* \varepsilon_{3'}\ \varepsilon_3^* k_{43}/y_3 - \varepsilon_3^* \varepsilon_{3'}\ \varepsilon_4^* k_{43}/y_4\right)\ , \qquad (588) \\
Y_{43'3} &= i f^{c_4 c_{3'} c_3} \left(\varepsilon_4^* \varepsilon_{3'}^*\ \varepsilon_3 p_4 - \varepsilon_4^* \varepsilon_3\ \varepsilon_{3'}^* p_3 + \varepsilon_{3'}^* \varepsilon_3\ \varepsilon_4^* p_3\right)\ , \qquad (589) \\
Y^*_{43'3} &= -i f^{c_4 c_{3'} c_3} \left(\varepsilon_4 \varepsilon_{3'}\ \varepsilon_3^* p_4 - \varepsilon_4 \varepsilon_3^*\ \varepsilon_{3'} p_3 + \varepsilon_{3'} \varepsilon_3^*\ \varepsilon_4 p_3\right)\ . \qquad (590)
\end{aligned}$$

I can now pool out $if^{c_4 c_3 c_{3'}}$ and write (remember the regularization factors are included)

$$\begin{aligned}
\mathcal{H}_{13\,\mathrm{exch}} &= g^2 \sum_{\mathrm{tr}} \int [11'33']\, \tilde{\delta}_{13.1'3'}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1 \gamma_\nu u_{1'}\ \left(\mathcal{J}^\nu_{\mathrm{tr}}/p_4^+\right)\ i f^{c_4 c_3 c_{3'}}\ a_3^\dagger b_1^\dagger b_{1'} a_{3'}\ , \quad p_4 = p_{\bar{1}} - p_{\underline{1}} \\
&+ g^2 \int [11'33']\, \tilde{\delta}_{13.1'3'}\ \chi_1^\dagger T^4 \chi_{1'}\ \bar{u}_1 \gamma_\nu u_{1'}\ \left(\mathcal{J}^\nu_{\eta}/p_4^+\right)\ i f^{c_4 c_3 c_{3'}}\ a_3^\dagger b_1^\dagger b_{1'} a_{3'}\ , \quad p_4 = p_{\bar{1}} - p_{\underline{1}}\ . \qquad (591)
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}^\nu_{\mathrm{tr}} &= 2\theta_{1-1'}\, \varepsilon_4^\nu Z_{433'} + 2\theta_{1'-1}\, \varepsilon_4^{\nu *} Z^*_{43'3}\ , \qquad (592) \\
\mathcal{J}^\nu_{\eta} &= \eta^\nu\ S_\eta\ , \qquad (593) \\
Z_{433'} &= \varepsilon_4^{*\mu} S_{433'\mu}\ , \qquad (594) \\
S_{0433'\mu} &= \left(\varepsilon^*_{3\mu}\ \varepsilon_{3'} p_4 - \varepsilon_{3'\mu}\ \varepsilon_3^* p_{3'} + \varepsilon_3^* \varepsilon_{3'}\ p_{3'\mu}\right) \qquad p_4^- \text{ is absent}\ , \qquad (595) \\
Z^*_{43'3} &= \varepsilon_4^\mu S^*_{43'3\mu}\ , \qquad (596) \\
S^*_{043'3\mu} &= \left(\varepsilon_{3'\mu}\ \varepsilon_3^* p_4 - \varepsilon^*_{3\mu}\ \varepsilon_{3'} p_3 + \varepsilon_{3'} \varepsilon_3^*\ p_{3\mu}\right) \qquad p_4^- \text{ is absent}\ , \qquad (597) \\
S_\eta &= \frac{m_g^2}{p_4^{+2}}\ (p_3^+ + p_{3'}^+)\ \varepsilon_3^* \varepsilon_{3'}\ . \qquad (598)
\end{aligned}$$

0 refers to the gluon mass 0, although the transverse polarization vectors do npot depend on the gluon mass. Then

$$\mathcal{J}^\nu_{\mathrm{tr}} = 2\theta_{1-1'} \sum_{\sigma_4} \varepsilon_4^\nu \varepsilon_4^{*\mu} S_{0433'\mu} + 2\theta_{1'-1} \sum_{\sigma_4} \varepsilon_4^{\nu *} \varepsilon_4^\mu S^*_{043'3\mu}\ . \qquad (599)$$

Sum over spins of the transverse gluons yields

$$\mathcal{J}^\nu_{\mathrm{tr}} = 2\theta_{1-1'}\, d_{04}^{\nu\mu}\, S_{0433'\mu} + 2\theta_{1'-1}\, d_{04}^{\nu\mu}\, S^*_{043'3\mu}\ , \qquad (600)$$

where

$$d_{04}^{\nu\mu} = -g^{\nu\mu} + \frac{p_{04}^\nu \eta^\mu + \eta^\nu p_{04}^\mu}{p_4^+}\ , \qquad (601)$$

and 0 refers again to the gluon mass set to 0. I recall Eq. (342) for the quark-gluon seagull, not showing $f^r_{\bar{1},\underline{1}(\bar{1}-\underline{1})}$ $f^r_{\bar{3},\underline{3}(\bar{3}-\underline{3})}$,

$$H_{sqg} = -g^2 \int [33'11']\ \tilde{\delta}_{c.a}\ \delta_{\mathrm{spins}}\ \frac{(p_3^+ + p_{3'}^+)\ 2\sqrt{p_1^+ p_{1'}^+}}{(p_{3'}^+ - p_3^+)^2}\ \chi_1^\dagger T^{c_4} \chi_{1'}\ i f^{c_4 c_3 c_{3'}}\ a_3^\dagger b_1^\dagger b_{1'} a_{3'}\ , \qquad (602)$$

while the full expression for the gluon exchange, including regulating factors f that are not indicated,

$$\mathcal{H}^{(2)} \;=\; f_{LR}\,\mathcal{H}_0^{(2)} + (f_{LR} - f_{LI} f_{IR})\,\Delta_{LIR}\,\mathcal{H}_0^{(1)}\,\mathcal{H}_0^{(1)}\;, \tag{603}$$

reads, see Eq. (591),

$$\text{Exch} \;=\; (f_{LR} - f_{LI} f_{IR})\,\Delta_{LIR}\,g^2 \int [11'33']\,\tilde{\delta}_{c.a}\;\chi_1^\dagger T^{c_4}\chi_{1'}\;\bar{u}_1\gamma_\nu u_{1'}\,(d_{04}^{\nu\mu}\mathcal{E}_{\mu\mathrm{tr}} + \eta^\nu S_\eta)\,\frac{1}{p_4}\,i f^{c_4 c_3 c_{3'}}\quad a_3^\dagger b_1^\dagger b_{1'} a_{3'}\;, \tag{604}$$

where

$$\begin{aligned}
\mathcal{E}_{\mu\mathrm{tr}} &= 2\theta_{1-1'}\,S_{0433'\mu} + 2\theta_{1'-1}\,S^*_{043'3\mu}\;, &(605)\\
S_\eta &= \frac{m_g^2}{p_4^{+2}}\,(p_3^+ + p_{3'}^+)\;\varepsilon_3^*\varepsilon_{3'}\;, &(606)\\
\frac{1}{p_4^+}\Delta_{LIR} &= \frac{1}{2p_4^+}\left(\frac{1}{p_3^- + p_1^- - P^-_{\rm three}} + \frac{1}{p_{3'}^- + p_{1'}^- - P^-_{\rm three}}\right) = -\,\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_3}\right)\;. &(607)
\end{aligned}$$

The gluon four-momenta in the denominators come from $\mathcal{H}_f$ and have mass m_g. Therefore, they do not carry subscript 0 indicating mass 0.

$$\begin{aligned}
\theta_{1-1'}\;\to\qquad & p_4^+\,(p_3^- + p_1^- - P^-_{\rm three}) \;=\; (p_1^+ - p_{1'}^+)\,(p_3^- + p_1^- - p_3^- - p_4^- - p_{1'}^-) &(608)\\
&= (p_1^+ - p_{1'}^+)\,(p_1^- - p_{1'}^- - p_4^-) \;=\; (p_1^+ - p_{1'}^+)(p_1^- - p_{1'}^-) - (p_1^\perp - p_{1'}^\perp)^2 - m_g^2 &(609)\\
&= q_1^2 - m_g^2 \;=\; -\rho_1\;, &(610)\\
& p_4^+\,(p_{3'}^- + p_{1'}^- - P^-_{\rm three}) \;=\; (p_1^+ - p_{1'}^+)\,(p_{3'}^- + p_{1'}^- - p_3^- - p_4^- - p_{1'}^-) &(611)\\
&= (p_{3'}^+ - p_3^+)\,(p_{3'}^- - p_3^- - p_4^-) \;=\; (p_{3'}^+ - p_3^+)(p_{3'}^- - p_3^-) - (p_{3'}^\perp - p_3^\perp)^2 - m_g^2 &(612)\\
&= q_3^2 - m_g^2 \;=\; -\rho_3\;, &(613)\\
\theta_{1'-1}\;\to\qquad & p_4^+\,(p_3^- + p_1^- - P^-_{\rm three}) \;=\; (p_{1'}^+ - p_1^+)\,(p_3^- + p_1^- - p_{3'}^- - p_4^- - p_1^-) &(614)\\
&= (p_3^+ - p_{3'}^+)\,(p_3^- - p_{3'}^- - p_4^-) \;=\; (p_3^+ - p_{3'}^+)(p_3^- - p_{3'}^-) - (p_3^\perp - p_{3'}^\perp)^2 - m_g^2 &(615)\\
&= q_3^2 - m_g^2 \;=\; -\rho_3\;, &(616)\\
& p_4^+\,(p_{3'}^- + p_{1'}^- - P^-_{\rm three}) \;=\; (p_{1'}^+ - p_1^+)\,(p_{3'}^- + p_{1'}^- - p_{3'}^- - p_4^- - p_1^-) &(617)\\
&= (p_{1'}^+ - p_1^+)\,(p_{1'}^- - p_1^- - p_4^-) \;=\; (p_{1'}^+ - p_1^+)(p_{1'}^- - p_1^-) - (p_{1'}^\perp - p_1^\perp)^2 - m_g^2 &(618)\\
&= q_1^2 - m_g^2 \;=\; -\rho_1\;. &(619)
\end{aligned}$$

C. Products in gluon exchange between a quark and a gluon in the 3-body space

Added first subscript 0 indicates that the relevant mass-squared in minus component is 0.

$$\begin{aligned}
\theta_{1-1'}\;\to\qquad & q_{03}^\mu S_{0433'\mu} \;=\; (p_{03'} - p_{03})^\mu S_{0433'\mu} &(620)\\
&= (p_{03'} - p_{03})^\mu\left(\varepsilon^*_{3\mu}\;\varepsilon_{3'}p_{04} - \varepsilon_{3'\mu}\;\varepsilon_3^* p_{03'} + \varepsilon_3^*\varepsilon_{3'}\;p_{03'\mu}\right) &(621)\\
&= (p_{03'} - p_{03})^\mu\varepsilon^*_{3\mu}\;\varepsilon_{3'}p_{04} - (p_{03'} - p_{03})^\mu\varepsilon_{3'\mu}\;\varepsilon_3^* p_{03'} + \varepsilon_3^*\varepsilon_{3'}\;(p_{03'} - p_{03})^\mu p_{03'\mu} &(622)\\
&= p_{03'}^\mu\varepsilon^*_{3\mu}\;\varepsilon_{3'}p_{04} + p_{03}^\mu\varepsilon_{3'\mu}\;\varepsilon_3^* p_{03'} - \varepsilon_3^*\varepsilon_{3'}\;p_{03}^\mu p_{03'\mu} &(623)\\
&= p_{03'}\varepsilon_3^*\;\varepsilon_{3'}p_{04} + p_{03}\varepsilon_{3'}\;\varepsilon_3^* p_{03'} - \varepsilon_3^*\varepsilon_{3'}\;p_{03}p_{03'} &(624)\\
&= -p_{03'}\varepsilon_3^*\;\varepsilon_{3'}p_{03} + p_{03}\varepsilon_{3'}\;\varepsilon_3^* p_{03'} - \varepsilon_3^*\varepsilon_{3'}\;p_{03}p_{03'}\;. &(625)
\end{aligned}$$

$$\theta_{1-1'}\;\to\qquad q_{03}^\mu S_{0433'\mu} \;=\; -\varepsilon_3^*\varepsilon_{3'}\;p_{03}p_{03'} \;=\; \delta_{3\mathrm{spin}}\;p_{03}p_{03'}\;. \tag{626}$$

$$\begin{aligned}
\theta_{1'-1}\;\to\qquad & q_{03}^\mu S^*_{043'3\mu} \;=\; (p_{03} - p_{03'})^\mu S_{043'3\mu} &(627)\\
&= (p_{03} - p_{03'})^\mu\left(\varepsilon_{3'\mu}\;\varepsilon_3^* p_{04} - \varepsilon^*_{3\mu}\;\varepsilon_{3'}p_{03} + \varepsilon_{3'}\varepsilon_3^*\;p_{03\mu}\right) &(628)\\
&= (p_{03} - p_{03'})^\mu\varepsilon_{3'\mu}\;\varepsilon_3^* p_{04} - (p_{03} - p_{03'})^\mu\varepsilon^*_{3\mu}\;\varepsilon_{3'}p_{03} + \varepsilon_{3'}\varepsilon_3^*\;(p_{03} - p_{03'})^\mu p_{03\mu} &(629)\\
&= p_{03}\varepsilon_{3'}\;\varepsilon_3^* p_{04} + p_{03'}\varepsilon_3^*\;\varepsilon_{3'}p_{03} - \varepsilon_{3'}\varepsilon_3^*\;p_{03'}p_{03} &(630)\\
&= p_{03}\varepsilon_{3'}\;\varepsilon_3^* p_{04} + (p_{03} - p_{04})\varepsilon_3^*\;\varepsilon_{3'}p_{03} - \varepsilon_{3'}\varepsilon_3^*\;p_{03'}p_{03}\;\;\textit{error?} &(631)\\
&= p_{03}\varepsilon_{3'}\;\varepsilon_3^* p_{04} - p_{04}\varepsilon_3^*\;\varepsilon_{3'}p_{03} - \varepsilon_{3'}\varepsilon_3^*\;p_{03'}p_{03}\;\;\textit{no error} &(632)\\
&= -\varepsilon_{3'}\varepsilon_3^*\;p_{03'}p_{03}\;. &(633)
\end{aligned}$$

$$\theta_{1'-1} \;\rightarrow\; \qquad q^\mu_{03} S^*_{043'3\mu} \;=\; -\varepsilon^*_3 \varepsilon_{3'}\; p_{03} p_{03'} \;=\; \delta_{3\text{spin}}\; p_{03} p_{03'}\;. \tag{634}$$

Now, 20250921 09:51 nie San Dimas, I can work out the $d^{\nu\mu}_{04}$. The factor $-\varepsilon^*_3\varepsilon_{3'}$ is the gluon spin conservation factor $\delta_{\sigma_3 \sigma_{3'}}$. The indices ν and μ are contracted with the quark and gluon currents, respectively.

$$d^{\nu\mu}_{04} \;=\; -g^{\nu\mu} + \frac{p^\nu_{04}\eta^\mu + \eta^\nu p^\mu_{04}}{p^+_4}\;, \tag{635}$$

$$p^\nu_{04} \;=\; q^\nu_1 + \frac{1}{2}\,\eta^\nu (p^-_{04} - q^-_1)\;,\; p^\mu_{04} \;=\; q^\mu_{03} + \frac{1}{2}\,\eta^\mu(p^-_{04} - q^-_{03})\;. \tag{636}$$

I need to write down the sum of the exchange and seagull terms.

$$\text{Seagull} \;=\; -g^2 f_{LR} \int [33'11']\; \tilde{\delta}_{c.a}\; \delta_{\text{spins}}\; \frac{(p^+_3 + p^+_{3'})\; 2\sqrt{p^+_1 p^+_{1'}}}{(p^+_{3'} - p^+_3)^2}\; \chi^\dagger_1 T^{c_4} \chi_{1'}\, i f^{c_4 c_3 c_{3'}}\; a^\dagger_3 b^\dagger_1 b_{1'} a_{3'}\;, \tag{637}$$

$$\text{Exchange} \;=\; (f_{LR} - f_{LI} f_{IR})\; \Delta_{LIR}\; g^2 \int [11'33']\, \tilde{\delta}_{c.a}\; \chi^\dagger_1 T^{c_4}\chi_{1'}\; \bar{u}_1 \gamma_\nu u_{1'}\, (d^{\nu\mu}_{04}\mathcal{E}_{\mu\text{tr}} + \eta^\nu S_\eta)\, \frac{1}{p_4}\, i f^{c_4 c_3 c_{3'}}\;\; a^\dagger_3 b^\dagger_1 b_{1'} a_{3'}\;. \tag{638}$$

Since $f_{LI} f_{IR}$ makes the terms finite, I focus on terms f_{LR}. I extract common factors

$$CF \;=\; g^2\; f_{LR} \int [11'33']\; \tilde{\delta}_{c.a}\; \chi^\dagger_1 T^{c_4} \chi_{1'}\; i f^{c_4 c_3 c_{3'}}\; a^\dagger_3 b^\dagger_1 b_{1'} a_{3'}\;. \tag{639}$$

The sum exchange plus seagull is

$$CF\,(\text{Seagull} + \text{Exchange})\;, \tag{640}$$

where the remaining factors are

$$\text{Seagull} \;=\; -\delta_{\text{spins}}\; \frac{(p^+_3 + p^+_{3'})\; 2\sqrt{p^+_1 p^+_{1'}}}{(p^+_{3'} - p^+_3)^2}\;, \tag{641}$$

$$\text{Exchange} \;=\; \bar{u}_1 \gamma_\nu u_{1'}\, (d^{\nu\mu}_{04}\mathcal{E}_{\mu\text{tr}} + \eta^\nu S_\eta) \left[\frac{\Delta_{LIR}}{p_4} = -\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_3}\right)\right]\;. \tag{642}$$

$$\mathcal{E}_{\mu\text{tr}} \;=\; 2\theta_{1-1'}\, S_{0433'\mu} + 2\theta_{1'-1}\, S^*_{043'3\mu}\;, \tag{643}$$

$$S_\eta \;=\; \frac{m^2_g}{p^{+2}_4}\; (p^+_3 + p^+_{3'})\; \varepsilon^*_3 \varepsilon_{3'}\;, \tag{644}$$

$$d^{\nu\mu}_{04} \;=\; -g^{\nu\mu} + \frac{p^\nu_{04}\eta^\mu + \eta^\nu p^\mu_{04}}{p^+_4}\;, \tag{645}$$

$$p^\nu_{04} \;=\; q^\nu_1 + \frac{1}{2}\,\eta^\nu (p^-_{04} - q^-_1)\;,\; p^\mu_{04} \;=\; q^\mu_{03} + \frac{1}{2}\,\eta^\mu(p^-_{04} - q^-_{03})\;. \tag{646}$$

I can write

$$d^{\nu\mu}_{04} \;=\; -g^{\nu\mu} + \frac{1}{p^+_4}\left[q^\nu_1 + \frac{1}{2}\,\eta^\nu(p^-_{04} - q^-_1)\right]\eta^\mu + \frac{1}{p^+_4}\eta^\nu\left[q^\mu_{03} + \frac{1}{2}\,\eta^\mu(p^-_{04} - q^-_{03})\right] \tag{647}$$

Contraction of q_1 with $\bar{u}_1\gamma_\nu u_{1'}$ yields 0. Contraction of q_{03} with $\mathcal{E}_{\mu\text{tr}}$ gives

$$q^\mu_{03}\mathcal{E}_{\mu\text{tr}} \;=\; 2\theta_{1-1'}\, q^\mu_{03} S_{0433'\mu} + 2\theta_{1'-1}\, q^\mu_{03} S^*_{043'3\mu} \;=\; -2\varepsilon^*_3\varepsilon_{3'}\; p_{03}p_{03'} \;=\; 2\delta_{\text{spin}3}\, p_{03}p_{03'}\;, \tag{648}$$

$$-2p_{03}p_{03'} \;=\; (p_{03} - p_{03'})^2 \;=\; [(p_3 - p_{3'}) + \eta m^2_g/(2p^+_{3'}) - \eta m^2_g/(2p^+_3)]^2 \tag{649}$$

$$= (p_3 - p_{3'})^2 + m^2_g (p^+_3 - p^+_{3'})(1/p^+_{3'} - /p^+_3) \tag{650}$$

$$= (p_3 - p_{3'})^2 + \frac{m^2_g}{p^+_{3'}p^+_3}(p^+_3 - p^+_{3'})^2 \;=\; q^2_3 + \frac{m^2_g}{p^+_{3'}p^+_3}(p^+_3 - p^+_{3'})^2\;. \tag{651}$$

Therefore, using $j_{1\nu} = \bar{u}_1 \gamma_\nu u_{1'}$,

$$
\begin{aligned}
j_{1\nu}\,(d_{04}^{\nu\mu}\mathcal{E}_{\mu\rm tr} + \eta^\nu S_\eta) &= -j_{1\nu} g^{\nu\mu}\mathcal{E}_{\mu\rm tr} + j_{1\nu}\frac{1}{p_4^+}\left[\frac{1}{2}\,\eta^\nu(p_{04}^- - q_1^-)\right]\eta^\mu\mathcal{E}_{\mu\rm tr} \\
&+\; j_{1\nu}\frac{1}{p_4^+}\eta^\nu\left[q_{03}^\mu + \frac{1}{2}\,\eta^\mu(p_{04}^- - q_{03}^-)\right]\mathcal{E}_{\mu\rm tr} + j_{1\nu}\eta^\nu S_\eta \qquad (652)\\
&= -j_{1\nu} g^{\nu\mu}\mathcal{E}_{\mu\rm tr} + \frac{j_1^+}{p_4^+}\,\frac{1}{2}\,(p_{04}^- - q_1^-)\mathcal{E}_{\rm tr}^+ + \frac{j_1^+}{p_4^+}\,\left[q_{03}^\mu\mathcal{E}_{\mu\rm tr} + \frac{1}{2}\,(p_{04}^- - q_{03}^-)\mathcal{E}_{\rm tr}^+\right] + j_1^+ S_\eta \qquad (653)\\
&= -j_{1\mu}\mathcal{E}_{\rm tr}^\mu + \frac{j_1^+}{p_4^+}T\;. \qquad (654)
\end{aligned}
$$

The term $-j_{1\mu}\mathcal{E}_{\rm tr}^\mu$ is free from small-x singularities.

$$
\begin{aligned}
\text{Exchange} &= j_{1\nu}\,(d_{04}^{\nu\mu}\mathcal{E}_{\mu\rm tr} + \eta^\nu S_\eta)\left[\frac{\Delta_{LIR}}{p_4} = -\,\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_3}\right)\right] \qquad (655)\\
&= \left(-j_{1\mu}\mathcal{E}_{\rm tr}^\mu + \frac{j_1^+}{p_4^+}T\right)\left[\frac{\Delta_{LIR}}{p_4} = -\,\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_3}\right)\right] \qquad (656)
\end{aligned}
$$

$$
T = \frac{1}{2}\,(p_{04}^- - q_1^-)\mathcal{E}_{\rm tr}^+ + q_{03}^\mu\mathcal{E}_{\mu\rm tr} + \frac{1}{2}\,(p_{04}^- - q_{03}^-)\mathcal{E}_{\rm tr}^+ + p_4^+ S_\eta\;. \qquad (657)
$$

$$
\begin{aligned}
S_{0433'}^+ &= \varepsilon_3^*\varepsilon_{3'}\; p_{3'}^+ \;=\; -\delta_{\rm spin3}\, p_{3'}^+\;,\;\; S_{043'3}^{+*} \;=\; \varepsilon_{3'}\varepsilon_3^*\; p_3^+ \;=\; -\delta_{\rm spin3}\, p_3^+\;, \qquad (658)\\
\mathcal{E}_{\rm tr}^+ &= 2\theta_{1-1'}\, S_{0433'}^+ + 2\theta_{1'-1}\, S_{043'3}^{+*} \;=\; -2\theta_{1-1'}\;\delta_{\rm spin3}\, p_{3'}^+ - 2\theta_{1'-1}\;\delta_{\rm spin3}\, p_3^+ \qquad (659)\\
&= -2p_3^+\;\delta_{\rm spin3}\;, \qquad (660)\\
q_{03}^\mu\mathcal{E}_{\mu\rm tr} &= 2\delta_{\rm spin3}\, p_{03}p_{03'}\;, \qquad (661)\\
p_4^+ S_\eta &= -\frac{m_g^2}{p_4^+}\;(p_3^+ + p_{3'}^+)\;\delta_{\rm spin3}\;. \qquad (662)
\end{aligned}
$$

$$
T = \frac{1}{2}\,(p_{04}^- - q_1^-)(-2p_3^+\;\delta_{\rm spin3}) + 2\delta_{\rm spin3}\, p_{03}p_{03'} + \frac{1}{2}\,(p_{04}^- - q_{03}^-)(-2p_3^+\;\delta_{\rm spin3}) - \frac{m_g^2}{p_4^+}\;(p_3^+ + p_{3'}^+)\;\delta_{\rm spin3}\;. \qquad (663)
$$

$$
\begin{aligned}
T &= T_s\;\delta_{\rm spin3}\;, \qquad (664)\\
T_s &= -(p_{04}^- - q_1^- + p_{04}^- - q_{03}^-)p_3^+ + 2p_{03}p_{03'} - \frac{m_g^2}{p_4^+}\;(p_3^+ + p_{3'}^+)\;. \qquad (665)
\end{aligned}
$$

*** Remaining

$$
\text{Seagull} = -\delta_{\rm spins}\;\frac{(p_3^+ + p_{3'}^+)\;2\sqrt{p_1^+ p_{1'}^+}}{(p_{3'}^+ - p_3^+)^2} = -\delta_{\rm spins}\;\frac{(p_3^+ + p_{3'}^+)\;j_1^+}{(p_{3'}^+ - p_3^+)^2}\;. \qquad (666)
$$

$$
\begin{aligned}
\text{Exchange} &= j_{1\nu}\,(d_{04}^{\nu\mu}\mathcal{E}_{\mu\rm tr} + \eta^\nu S_\eta)\left[\frac{\Delta_{LIR}}{p_4} = -\,\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_3}\right)\right] \qquad (667)\\
&= \left(-j_{1\mu}\mathcal{E}_{\rm tr}^\mu + \frac{j_1^+}{p_4^+}T\right)\left(\frac{1}{-2\rho_1} + \frac{1}{-2\rho_3}\right) \qquad (668)
\end{aligned}
$$

The term

$$
-j_{1\mu}\mathcal{E}_{\rm tr}^\mu\left(\frac{1}{-2\rho_1} + \frac{1}{-2\rho_3}\right) \qquad (669)
$$

is free from small-x singularities. It does involve transverse momenta, limited by the RGPEP factor f_{LR} and the test wave functions. I include $\delta_{\rm spins}$ and j_1^+ in the common factors

$$CF \;\to\; g^2\; f_{LR} \;\int [11'33']\; \tilde{\delta}_{c.a}\; \chi_1^\dagger T^{c_4} \chi_{1'}\; i f^{c_4 c_3 c_{3'}}\; \delta_{\rm spins}\; j_1^+\; a_3^\dagger b_1^\dagger b_{1'} a_{3'}\; . \tag{670}$$

The terms that diverge for $p_4 \to 0$ in the sum Seagull + Exchange have the form

$$CF\,({\rm Seagull + Exchange}) \;=\; CF \left[-\frac{p_3^+ + p_{3'}^+}{p_4^{+2}} + \frac{1}{p_4^+} T_s \left(\frac{1}{-2\rho_1} + \frac{1}{-2\rho_3} \right) \right] \tag{671}$$

$$= \; CF \left\{ -\frac{p_3^+ + p_{3'}^+}{p_4^{+2}} + \frac{1}{p_4^+} \left[-(p_{04}^- - q_1^- + p_{04}^- - q_{03}^-) p_3^+ + 2 p_{03} p_{03'} - \frac{m_g^2}{p_4^+}\,(p_3^+ + p_{3'}^+) \right] \left(\frac{1}{-2\rho_1} + \frac{1}{-2\rho_3} \right) \right\} . \tag{672}$$

D. Evaluation of quark-gluon interaction in Eqs. (671) and (672)

I rewrite {...} in Eq.(672),

$$\{...\} \;=\; -\frac{p_3^+ + p_{3'}^+}{p_4^{+2}} + \left[(p_{04}^- - q_1^- + p_{04}^- - q_{03}^-) p_3^+ - 2 p_{03} p_{03'} + \frac{m_g^2}{p_4^+}\,(p_3^+ + p_{3'}^+) \right] \frac{\rho_1 + \rho_3}{2\rho_1 \rho_3 p_4^+} \tag{673}$$

$$= \; \theta_{1-1'} A + \theta_{1'-1} B \; . \tag{674}$$

$$A \;=\; -\frac{p_3^+ + p_{3'}^+}{p_4^{+2}} + \left[(p_{04}^- - q_1^- + p_{04}^- - q_{03}^-) p_{3'}^+ - 2 p_{03} p_{03'} + \frac{m_g^2}{p_4^+}\,(p_3^+ + p_{3'}^+) \right]_A \frac{\rho_1 + \rho_3}{2\rho_1 \rho_3 p_4^+} \; , \tag{675}$$

$$B \;=\; -\frac{p_3^+ + p_{3'}^+}{p_4^{+2}} + \left[(p_{04}^- - q_1^- + p_{04}^- - q_{03}^-) p_3^+ - 2 p_{03} p_{03'} + \frac{m_g^2}{p_4^+}\,(p_3^+ + p_{3'}^+) \right]_B \frac{\rho_1 + \rho_3}{2\rho_1 \rho_3 p_4^+} \; . \tag{676}$$

$$[...]_A \;=\; \left(2p_4^- - 2m_g^2/p_4^+ - q_1^- - p_{3'}^- + m_g^2/p_{3'}^+ + p_3^- - m_g^2/p_3^+ \right) p_{3'}^+ - 2 p_{03} p_{03'} + \frac{m_g^2}{p_4^+}\,(p_3^+ + p_{3'}^+) \tag{677}$$

$$= \; (2p_4^- - q_1^- - q_3^-) p_{3'}^+ - 2 m_g^2 p_{3'}^+ / p_4^+ + (m_g^2/p_{3'}^+ - m_g^2/p_3^+) p_{3'}^+ - 2 p_{03} p_{03'} + \frac{m_g^2}{p_4^+}\,(p_3^+ + p_{3'}^+) \tag{678}$$

$$= \; (2p_4^- - q_1^- - q_3^-) p_{3'}^+ + \frac{m_g^2}{p_3^+} (p_3^+ - p_{3'}^+) - 2 p_{03} p_{03'} - \frac{m_g^2}{p_4^+} p_4^+ \tag{679}$$

$$= \; (p_4^- - q_1^- + p_4^- - q_3^-) p_4^+ \frac{p_{3'}^+}{p_4^+} - m_g^2 \frac{p_{3'}^+}{p_3^+} - 2 p_{03} p_{03'} \tag{680}$$

$$= \; \left(m_g^2 + q_1^{\perp 2} - q_1^- q_1^+ + m_g^2 + q_3^{\perp 2} - q_3^- q_3^+ \right) \frac{p_{3'}^+}{p_4^+} - m_g^2 \frac{p_{3'}^+}{p_3^+} - 2 p_{03} p_{03'} \tag{681}$$

$$= \; (\rho_1 + \rho_3) \frac{p_{3'}^+}{p_4^+} - m_g^2 \frac{p_{3'}^+}{p_3^+} - 2 p_{03} p_{03'} \tag{682}$$

$$[...]_B \;=\; (p_{04}^- - q_1^- + p_{04}^- - q_{03}^-)p_3^+ - 2p_{03}p_{03'} + \frac{m_g^2}{p_4^+}\,(p_3^+ + p_{3'}^+) \tag{683}$$

$$= \left(2p_4^- - 2m_g^2/p_4^+ - q_1^- - p_3^- + m_g^2/p_3^+ + p_{3'}^- - m_g^2/p_{3'}^+\right)p_3^+ - 2p_{03}p_{03'} + \frac{m_g^2}{p_4^+}\,(p_3^+ + p_{3'}^+) \tag{684}$$

$$= \left(2p_4^- - q_1^- - p_3^- + p_{3'}^-\right)p_4^+ p_3^+/p_4^+ + m_g^2(p_{3'}^+ - p_3^+)/p_{3'}^+ - 2p_{03}p_{03'} + \frac{m_g^2}{p_4^+}\,(-p_3^+ + p_{3'}^+) \tag{685}$$

$$= \left(2p_4^- - q_1^- - q_3^-\right)p_4^+ p_3^+/p_4^+ + m_g^2(p_{3'}^+ - p_3^+)/p_{3'}^+ - 2p_{03}p_{03'} - \frac{m_g^2}{p_4^+}p_4^+ \tag{686}$$

$$= (\rho_1 + \rho_3)\,p_3^+/p_4^+ + m_g^2(p_{3'}^+ - p_3^+)/p_{3'}^+ - 2p_{03}p_{03'} - \frac{m_g^2}{p_4^+}p_4^+ \tag{687}$$

$$= (\rho_1 + \rho_3)\,\frac{p_3^+}{p_4^+} - m_g^2\frac{p_3^+}{p_{3'}^+} - 2p_{03}p_{03'}\ . \tag{688}$$

With these results for the brackets,

$$A \;=\; -\frac{p_3^+ + p_{3'}^+}{p_4^{+2}} + \left[(\rho_1+\rho_3)\,\frac{p_{3'}^+}{p_4^+} - m_g^2\frac{p_{3'}^+}{p_3^+} - 2p_{03}p_{03'}\right]_A \frac{\rho_1+\rho_3}{2\rho_1\rho_3 p_4^+}\ , \tag{689}$$

$$B \;=\; -\frac{p_3^+ + p_{3'}^+}{p_4^{+2}} + \left[(\rho_1+\rho_3)\,\frac{p_3^+}{p_4^+} - m_g^2\frac{p_3^+}{p_{3'}^+} - 2p_{03}p_{03'}\right]_B \frac{\rho_1+\rho_3}{2\rho_1\rho_3 p_4^+}\ . \tag{690}$$

In A, with $\theta_{1-1'}$,

$$-2p_{03}p_{03'} \;=\; -2[p_3 - \eta m_g^2/2p_3^+)][(p_{3'} - \eta m_g^2/(2p_{3'}^+)] \;=\; -2p_3p_{3'} + m_g^2\frac{p_3^+}{p_{3'}^+} + m_g^2\frac{p_{3'}}{p_3^+} \tag{691}$$

$$= (p_{3'} - p_3)^2 - 2m_g^2 + m_g^2\frac{p_3^+}{p_{3'}^+} + m_g^2\frac{p_{3'}}{p_3^+} \;=\; (p_{3'} - p_3)^2 + m_g^2\frac{(p_{3'}^+ - p_3^+)^2}{p_3^+p_{3'}^+} \tag{692}$$

$$= (p_{3'} - p_3)^2 + m_g^2\frac{p_4^{+2}}{p_3^+p_{3'}^+} \;=\; q_3^2 + m_g^2\frac{p_4^{+2}}{p_3^+p_{3'}^+}\ . \tag{693}$$

In B, with $\theta_{1'-1}$,

$$-2p_{03}p_{03'} \;=\; (p_3 - p_{3'})^2 + m_g^2\frac{p_4^{+2}}{p_3^+p_{3'}^+} \;=\; q_3^2 + m_g^2\frac{p_4^{+2}}{p_3^+p_{3'}^+}\ . \tag{694}$$

Thus,

$$A \;=\; -\frac{p_3^+ + p_{3'}^+}{p_4^{+2}} + \left[(\rho_1+\rho_3)\,\frac{p_{3'}^+}{p_4^+} - m_g^2\frac{p_{3'}^+}{p_3^+} + q_3^2 + m_g^2\frac{p_4^{+2}}{p_3^+p_{3'}^+}\right]_A \frac{\rho_1+\rho_3}{2\rho_1\rho_3 p_4^+}\ , \tag{695}$$

$$B \;=\; -\frac{p_3^+ + p_{3'}^+}{p_4^{+2}} + \left[(\rho_1+\rho_3)\,\frac{p_3^+}{p_4^+} - m_g^2\frac{p_3^+}{p_{3'}^+} + q_3^2 + m_g^2\frac{p_4^{+2}}{p_3^+p_{3'}^+}\right]_B \frac{\rho_1+\rho_3}{2\rho_1\rho_3 p_4^+}\ . \tag{696}$$

****
****

****

E. Old text discussing the 3-body space

So far I developed an equation in the dominant 2-body component alone. But there also exist terms of 2nd order due to the components with one effective gluon. If I used perturbative expansion up to 2nd order only, I would return to the 2body equation like in QED with the coupling constant α multiplied by the color factor. But gluons are different from photons. They are blocked already in 2nd order by the gluon-quark interactions and more interactions . This is the same as if one omitted other components than the dominant one. But if we want terms of second order, we can do so with mass $m_g \to m_G$ in 3-body sector. I think I finally understand what I am doing.

The interaction terms in H_s change the number of effective particles with much less strength than indicated solely by the smallness of the coupling constant. The reason is that the vertex form factors limit the range of interactions in the FF energy. Nevertheless, in addition to the dominant quark-antiquark component, a quarkonium state includes also the components containing a quark-antiquark pair and 1 effective gluon, 2 gluons, and so on. Components with more heavy quarks than 2 are insignificant when the quark mass is large. Together, all these components form a vector in an *a priori* infinite-dimensional space. Corrections to the dynamics of the dominant effective 2-body component due to these additional components can be computed using the general expression for an effective Hamiltonian acting only in a subspace of states [36]. We discuss the lowest order, using the 2nd-order of the weak-coupling expansion. **Insert here a comment on m_G**

$$ {}_s\langle 12|H_s|1'2'\rangle_s \;=\; {}_s\langle 12| \left\{ H_f + H_s^{(2)} + \frac{1}{2} H_s^{(1)} \left[(P_{12}^- - H_f)^{-1} + (P_{1'2'}^- - H_f)^{-1} \right] H_t^{(1)} \right\} |1'2'\rangle_s \; . \tag{697} $$

The same matrix is obtained using the operator $\mathcal{H}$ and bare states $|12\rangle = b_1^\dagger d_2^\dagger |0\rangle$,

$$ \langle 12|\mathcal{H}|1'2'\rangle \;=\; \langle 12| \left\{ \mathcal{H}_f + \mathcal{H}^{(2)} + \frac{1}{2} \mathcal{H}^{(1)} \left[(P_{12}^- - \mathcal{H}_f)^{-1} + (P_{1'2'}^- - \mathcal{H}_f)^{-1} \right] \mathcal{H}^{(1)} \right\} |1'2'\rangle \; . \tag{698} $$

XI. EIGENVALUE PROBLEMS FOR $m_g \to 0$

The RGPEP form factor $f^t_{\tilde{p},\tilde{q}}$ in Eq. (**??**), associated with emission or absorption of a gluon with the FF energy p_g^-, behaves for extremely small p_g^+ as

$$ f^t_{\tilde{p},\tilde{q}} \;\sim\; e^{-t^2\,(p_g^-\,\pi_{ca})^2} \;=\; e^{-t^2\,[(p_g^{\perp\,2}+m_g^2)\,\pi_{ca}/p_g^+]^2} \; . \tag{699} $$

For $|p_g^\perp| \gg m_g$, the limit $m_g \to 0$ presents no problem. But when for $|p_g^\perp| \lesssim m_g$, which is the infrared regime, the form factor suppresses contributions from gluons with small momentum p_g^+ by a factor of the type

$$ f^t_{\tilde{p},\tilde{q}} \;\sim\; e^{-\epsilon^2/x^2} \; , \; \epsilon \;=\; t\, m_g^2\, \pi_{ca}/P^+ \; , \tag{700} $$

where P^+ is the momentum of a system under consideration and x is the fraction of that momentum carried by the gluon. The fraction x is thus effectively limited from below by a number on the order of ϵ. For a fixed gluon mass parameter m_g, the severe divergences due to the gluon factors $1/x^2$ are thus tamed in a way that is qualitatively illustrated by the formula

$$ \int_\epsilon^{x_0} dx/x^2 \;=\; 1/\epsilon - 1/x_0 \; , \tag{701} $$

where x_0 is some positive number smaller than 1. This result is finite but diverges as $1/m_g^2$ when $m_g \to 0$. When the divergences due to the factors $1/x^2$ cancel out, one is left with a logarithmic dependence on ϵ,

$$ \int_\epsilon^{x_0} dx/x \;=\; \log(x_0/\epsilon) \; . \tag{702} $$

We inspect the logarithmic dependence of the quarkonium eigenvalue problem for the Hamiltonian H_{LR} given in Eq. (**??**). In particular, we examine the relationship between the quarkonium and single-quark dynamics. The relationship holds through the quark self-interactions.

XII. SINGLE-QUARK EIGENVALUE PROBLEM FOR $m_g \to 0$

The single-quark eigenvalue problem involes self-interaction terms. As observed in discussion of Eqs. (86) and Eq. (**??**), the counterterm integrand $\sigma^{(2)} = 1/x^2$ is sufficient to remove the 2nd-order severe small-x divergence from the self-interactions in scattering amplitudes and bound-state eigenvalue problems when $m_g \neq 0$. Moreover, the modification $\sigma^{(2)} \to \sigma^{(2)} + 1/[2(1-x)]$ yields simultaneous cancellation of the ultraviolet transverse quadratic divergence. Thus the divergences $\sim 1/r$ in the quark self-interactions for $r \to 0$ are removed by the counterterm $\delta m^2_{i\,\rm sev}$ given in Eq. (**??**). The remaining logarithmic divergence requires an additional counterterm $\delta m^2_{i\,\ln}$, see Sec. **??**.

If free quarks with a finite, definite masses existed, one could demand that these masses solve the eigenvalue Eq. (101) for states with the single-quark quantum numbers. But the single quarks are not observed, which might be explained if the quark mass eigenvalues are sufficiently large. The single-quark eigenstates must have the form

$$|\psi\rangle \ = \ |1\rangle + |2\rangle + ... \ , \tag{703}$$

where the component $|1\rangle$ is a state of one effective quark, $|2\rangle$ is a state with addition of one effective gluon and the dots indicate components with more effective particles. Using the single-quark counterpart of Eq. (**??**), one obtains the eigenvalue equation for the component $|1\rangle$,

$$H_1 |1\rangle \ = \ p^-_E |1\rangle \ , \tag{704}$$

where

$$H_1 \ = \ H_f + H^{(2)}_{t11} + H^{(1)}_{t12}\,(p^- - H_f)^{-1} H^{(1)}_{t21} \ . \tag{705}$$

The effective quark in the state $|1\rangle$ carries the kinematic momentum with components p^+ and $p^\perp$. So, the eigenvalue is written in the form

$$p^-_E \ = \ (p^{\perp 2} + m^2_E)/p^+ \ . \tag{706}$$

Given the RGPEP solution for the Hamiltonian, one obtains the eigenvalue condition

$$\begin{aligned} &\mathcal{H}_{f1} + f_{11}\left[\mathcal{H}^{(2)}_{0,11} + \Delta_{121}\right] \mathcal{H}^{(1)}_{0,12}\,\mathcal{H}^{(1)}_{0,21} \\ &+ \ f_{12}\mathcal{H}^{(1)}_{0,12}\left[(p^- - \mathcal{H}_{f2})^{-1} - \Delta_{121}\right] f_{21}\mathcal{H}^{(1)}_{0,21} \ = \ p^-_E \ , \end{aligned} \tag{707}$$

where

$$\mathcal{H}_{f1} \ = \ p^-_1 \ = \ (m^2_i + p^{\perp 2})/p^+ \qquad f_{11} \ = \ 1 \ , \tag{708}$$

$$\mathcal{H}^{(2)}_{0,11} \ = \ \delta m^2_{i\,\rm tot}/p^+ \quad \text{see Eq. (??)} \ , \tag{709}$$

$$f_{12}f_{21} \ = \ f^2_{12} \quad , \quad \Delta_{121} \ = \ (p^-_1 - p^-_q - p^-_g)^{-1} \ , \tag{710}$$

$$(p^- - \mathcal{H}_{f2})^{-1} \ = \ \left[(p^-_E \text{ or } p^-_1) - p^-_q - p^-_G\right]^{-1} \ . \tag{711}$$

The eigenvalue p^-_E in the denominator results from the TD procedure. If the eigenvalue were infinite, the corresponding term would vanish and the other terms would have to yield the infinite result for the eigenvalue. One can also consider the operation R, *e.g.* see [Wilson], which in second order of expansion in powers of the coupling constant yields p^-_1 in place of p^-_E in the denominator. We pursue the latter case since the operation R is precisely defined order-by-order and does not require knowledge of the eigenvalue. The notation p_g and p_G for the gluon momenta indicates that $p^2_g = m^2_g$ and $p^2_G = m^2_{iG}$. The gluon-mass ansatz m_{iG} carries the subscript i to indicate its dependence on the quark mass m_i.

A. Gluon mass ansatz in single-quark eigenstates

The single-quark eigenvalue condition of Eq. (707) implies

$$m^2_i + \delta m_{i\,\rm tot} + \Sigma^{(2)}_q + p^+ \Delta p^-_E \ = \ m^2_E \ , \tag{712}$$

where $\delta m^2_{i\,\rm tot}$ is the counterterm given in Eq. (**??**), $\Sigma^{(2)}_q$ is the self-interaction term of Eq. (**??**), and

$$\Delta p^-_E \ = \ f_{12}\mathcal{H}^{(1)}_{0,12}\left(\frac{1}{p^- - p^-_q - p^-_G} - \frac{1}{p^- - p^-_q - p^-_g}\right) f_{21}\,\mathcal{H}^{(1)}_{0,21} \ . \tag{713}$$

B. Computation of Δp_E^- dependence on m_g

In detail,

$$\Delta p_E^- \;=\; \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3 x(1-x)p^+} \left(\frac{N\; f^{t2}_{p_q p_g, p}}{p^- - p_q^- - p_g^- - A/p^+} - \frac{N\; f^{t2}_{p_q p_g, p}}{p^- - p_q^- - p_g^-} \right) \;, \tag{714}$$

$$A \;=\; (m_{iG}^2 - m_g^2)/x \;,\;\; B \;=\; (1-x)(m_{iG}^2 - m_g^2) \;=\; x(1-x)A \;, \tag{715}$$

$$N \;\equiv\; N_g \;=\; 4[k^{\perp 2} + (1-x)m_g^2]/x^2 + 2(k^{\perp 2} + x^2 m_i^2)/(1-x) \;, \tag{716}$$

$$N_G \;\equiv\; 4[k^{\perp 2} + (1-x)m_{iG}^2]/x^2 + 2(k^{\perp 2} + x^2 m_i^2)/(1-x) \;, \tag{717}$$

$$N_g \;=\; N_G - 4(1-x)(m_{iG}^2 - m_g^2)/x^2 \;=\; N_G - 4B/x^2 \;, \tag{718}$$

$$D \;\equiv\; D_g \;=\; x(1-x)p^+(p^- - p_q^- - p_g^-) \;=\; x(1-x)(m_i^2 - \mathcal{M}_{qg}^2) \;, \tag{719}$$

$$D_g \;=\; -[k^{\perp 2} + x^2 m_i^2 + (1-x)m_g^2] \;, \tag{720}$$

$$D_G \;\equiv\; x(1-x)p^+(p^- - p_q^- - p_G^-) \;=\; x(1-x)(m_i^2 - \mathcal{M}_{qG}^2) \tag{721}$$

$$\;=\; D_g + x(1-x)p^+(p_g - p_G) \;=\; D_g - B \;, \tag{722}$$

$$D_G \;=\; -[k^{\perp 2} + x^2 m_i^2 + (1-x)m_{iG}^2] \;. \tag{723}$$

Then

$$\Delta p_E^- \;=\; \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3} \left(\frac{N_G - 4B/x^2}{D_G} - \frac{N_g}{D_g} \right) f^{t2}_{p_q p_g, p} \tag{724}$$

$$\;=\; \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3} \left(\frac{N_G}{D_G} - \frac{N_g}{D_g} - \frac{4}{x^2}\,\frac{B}{D_G} \right) f^{t2}_{p_q p_g, p} \;. \tag{725}$$

Now I can use the formulas

$$\sigma_q^{(2)} \;=\; \frac{1}{x^2} + \frac{1}{2(1-x)} \;, \tag{726}$$

$$\frac{N_g}{D_g} + 4\sigma_q^{(2)} \;=\; \frac{4m_i^2 + 2m_g^2}{-D_g} \;, \tag{727}$$

$$\frac{N_G}{D_G} + 4\sigma_q^{(2)} \;=\; \frac{4m_i^2 + 2m_{iG}^2}{-D_G} \;. \tag{728}$$

I write

$$\Delta p_E^- \;=\; \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3} \left(\frac{4m_i^2 + 2m_{iG}^2}{-D_G} - \frac{4m_i^2 + 2m_g^2}{-D_g} - \frac{4}{x^2}\,\frac{B}{D_G} \right) f^{t2}_{p_q p_g, p} \;, \tag{729}$$

combined with the change of integration variables in Eq. (**??**),

$$\int [xk] \;=\; \int \frac{dx\, d^2k^\perp}{2(2\pi)^3 x(1-x)} \;=\; \int \frac{d^3k}{2(2\pi)^3} \left(\frac{1}{E_g} + \frac{1}{E_q} \right) \;=\; \int \frac{d\Omega\, k\, dE}{2(2\pi)^3} \;, \tag{730}$$

$$\tag{731}$$

$$\int \frac{dx\, d^2k^\perp}{2(2\pi)^3} \;=\; \int \frac{d\Omega\, k\, dE}{2(2\pi)^3}\; x(1-x) \;. \tag{732}$$

The change of variables is made only for gluons with m_g, even though a more advanced calculations with running gluon mass may lead to variables proper for the gluons with effective mass. The factor $x(1-x)$ cancels out and the eigenvalue contribution is

$$\begin{aligned}\Delta p_E^- &= \frac{g^2 C_F}{p^+}\int_{m_i+m_g}^{\infty}\frac{d\Omega\, k\, dE}{2(2\pi)^3}\; e^{-2t^2(E^2-m_i^2)^2(\pi^+/p^+)^2} \\ &\times \left[\frac{4m_i^2+2m_{iG}^2}{\mathcal{M}_{qG}^2-m_i^2}-\frac{4m_i^2+2m_g^2}{\mathcal{M}_{qg}^2-m_i^2}+\frac{4}{x^2}\,\frac{B}{\mathcal{M}_{qG}^2-m_i^2}\right] . \end{aligned} \tag{733}$$

In terms of E, with $M^2 = m_{iG}^2 - m_g^2$,

$$\begin{aligned}\Delta p_E^- &= \frac{g^2 C_F}{p^+}\int_{m_i+m_g}^{\infty}\frac{d\Omega\, k\, dE}{2(2\pi)^3}\; e^{-2t^2(E^2-m_i^2)^2(\pi^+/p^+)^2} \\ &\times \left[\frac{4m_i^2+2m_{iG}^2}{E^2-m_i^2+M^2/x}-\frac{4m_i^2+2m_g^2}{E^2-m_i^2}+\frac{4}{x^2}\,\frac{(1-x)M^2}{E^2-m_i^2+M^2/x}\right] . \end{aligned} \tag{734}$$

The question is how this correction depends on $m_g \to 0$. Does Δp_E^- diverge?

$$\begin{aligned}\Delta p_E^- &= \frac{g^2 C_F}{p^+}\int_{m_i+m_g}^{\infty}\frac{d\Omega\, k\, dE}{2(2\pi)^3}\; e^{-2t^2(E^2-m_i^2)^2(\pi^+/p^+)^2} \\ &\times \left[\frac{x(4m_i^2+2m_{iG}^2)}{M^2+x(E^2-m_i^2)}-\frac{4m_i^2+2m_g^2}{E^2-m_i^2}+\frac{4M^2(1-x)/x}{M^2+x(E^2-m_i^2)}\right] . \end{aligned} \tag{735}$$

The key are the relations between x, k and E.

$$x = (E_g + k\cos\theta)/E = \frac{\sqrt{m_g^2+k^2}+k\cos\theta}{E_g+E_q} = \frac{\sqrt{m_g^2+k^2}+k\cos\theta}{\sqrt{m_g^2+k^2}+\sqrt{m_i^2+k^2}} . \tag{736}$$

The momentum $k = |\vec{k}\,|$ is limited because E is limited by the RGPEP form factor. There is also the bound $E \geq m_i + m_g$.

The first term in the bracket has a finite limit when $m_g \to 0$ because $0 < x < 1$. The last term can be simplified by dropping $-4M^2$ in its numerator as not diverging. If there is a singularity it must come from the part of the square bracket,

$$\begin{aligned}\Delta p_{E2}^- &= \frac{g^2 C_F}{p^+}\int_{m_i+m_g}^{\infty}\frac{d\Omega\, k\, dE}{2(2\pi)^3}\; e^{-2t^2(E^2-m_i^2)^2(\pi^+/p^+)^2} \\ &\times \left[-\frac{4m_i^2+2m_g^2}{E^2-m_i^2}+\frac{4M^2/x}{M^2+x(E^2-m_i^2)}\right] . \end{aligned} \tag{737}$$

The first term in the bracket can get large when $E \to m_i + m_g$. This happens only when $k \to 0$, where

$$E^2 - m_i^2 = (E+m_i)(E-m_i) \sim (2m_i + E_g)(E_g + E_q - m_i) . \tag{738}$$

The leading denominator term for small k is then $2m_i E_g$, while in the numerator of the integral there is k. The ratio $k/\sqrt{m_g^2+k^2} = 1/\sqrt{1+m_g^2/k^2} < 1$ for finite k. One is left only with the term

$$\begin{aligned}\Delta p_{E1}^- &= \frac{g^2 C_F}{p^+}\int_{m_i+m_g}^{\infty}\frac{d\Omega\, k\, dE}{2(2\pi)^3}\; e^{-2t^2(E^2-m_i^2)^2(\pi^+/p^+)^2} \\ &\times \frac{4M^2/x}{M^2+x(E^2-m_i^2)} . \end{aligned} \tag{739}$$

The only source of divergence can be small x. For sizable M, actually, only in the limit $M \to \infty$, the only source of divergence is

$$\Delta p_{E0}^- = \frac{g^2 C_F}{p^+}\int_{m_i+m_g}^{\infty}\frac{d\Omega\, k\, dE}{2(2\pi)^3}\; e^{-2t^2(E^2-m_i^2)^2(\pi^+/p^+)^2}\;\left[\frac{4}{x}=\frac{4E}{E_g+k\cos\theta}\right] . \tag{740}$$

Carrying out the integrals over angles I get

$$\Delta p^-_{E0} = \frac{g^2 C_F}{p^+} \int_{m_i+m_g}^\infty \frac{2\pi\, 4E dE}{2(2\pi)^3}\; e^{-2t^2(E^2-m_i^2)^2(\pi^+/p^+)^2} \left[\int_{-1}^1 \frac{k dz}{E_g+kz} = \ln\frac{E_g+k}{E_g-k}\right] . \tag{741}$$

$$\ln\frac{E_g+k}{E_g-k} = \ln\frac{(E_g+k)^2}{m_g^2} \;,\; dE = (k/E_g+k/E_q)dk \to (1+k/E_q)dk \;. \tag{742}$$

In the limit $m_g \to 0$,

$$\Delta p^-_{E0} = \frac{g^2 C_F}{p^+} \int_{m_i}^\infty \frac{2\pi\, 4E dE}{2(2\pi)^3}\; e^{-2t^2(E^2-m_i^2)^2(\pi^+/p^+)^2} 2\ln\frac{2k}{m_g} \;. \tag{743}$$

In the limit I have

$$2k = (E^2-m_i^2)/E \;,\; u = E^2 - m_i^2 \;. \tag{744}$$

$$\Delta p^-_{E0} = \frac{g^2 C_F}{2\pi^2 p^+} \int_0^\infty du\, e^{-2t^2u^2(\pi^+/p^+)^2} \ln\frac{u}{m_g\sqrt{u+m_i^2}} \;. \tag{745}$$

Using $u = v m_i^2$

$$\Delta p^-_{E0} = \frac{g^2 C_F}{2\pi^2 p^+} m_i^2 \int_0^\infty dv\, e^{-2t^2 m_i^4(\pi^+/p^+)^2 v^2}\; \ln\frac{m_i v}{m_g\sqrt{1+v^2}} \;. \tag{746}$$

Using $a = \sqrt{2}\, t m_i^2 \pi^+/p^+$,

$$\Delta p^-_{E0} = \frac{g^2 C_F}{2\pi^2 p^+} \frac{m_i^2}{a} \int_0^\infty a\, dv\, e^{-(av)^2}\; \ln\frac{m_i v}{m_g\sqrt{1+v^2}} \;. \tag{747}$$

Finally, using $w = av$,

$$\Delta p^-_{E0} = \frac{g^2 C_F}{2\pi^2 p^+} \frac{p^+/\pi^+}{\sqrt{2}\, t} \int_0^\infty dw\, e^{-w^2} \left[\ln\frac{m_i}{m_g} + \ln\frac{w}{\sqrt{a^2+w^2}}\right] . \tag{748}$$

There is a divergence when $m_g \to 0$ of the form

$$\Delta p^-_{E0} = \frac{g^2 C_F}{2\pi^2 p^+} \frac{p^+/\pi^+}{\sqrt{2}\, t} \frac{\sqrt{\pi}}{2}\; \ln\frac{m_i}{m_g} \;. \tag{749}$$

Simplified,

$$p^+\Delta p^-_{E0} = \frac{g^2 C_F}{(2\pi)^2} \frac{p^+}{\pi^+ t} \sqrt{\frac{\pi}{2}}\; \ln\frac{m_i}{m_g} \;. \tag{750}$$

This looks almost the same as Eq. (51) for $\tilde{\mathcal{M}}_i^2$ in [K. Serafin et al., Phys. Rev. D 109, 016017 (2024)], where the tilde stands for the slanted $\mathcal{M}$ in Eq. (51). But in a product under the logarithm one can have various factors depending on the arrangement of other terms.

C. Other single-particle eigenvalue problems

It is shown in previous sections that in the case of a heavy single-quark eigenvalue problem, the eigenmass can tend to infinity when $m_g \to 0$. The divergence is logarithmic, $\ln(m_i/m_g)$. In the cases of a single-light quark and a single-gluon eigenvalue problems one faces the complication that the scale of $\Lambda_{\rm QCD}$ is much greater than the masses in H_f. In distinction from the heavy quark case, the assumption that a state of a single effective particle is the dominant component of a single-quark or single-gluon eigenstate is in doubt. The 2nd-order computation of the Hamiltonian that is reasonable for heavy quarkonia and the 3rd-order computation that demonstrates asymptotic freedom in the

Hamiltonian, do not provide the required understanding of the binding dynamics. To address the issue of how the quark and gluon operators and their interactions "dress up" in the RGPEP evolution, to find out if for large s a constituent picture emerges, one needs to solve Eq. (**??**) including at least the 4th order terms. This is the lowest order in which the running coupling enters the Hamiltonian interaction terms that count in the eigenvalue problems. The need to carry out the 4th-order computations is precisely the reason for developing a method for handling the severe small-x divergences in the presence of mass for gluons described in this paper.

The self-interaction terms in the single-quark eigenvalue problems can be adjusted to remove the ultraviolet transverse divergences. Then one can subtract the ultraviolet logarithmic divergences, to be left with a finite eigenvalue condition that depends on m_g. The issue may appear similar to the Bloch-Nordsieck issue of photon divergence in the scattering amplitudes involving electrons, whose cancellation requires the inclusion of the electron-photon contributions. **Check Bloch-Nordsieck argument** Correspondingly, in the electron eigenvalue problem the contribution with one photon is needed. But it does not include the photon mass and the term that diverges here is absent. There is no blocking of the photon. In contrast with QED, the effective gluons are assumed blocked by their non-Abelian interactions. The blocking is modeled here using the sizable gluon mass m_G in the component with one gluon, possibly representing the effects of non-Abelian interactions in all components with more quanta. Therefore, the RGPEP and Wilsonian elimination no longer cancel each other in the quark self-interaction. The resulting term diverges in the limit $m_g \to 0$. Could this be a signal of how to handle confinement, visible already in 2nd order? One can assume this is the case and proceed to the limit $m_g \to 0$ in the case of quarkonium.

XIII. ELEMENTS OF THE QUARKONIUM EIGENVALUE PROBLEM

The quark-antiquark Hamiltonian H_{LR} in Eq. (**??**) has 6 parts,

$$H_{LR} \ = \ A + B + C \ . \tag{751}$$

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$$H_{LR} \ = \ [m^2]_A + [f\delta S + fS + f\delta\Sigma + f\delta G + (f - ff)\Delta\,(\Sigma + G)]_B + [ffR\,(\Sigma + G)]_C \ . \tag{752}$$

Here Δ does not include $\rho_1 + \rho_2$. That factor comes along with $\eta^\mu\eta^\nu$.
$f\delta S$ cancels the r-divergence in fS,
$f\delta\Sigma$ cancels the r-divergence in $(f - ff)\Delta\,\Sigma$,
$f\delta G$ cancels the r-divergence in $(f - ff)\Delta\,G$,
I established that $\delta G = -\delta S$.

$$H_{LR} \ = \ m^2 + fS + f\delta\Sigma + (f - ff)\Delta\,(\Sigma + G) + ffR\,(\Sigma + G) \ . \tag{753}$$

I change the order of terms. In case of Σ there is no zero in denominator because the RGPEP Δ equals the inverse of the old-fashioned denominator that has one sign and is separated by at least m_g^2 from 0. The minimal separation only happens when $k^\perp$ and x_g tend to 0. In the absence of 0 in the denominator, I can separate ff from f in the self-interaction terms,

$$\Sigma_{\rm tot} \ = \ f\delta\Sigma + (f - ff)\Delta\,\Sigma + ffR\,\Sigma \ = \ f(\delta\Sigma + \Delta\,\Sigma) + ff(-\Delta + R)\,\Sigma \ . \tag{754}$$

So,

$$H_{LR} \ = \ m^2 + f(\delta\Sigma + \Delta\,\Sigma) + ff(-\Delta + R)\,\Sigma + fS + (f - ff)\Delta\,G + ffR\,G \ . \tag{755}$$

In the exchange terms, the issue is that $(f - ff)\Delta$ is free from the denominator 0 while Δ itself is not. By adding and subtracting fRG to the exchange terms,

$$G_{\rm tot} \ = \ fS + (f - ff)\Delta\,G + ffR\,G \ , \tag{756}$$

one obtains

$$G_{\rm tot} \ = \ f(S + RG) + (ff - f)(-\Delta + R)\,G. \tag{757}$$

Now the denominator 0 in Δ is always compensated by the numerator 0 in $ff - f$. I can write $G = G_g + G_\eta$, where G_g is the part with $-g^{\mu\nu}$ and G_η is the part with $\eta^\mu \eta^\nu$.

$$G_{\rm tot} = f(S + RG_g + RG_\eta) + (ff - f)(-\Delta + R)\,(G_g + G_\eta) \quad (758)$$

$$= fRG_g + (ff - f)(-\Delta + R)\,G_g + f(S + RG_\eta) + (ff - f)(-\Delta + R)\,G_\eta \;. \quad (759)$$

So,

$$H_{LR} = m^2 + \Sigma_{\rm tot} + G_{\rm tot} \quad (760)$$

$$= m^2 + f(\delta\Sigma + \Delta\,\Sigma) + ff(-\Delta + R)\,\Sigma$$
$$+ \; fRG_g + (ff - f)(-\Delta + R)\,G_g + f(S + RG_\eta) + (ff - f)(-\Delta + R)\,G_\eta \;. \quad (761)$$

Hypotheses:
In $G_{\rm tot}$, the first term is the Coulomb, not sensitive to $m_g \to 0$ because G_g is not;
The second term in $G_{\rm tot}$ is small because G_g is not singular;
The third term shows how the singular seagull cancels out with G_η;
The fourth term is the exchange whose logarithmic dependence on m_g is suspected to cancel with that of the second term in $\Sigma_{\rm tot}$, which is $ff(-\Delta + R)\,\Sigma$.

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The hoped-for cancellation of $\ln m_g$ between the self-energy and exchange concerns the sum of singular terms with η,

$$(\Sigma G)_\eta = ff(-\Delta + R)\,\Sigma + (ff - f)(-\Delta + R)\,G_\eta \;. \quad (762)$$

The factor $ff - f$ cancels the denominator 0 in Δ. All I have done here is merely to write the terms in such an order that the RGPEP small denominator cancellation is explicit. The difference between Eqs. (762) and (804), the latter written in full detail in Eqs. (837) and (838), is that now the denominator 0 in Δ is canceled by the corresponding numerator 0.

A. Analysis of the part A

The first part, A, describes the free p^- of quarks. Their mass parameters need to be fitted to data [KSerafin, heavy quarkonia and heavy baryons]. Part A does not depend on m_g.

B. Analysis of the part B

The second part,

$$B = f_{LR} H_0^{(2)} + (f_{LR} - f_{L3} f_{3R}) \Delta_{L3R} \left\{ [H_0^{(1)}\, H_0^{(1)}]_\Sigma + [H_0^{(1)}\, H_0^{(1)}]_{\rm exch} \right\} \;, \quad (763)$$

results from solving the RGPEP equation. The term $H_0^{(2)}$ contains the RGPEP-tansformed seagull term and its counterterm, see Sec. **??**, whose contributions to the eigenvalue equation for the $Q\bar{Q}$ quarkonium-component wave function are denoted below in Eq. (823) by $B_{\rm s}$ and $\delta B_{\rm s}$, respectively. The terms $[H_0^{(1)}\, H_0^{(1)}]_\Sigma$ and $[H_0^{(1)}\, H_0^{(1)}]_{\rm exch}$ contribute the quark self-interactions and the gluon exchange, respectively. These contributions are denoted in Eq. (823) by $\Sigma_q^{(2)}$ and $B_{\rm exch}$. Both require counterterms, contained in $H_0^{(2)}$ and denoted by $\delta m_{i{\rm tot}}$ and $\delta B_{\rm exch}$ in Eq. (823), correspondingly.

1. *Quark self-interactions in the part B*

The bare 2nd-order quark self-interaction is given by Eq. (**??**). It requires the counterterm $\delta m_{i\,\rm tot}$ of Eq. (**??**), which is

$$\delta m_{i\,\rm tot} = \delta m^2_{i\,\rm sev} + \delta m^2_{i\,\ln} \;, \quad (764)$$

where $\delta m^2_{i\,\rm sev}$ counters the severe small-x and transverse quadratic divergences, and $\delta m^2_{i\,\ln}$ counters the remaining logarithmic dependence on r. The counterterm $\delta m_{i\,\rm tot}$ appears the same way in the single-quark eigenvalue problem (in 2nd order there is no Lamb-shift effect, besides a variation of the gluon mass ansatz to be discussed later on) and in the quarkonium eigenvalue problem, as befits the RGPEP universal solution for a term in the Hamiltonian. There is no issue of zero denominator in Δ in the self-interactions and one can separate the diverging term with f_{LR} from the r-independent term with $f_{L3}f_{3R}$ since $L = R$ in this case.

The full self-interaction term in part B reads

$$B_\Sigma \;=\; (f_{LR} - f_{L3}f_{3R})\Delta_{L3R}\, [H_0^{(1)}\, H_0^{(1)}]_\Sigma \;, \tag{765}$$

where, according to Eq. (??), the operator B_Σ leads to

$$\frac{\Sigma_q^{(2)}}{p^+} \;=\; \frac{g^2 C_F}{p^+}\int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\,\frac{N}{D}\, f^{r2}_{p_q p_g, p}(1 - f^{t2}_{p_q p_g, p}) \;, \tag{766}$$

$$N \;=\; 4[k^{\perp 2} + (1-x)m_g^2]/x^2 + 2(k^{\perp 2} + x^2 m_i^2)/(1-x) \;, \tag{767}$$

$$D \;=\; x(1-x)p^+(p^- - p_q^- - p_g^-) \;=\; x(1-x)(m_i^2 - \mathcal{M}_{qg}^2) \;. \tag{768}$$

The calculation starts with Eq. (E11),

$$\mathcal{H}^{\rm div}_{\Sigma\; p\sigma} \;=\; \frac{[H_1^r H_1^r]_{\Sigma\; p\sigma}}{p_q^- - p_{qg}^-} \tag{769}$$

$$= \frac{g^2 C_F}{p^+}\int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\,\frac{\bar u_{p\sigma}\gamma_\mu(\not p_q + m)\gamma_\nu u_{p\sigma}}{(1-x)p_g^+(p^- - p_q^- - p_g^-)}\,\left[-g^{\mu\nu} + \eta^\mu\eta^\nu(p_g^- + p_q^- - p^-)/p_g^+\right] f^{r2}_{p_q p_g, p}$$

$$= \frac{g^2 C_F}{p^+}\int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\,\bar u_p\left\{\frac{2\not p_q - 4m}{(1-x)p_g^+(p^- - p_q^- - p_g^-)} - \frac{2p_q^+\gamma^+}{(1-x)p_g^{+2}}\right\} u_p f^{r2}_{p_q p_g, p} \tag{770}$$

$$= \frac{g^2 C_F}{p^+}\int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\left[\frac{4p_q p - 8m^2}{(1-x)p_g^+(p^- - p_q^- - p_g^-)} - \frac{4p^+p_q^+}{(1-x)p_g^{+2}}\right] f^{r2}_{p_q p_g, p} \;, \tag{771}$$

where the first term is from $-g^{\mu\nu}$ and the second from $\eta^\mu\eta^\nu$. Together, they yield N/D, see Eq. (E15). After addition of the counterterm $\delta m^2_{i\,\rm sev}$, see Eqs. (??), (??), (??), and (??), the factor N/D is replaced by the remaining

$$\frac{4m_i^2 + 2m_g^2}{k^{\perp 2} + m_i^2 x^2 + m_g^2(1-x)} \;, \tag{772}$$

which leads to the logarithmic divergence in Eq. (??),

$$\Delta_i^2 \;=\; g^2 C_F \int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\,\frac{4m_i^2 + 2m_g^2}{k^{\perp 2} + m_i^2 x^2 + m_g^2(1-x)}\, f^{r2}_{p_q p_g, p} \;, \tag{773}$$

and the logarithmic counterterm $\delta m^2_{i\,\ln}$ in Eq. (??), all independent of m_g. Thus, according to Eqs. (??) to (??), the counterterm $\delta m^2_{i\,\rm tot}$ removes dependence on $r \to 0$ without producing any dependence on m_g from 1 in $(1 - f^{t2}_{p_q p_g, p})$. The reason is that the integral is 3-dimensional and its integration measure makes it finite for $m_g = 0$ after the severe divergences are countered by $\delta m^2_{i\,\rm sev}$. I think about an analogy with the case of the Coulomb potential, where d^3q/q^2 is finite for $q \to 0$. So, the self-interaction terms with $f = 1$ in $f - ff$ are all eliminated and only m_i^2 is left in the eigenvalue equation for the $Q\bar Q$ component of the quarkonium.

The self-interaction due to the second term in the bracket, $-f^{t2}_{p_q p_g, p}$, does not depend on $r \to 0$. Instead it depends on s and does not induce any counterterms. The gluon mass in it is m_g. This contribution combines with the effective-particle contribution to the self-interaction that includes the gluon mass ansatz m_G in the part C, see below. Such ansatz mass, m_G, is not acceptable in QED because photons do not interact with electrons like gluons do with quarks. The combination of the self-interaction terms with ff, one in the part B and the other one in part C, does not vanish because $m_G \neq m_g$. If m_G were equal to m_g, which would be the case in QED, a theory without confinement, the combination would contribute 0 and the ith quark mass-squared eigenvalue would be just m_i^2. For $m_G \neq m_g$, the self-interaction cancellation does not occur. The non-cancellation leads to the divergent logarithmic dependence of the single-quark eigenvalue on m_g, shown in Eq. (750). The self-interactions in the quarkonium eigenvalue problem may differ from the self-interactions in the single-quark due to a different value of the gluon mass ansatz, corresponding to different color dynamics in triplet and singlet states. **Irrespective of that ansatz difference, the question is if the self-interaction logarithm, $\ln m_g$, can be canceled by the combination of the gluon exchange interactions in parts B and C.**

2. Gluon exchange in the part B

Start 20250128 01:16 wto Ekologiczna The gluon-exchange term in the part B contains the factor Δ_{L3R} generated by the RGPEP and requires a counterterm placed in $\mathcal{H}_0^{(2)}$. To compute the divergence and the counterterm we proceed as in Sec. **??**, see Sec. **??**. The exchange term matrix elements between quark-antiquark states of Eq. (**??**),

$$|Pn\rangle_s = \sum_{12} \int [12]\, \psi^n_P(1,2)\, b^\dagger_{t1} d^\dagger_{t2} |0\rangle \ , \tag{774}$$

read

$$_s\langle Pn|\mathcal{U}_s^\dagger [(f_{LR} - f_{L3} f_{3R}) \Delta_{L3R} \mathcal{H}_0^{(1)} \mathcal{H}_0^{(1)}]_{\rm exch}\, \mathcal{U}_s |P'n'\rangle_s = P^+ \tilde\delta_{P.P'} \sum_{122'1'} Y_{nn'122'1'} \ , \tag{775}$$

where

$$\begin{aligned} Y_{nn'122'1'} &= \int [x_1 k^\perp_{12}] \int [x_{1'} k^\perp_{1'2'}]\ (f - ff) \\ &\quad \times\ \psi^{*n}_{12}(x_1, k^\perp_{12})\ \{\ \}_{\rm exch}\ \psi^{n'}_{1'2'}(x_{1'}, k^\perp_{1'2'}) \ , \end{aligned} \tag{776}$$

$$(f - ff) = (f^t_{12,2'1'} - f^t_{1,g\underline{1}} f^t_{2,g\underline{2}}) \ , \tag{777}$$

and

$$\{\ \}_{\rm exch} = \Delta_{12g2'1'} \sum_{c_g=1}^{8} j^{c_g}_{11'\mu} j^{c_g}_{22'\nu}\ d^{\mu\nu} \frac{1}{q^+_{m_g}}\ f^r_{\overline{1},g\underline{1}} f^r_{\overline{2},g\underline{2}} \tag{778}$$

$$= \sum_{c_g=1}^{8} j^{c_g}_{11'\mu} j^{c_g}_{22'\nu}\ d^{\mu\nu}\ \Delta\ f^r_{\overline{1},g\underline{1}} f^r_{\overline{2},g\underline{2}} \ , \tag{779}$$

$$d^{\mu\nu} = -g^{\mu\nu} + \frac{\eta^\mu q^\nu_{m_g} + q^\mu_{m_g} \eta^\nu}{q^+_{m_g}} = -g_{\mu\nu} + \eta_\mu \eta_\nu \frac{\rho_1 + \rho_2}{2 q^{+2}_{m_g}} \ , \tag{780}$$

$$\Delta = \frac{\Delta_{12g2'1'}}{q^+_{m_g}} = -\frac{\rho_1 \pi_{\overline{1}}^2 + \rho_2 \pi_{\overline{2}}^2}{\rho_1^2 \pi_{\overline{1}}^2 + \rho_2^2 \pi_{\overline{2}}^2 - (\rho_1 - \rho_2)^2 \pi_{12}^2} \ , \tag{781}$$

while $j^{c_g}_{11'\mu} = -\bar u_1 \gamma_\mu u_{1'} T^{c_g}_{c_1 c_{1'}}$, $j^{c_g}_{22'\nu} = \bar v_{2'} \gamma_\nu v_2 T^{c_g}_{c_{2'} c_2}$, $\rho_i = m_g^2 - q_i^2$, and $q_i = p_{\bar i} - p_{\underline{i}}$.

The regulating factors have the form $f^r_{\overline{1}} f^r_{\overline{2}} (f^t - f^t_1 f^t_2)$. One can keep a factor formally as different from 1 even if it is negligibly different from 1. In the limit $r \to 0$, the form factors $f^r_{\overline{1},g\underline{1}} f^r_{\overline{2},g\underline{2}}$ tend to 1 everywhere where the factors $f^t_{\overline{1},g\underline{1}} f^t_{\overline{2},g\underline{2}}$ are not equivalent to 0. Dependence on r due to $f^r f^r$ is removed from all the terms with $f^t f^t$ for fixed, positive s.

I need to understand ρ_i in terms of the variables defined in Eqs. (B1) - (B14). I need formulas for q_i^2. In the square, the sign of q_i does no matter, which implies I can simply use the difference between momenta before and after exchange.

$$q_i^2 = (p_{i'} - p_i)^2 = (p^+_{i'} - p^+_i)(p^-_{i'} - p^-_i) - (p_{i'} - p_i)^{\perp 2} \tag{782}$$

$$\begin{aligned} &= (x_{i'} - x_i) \left[\frac{(x_{i'} P^\perp + k^\perp_{i'})^2 + m_i^2}{x_{i'}} - \frac{(x_i P^\perp + k^\perp_i)^2 + m_i^2}{x_i} \right] - (x_{i'} P^\perp + k^\perp_{i'} - x_i P^\perp - k^\perp_i)^2 \\ &= (x_{i'} - x_i) \left[\frac{k^{\perp 2}_{i'} + m_i^2}{x_{i'}} - \frac{k^{\perp 2}_i + m_i^2}{x_i} \right] - (k^\perp_{i'} - k^\perp_i)^2 = -\frac{z^2 m_i^2 + (x_i k^\perp_{i'} - x_{i'} k^\perp_i)^2}{x_{i'} x_i} \ . \end{aligned} \tag{783}$$

Always true in $2 \leftarrow 2'$:

$$q_i^2 = -\frac{z^2 m_i^2 + (x_i k^\perp_{i'} - x_{i'} k^\perp_i)^2}{x_{i'} x_i} \ , \tag{784}$$

$$k_1 = k_{12} \ , \ k_{1'} = k_{1'2'} \ , \ k_2 = -k_{12} \ , \ k_{2'} = -k_{1'2'} \ . \tag{785}$$

$$q_1^2 \;=\; -\frac{4y^2m_1^2 + 4(xl^\perp - yk^\perp)^2}{x^2 - y^2} \;, \tag{786}$$

$$q_2^2 \;=\; -\frac{4y^2m_2^2 + 4[(1-x)l^\perp + yk^\perp]^2}{(1-x)^2 - y^2} \;, \tag{787}$$

$$\rho_1 \;=\; m_g^2 + \frac{4y^2m_1^2 + 4(xl^\perp - yk^\perp)^2}{x^2 - y^2} \;, \tag{788}$$

$$\rho_2 \;=\; m_g^2 + \frac{4y^2m_2^2 + 4[(1-x)l^\perp + yk^\perp]^2}{(1-x)^2 - y^2} \;. \tag{789}$$

In the non-relativistic limit:

$$\rho_1 \;=\; m_g^2 + z^2 m_1^2/x_1^2 + q^{\perp 2} \;,\;\; \rho_2 \;=\; m_g^2 + z^2 m_2^2/x_2^2 + q^{\perp 2} \;, \tag{790}$$

$$x_i \;=\; \frac{m_i}{m_1+m_2} \;,\;\; \rho_1 \;=\; \rho_2 \;=\; m_g^2 + z^2(m_1+m_2)^2 + q^{\perp 2} \;=\; \rho \;. \tag{791}$$

Note the sum $m_1 + m_2$! I have just observed the mass sum on 20250128 at 04:47 wto Ekologiczna. In the non-relativistic limit, $\Delta = -1/\rho$, $\tilde\Delta = -1$, $\tilde\Delta' = 0$, and 0 in the denominator of Δ does not appear in that limit, since $\rho_1 = \rho_2$. The factor $f^r_{1,g\underline{1}} f^r_{\underline{2},g\underline{2}}$ reduces to

$$f^t_{1,\underline{1}g} f^t_{\underline{2},\underline{2}g} \;=\; e^{-\left(t\,\rho_1\,\pi_{\underline{1}}/q^+_{m_g}\right)^2} e^{-\left(t\,\rho_2\,\pi_{\underline{2}}/q^+_{m_g}\right)^2} \;\sim\; e^{-t^2\rho^2(x_1^2+x_2^2)/z^2} \;. \tag{792}$$

For $t = 1/M^2$, where $M^2 = m_1^2 + m_2^2$, I obtain

$$f^t_{1,\underline{1}g} f^t_{\underline{2},\underline{2}g} \;\sim\; \exp\left[-\left(\frac{\vec q^{\,2}}{q_z M}\right)^2\right] \;=\; \exp\left[-\left(\frac{|\vec q\,|}{M\cos\theta}\right)^2\right] \;. \tag{793}$$

End of the non-relativistic limit.

Start with the term that involves $f^r f^r f^t$. The first term in $d^{\mu\nu}$, equal $-g^{\mu\nu}$, produces regular matrix elements and does not require any counterterms. It yields the Coulomb potential with the Breit-Fermi terms. In QED, the factor f^t in the potential could noticeably deviate from 1 only in the extreme atomic effects such as the ones associated with the measurement of the radius of proton in hydrogen. A few percent effect in the radius could occur when the value of the parameter s in the non-relativistic Schroedinger equation for hydrogen is set to the value on the order of the proton mass [Glazek proton radius].

The second term in $d^{\mu\nu}$, the one proportional to $\eta^\mu\eta^\nu$, produces a severe small-x divergence in the matrix element,

$$Y^{\rm div}_{nn'122'1'} \;=\; \int [x_1 k^\perp_{12}] \int [x_{1'} k^\perp_{1'2'}] \; (f - ff) \; \psi^{*n}_{12}(x_1, k^\perp_{12}) \; \{\;\;\}^{\rm div}_{\rm exch} \; \psi^{n'}_{1'2'}(x_{1'}, k^\perp_{1'2'}) \;, \tag{794}$$

$$\{\;\;\}^{\rm div}_{\rm exch} \;=\; -\,g^2\, C_{11'2'2}\,\delta_{sf}\, 4\sqrt{x_1 x_{1'} x_2 x_{2'}} \;\frac{\tilde\Delta}{z^2}\; f^r_{123} f^r_{31'2'} \;, \tag{795}$$

where

$$\tilde\Delta \;=\; \Delta\,\frac{\rho_1+\rho_2}{2} \;=\; -1 + \tilde\Delta' \;,\;\; \lim_{z\to 0} \tilde\Delta' \;=\; \frac{z}{2c}\,(b_1 - b_2)\,\frac{\pi_{\underline{1}}^2 - \pi_{\underline{2}}^2}{\pi_{\underline{1}}^2 + \pi_{\underline{2}}^2} \;, \tag{796}$$

$$c \;=\; 4l^{\perp 2} + m_g^2 \;,\;\; b_1 \;=\; -4k^\perp l^\perp/x \;,\;\; b_2 \;=\; 4k^\perp l^\perp/(1-x) \;. \tag{797}$$

Calculation of $\tilde\Delta'$:

$$\tilde\Delta \;=\; (\rho_1+\rho_2)\Delta/2 \;=\; -\,\frac{1}{2}\,\frac{\rho_1^2\pi_{\underline{1}}^2 + \rho_2^2\pi_{\underline{2}}^2 + \rho_1\rho_2(\pi_{\underline{1}}^2 + \pi_{\underline{2}}^2)}{\rho_1^2\pi_{\underline{1}}^2 + \rho_2^2\pi_{\underline{2}}^2 - (\rho_1-\rho_2)^2\pi_{12}^2} \;, \tag{798}$$

$$\tilde\Delta' \;=\; \tilde\Delta + 1 \;=\; -\,\frac{1}{2}\,\frac{\rho_1^2\pi_{\underline{1}}^2 + \rho_2^2\pi_{\underline{2}}^2 + \rho_1\rho_2(\pi_{\underline{1}}^2 + \pi_{\underline{2}}^2)}{\rho_1^2\pi_{\underline{1}}^2 + \rho_2^2\pi_{\underline{2}}^2 - (\rho_1-\rho_2)^2\pi_{12}^2} + 1 \;. \tag{799}$$

In the limit $y \to 0$ I have

$$\rho_1 = m_g^2 + \frac{4y^2 m_1^2 + 4(xl^\perp - yk^\perp)^2}{x^2 - y^2} \to m_g^2 + 4l^{\perp 2} - 8yl^\perp k^\perp/x \ , \tag{800}$$

$$\rho_2 = m_g^2 + \frac{4y^2 m_2^2 + 4[(1-x)l^\perp + yk^\perp]^2}{(1-x)^2 - y^2} \to m_g^2 + 4l^{\perp 2} + 8yl^\perp k^\perp/(1-x) \ , \tag{801}$$

and I confirm

$$\lim_{z\to 0} \tilde\Delta' = \frac{z}{2c}\,(b_1 - b_2)\,\frac{\pi_{\bar 1}^2 - \pi_{\bar 2}^2}{\pi_{\bar 1}^2 + \pi_{\bar 2}^2} \ . \tag{802}$$

The term -1 in $\tilde\Delta$ requires the counterterm opposite in sign to the one obtained in Sec. **??** for the homogeneous seagull term. The small-z correction $\tilde\Delta'$ depends on the choice of π_{ca} in the RGPEP generator in Eq. (**??**). For $\pi_{\bar i} = \pi^+$ for all i equally, one obtains $\tilde\Delta' \to O(z^2)$ and no need for a logarithmic counterterm. For $\pi_{\bar i} = p_{\bar i}^+$,

$$\lim_{z\to 0} \tilde\Delta' = -z\,\frac{8k^\perp l^\perp}{c}\,\frac{x - 1/2}{x(1-x)[x^2 + (1-x)^2]} \ , \tag{803}$$

which might lead to a logarithmic small-x divergence corresponding to the small-z behavior of the integrand here. The divergence would appear if the other factors in the integrand of $Y^{\rm div}_{nn'122'1'}$ in Eq. (794) were not smooth as functions of z near $z = 0$. However, the analysis of Sec. **??**, beginning with Eq. (**??**), applies and they are smooth. The result is that the logarithmic dependence on r is absent in the term B of Eq. (751). Hence no logarithmic small-x counterterm is required due to the term B in the bound-state eigenvalue problem of the effective Hamiltonian in Eq. (**??**). I do not see yet that a stronger argument is excluded: writing $l^\perp = \sqrt{|z|}\,\tilde l^\perp$, I obtain ρ/z in f^r that involves $\tilde l^{\perp 2}$ in the exponential without $1/z$ and the difference $b_1 - b_2 \sim \sqrt{|z|}$, implying reduction of the singularity from $1/z$ to the integrable one $1/\sqrt{|z|}$.

General explanation of how to treat 0 in the denominator of Δ in calculating the counterterms to small-x singularities is that, when the gluon $x \to 0$, the difference $\rho_1 - \rho_2 \to 0$ and cannot nullify the denominator because the other terms in the denominator tend to $(m_g^2 + q^{\perp 2})^2(\pi_{\bar 1}^2 + \pi_{\bar 2}^2) > 0$. In other words, one can handle the limits $z \to 0$ for terms with f and for terms with ff separately.

C. Analysis of the part C

The third part,

$$C = \frac{1}{2}\,f_{L3}\,H_0^{(1)}\left\{-2\Delta_{L3R} + \left[P_L^- - H_f'\right]^{-1} + \left[P_R^- - H_f'\right]^{-1}\right\} f_{3R}\,H_0^{(1)} \ . \tag{804}$$

is composed of two terms. The part with $-2\Delta_{L3R}$ is obtained by solving the RGPEP equation. The part with two denominators is obtained by reducing [Wilson R] the eigenvalue problem for H_s to the eigenvalue problem for the quark-antiquark states using the expansion in powers of the interaction to the 2nd order. The reduction is done using the simplifying assumption that one can replace all components besides the quark-antiquark component by the component with a quark, an antiquark and a gluon whose mass is set to a hypothetical value m_G, called the gluon-mass ansatz [Kamil etc.]. The mass parameter m_G is viewed as a multiple of $\Lambda_{\rm QCD}$ in the RGPEP scheme. As such, it would be 0 in the expansion in powers of the running coupling constant. The double action of the Hamiltonian term $H_0^{(1)}$ produces the quark self-interactions and the gluon exchange between the quarks. Both involve the singular expression p_g^-/x_g. The form factors f^t prevent any dependence on r from occurring. However, the resulting effective interaction depends on m_g. The question addressed below is how the self-interaction and gluon exchange terms in the part C behave in the limit $m_g \to 0$.

1. *Quark self-interactions in the part C*

$$fH\,\frac{1}{2}\,\{\ \ \}\,fH \ \to\ C_\Sigma = \int [1]\frac{1}{p_1^+}\int [23]\ \tilde\delta_{1.23}\ d_{\mu\nu}\ j_{12}^{c\mu} j_{21}^{c\nu}\ f^2\ \Delta_{1\Sigma}\ b_1^\dagger b_1$$

$$+ \int [2]\frac{1}{p_2^+}\int [13]\ \tilde\delta_{2.13}\ d_{\mu\nu}\ \bar j_{12}^{c\mu} \bar j_{21}^{c\nu}\ f^2\ \Delta_{2\Sigma}\ d_2^\dagger d_2 \ . \tag{805}$$

$$j_{12}^{c\mu} \;=\; -g\bar{u}_1\gamma^\mu u_2\;\chi_1^\dagger T^c\chi_2\;,\;\bar{j}_{12}^{c\mu}\;=\;g\bar{v}_1\gamma^\mu v_2\;\chi_1^\dagger T^c\chi_2\;, \tag{806}$$

$$q_1 \;=\; p_{\bar{1}} - p_{\underline{1}} \;=\; q_2 \;=\; p_{\bar{2}} - p_{\underline{2}}\;, \tag{807}$$

$$d^{\mu\nu} \;=\; -g^{\mu\nu} + \frac{1}{2q^+_{m_g}}\eta^\mu\eta^\nu\left(\frac{\mathcal{M}^2_{g\underline{1}} - m_1^2}{p^+_{\bar{1}}} + \frac{\mathcal{M}^2_{g\underline{2}} - m_2^2}{p^+_{\bar{2}}}\right) \tag{808}$$

$$\;=\; -g^{\mu\nu} + \frac{1}{2q^{+2}_{m_g}}\eta^\mu\eta^\nu\left(m_g^2 - q_1^2 + m_g^2 - q_2^2\right)\;. \tag{809}$$

Using the pattern of Eqs. (713) and (714), the factor $\Delta_{i\Sigma}$ is

$$\Delta_{i\Sigma} \;=\; \frac{1}{2}\left\{-2\Delta_{iqgi} + \left[p_i^- - p_{qG}^-\right]^{-1} + \left[p_i^- - p_{qG}^-\right]^{-1}\right\} \tag{810}$$

$$\;=\; -\Delta_{iqgi} + \left[p_i^- - p_{qG}^-\right]^{-1} \;=\; \frac{-1}{p_i^- - p_q^- - p_g^-} + \frac{1}{p_i^- - p_q^- - p_G^-}\;. \tag{811}$$

The self-interaction terms C_Σ in the part C depend on the assumed quarkonium parameter m_G, which is likely to differ from the single-quark ansatz parameters m_{iG}. In contrast, the calculation in Sec. XII B shows in Eq. (750) that the self-interaction logarithmic growth for $m_g \to 0$ does not depend on the single-quark gluon-mass ansatz m_{iG}. It thus occurs in the quarkonium eigenvalue problem in the same way as in a single quark eigenvalue problem even though m_G may differ from m_{iG}. The unlimited growth of the quark self-interactions in the $Q\bar{Q}$ quarkonium component for $m_g \to 0$ appears in a sum

$$C_\Sigma \;\to\; \frac{C_{1\Sigma}}{p_1^+} + \frac{C_{2\Sigma}}{p_2^+}\;, \tag{812}$$

which multiplies the quarkonium wave function in the eigenvalue equation. The parts of the self-interaction terms $C_{1\Sigma}$ and $C_{1\Sigma}$ that do not diverge logarithmically with $m_g \to 0$ are sensitive to the ansatz masses m_{iG} and m_G.

2. *Gluon exchange in the part C*

The gluon exchange interaction acts on the wave function $\psi_E(1,2)$ integrated in the eigenstate,

$$|E\rangle \;=\; \sum_{12}\int[12]\,\psi_E(1,2)\,b^\dagger_{t1}d^\dagger_{t2}|0\rangle\;. \tag{813}$$

For eigenvalues of the form $E = (P^{\perp 2} + M_E^2)/P^+$, the wave function may be set to have the form

$$\psi_E(1,2) \;=\; P^+\tilde{\delta}_{P.12}\;\psi_{12}(x_1, k_{12}^\perp)\;, \tag{814}$$

with the labels 1 and 2 including spins, flavors and colors as needed. The relative-motion variables x_1 and $k_{12}^\perp$ for the effective quarks 1 and 2 are defined in the same way as for the gluons in Eq. (45). The eigenvalue problem for the Hamiltonian in Eq. (751)

$$H_{LR} \;=\; A + B + C\;. \tag{815}$$

in terms of matrix elements in plane-wave $Q\bar{Q}$ basis states reads

$$\langle 12|H_{LR}|E\rangle \;=\; \langle 12|A+B+C|E\rangle \;=\; P_E^-\langle 12|E\rangle\;. \tag{816}$$

Using Eq. (G62), I get

$$\langle 12|C_{\rm exch}|E\rangle = \sum_c \int [1'2'] \; \tilde\delta_{12.1'2'} \; d_{\mu\nu} \; j^{\mu c}_{11'} \; \bar j^{\nu c}_{2'2} \; f^t_{\bar 1,\underline{1}q} f^t_{\bar 2,\underline{2}q}$$
$$\times \; \frac{1}{2q^+_{m_g}} \left\{ -2\Delta_{12g2'1'} + \left[P^-_{12} - P^-_3\right]^{-1} + \left[P^-_{1'2'} - P^-_3\right]^{-1} \right\} \psi_E(1',2') \; , \quad (817)$$

$$d_{\mu\nu} = -g_{\mu\nu} + \eta_\mu \eta_\nu \frac{\rho_1 + \rho_2}{2q^{+2}_{m_g}} \; , \quad (818)$$

$$\Delta = \frac{\Delta_{12g2'1'}}{q^+_{m_g}} = -\frac{\rho_1 \pi_1^2 + \rho_2 \pi_2^2}{\rho_1^2 \pi_1^2 + \rho_2^2 \pi_2^2 - (\rho_1 - \rho_2)^2 \pi_{12}^2} \; , \quad (819)$$

$$P^-_3 = p^-_{\underline 1} + p^-_{\underline 2} + q^-_{m_G} \; , \; \rho_i = m_g^2 - q_i^2 \; , \quad (820)$$

$$\frac{1}{q^+_{m_g}} \quad \left[\left(P^-_{12} - P^-_3\right)^{-1} + \left(P^-_{1'2'} - P^-_3\right)^{-1}\right] \; , \quad (821)$$

$$= -\left[\frac{1}{\rho_1 + (m_G^2 - m_g^2)} + \frac{1}{\rho_2 + (m_G^2 - m_g^2)}\right] \; , \quad \text{see Appendix.} \quad (822)$$

XIV. $Q\bar Q$ WAVE-FUNCTION EIGENVALUE EQUATION

20250107 23:05 wto San Dimas Start tomorrow od analizy rownania

The wave-function eigenvalue equation reads

$$\left[\frac{m_1^2 + \Sigma^{(2)}_{q1} + \delta m^2_{1\,\rm tot}}{p_1^+} + \frac{m_2^2 + \Sigma^{(2)}_{q2} + \delta m^2_{2\,\rm tot}}{p_2^+}\right] \psi_{12}(x_1, k^\perp_{12})$$
$$+ \int [x_{1'} k^\perp_{1'2'}] \, [B_{\rm s} + \delta B_{\rm s} + B_{\rm exch} + \delta B_{\rm exch}] \, (x_1, k^\perp_{12}; x_{1'}, k^\perp_{1'2'}) \; \psi_{1'2'}(x_{1'}, k_{1'2'})$$
$$+ \left[\frac{C_{1\Sigma}}{p_1^+} + \frac{C_{2\Sigma}}{p_2^+}\right] \psi_{12}(x_1, k_{12}) + \int [x_{1'} k^\perp_{1'2'}] C_{\rm exch}(x_1, k^\perp_{12}; x_{1'}, k^\perp_{1'2'}) \, \psi_{1'2'}(x_{1'}, k_{1'2'}) \; , \quad (823)$$

where

$$C_{i\Sigma} = \frac{1}{p_i^+} \int [x k^\perp] \; d_{\mu\nu} \; j^{c\mu}_{iq} j^{c\nu}_{qi} \; (f^t_{i,qg})^2$$
$$\times \; \frac{1}{2} \left\{ -2\Delta_{iqgi} + 2\left[p_i^- - p^-_{qG}\right]^{-1} \right\} \; , \quad (824)$$

$$C_{\rm exch}(x_1, k^\perp_{12}; x_{1'}, k^\perp_{1'2'}) = \frac{1}{q^+_{m_g} P^+} d_{\mu\nu} \; j^{c\mu}_{11'} \; \bar j^{c\nu}_{2'2} \; f^t_{\bar 1,\underline 1 q} f^t_{\bar 2,\underline 2 q}$$
$$\times \; \frac{1}{2} \left\{ -2\Delta_{12g2'1'} + \left[P^-_{12} - P^-_3\right]^{-1} + \left[P^-_{1'2'} - P^-_3\right]^{-1} \right\} . \quad (825)$$

The counterterms $\delta m^2_{i\,\rm tot}$ can be adjusted to cancel $\Sigma^{(2)}_{qi}$ and make m_i^2 the quark mass parameters that are fitted to the quarkonium spectra. The counterterms $\delta B_{\rm s}$ and $\delta B_{\rm exch}$ cancel each other, which is shown in Sec. VIII. They will be omitted in further equations. When $m_g \to 0$, the sum of terms $B_{\rm s} + B_{\rm exch}$ provides the Coulomb potential with Breit-Fermi spin dependence, in the non-relativistic approximation for the relative motion of quarks. The self-interaction terms $C_{1\Sigma}$ and $C_{2\Sigma}$ grow proportionally to $\ln(m_i/m_g)$ when $m_g \to 0$, as they do in the single-quark eigenvalue equations rendering infinite masses to single quarks, see Eq. (750), but perhaps using a potentially different gluon mass ansatz. Can the exchange cancel the infinite growth of the self-interactions in colorless quarkonia when $m_g \to 0$? Start 20250105 09:21 -¿ 19:06 nie San Dimas I focus on the cancellation issue.

The self-interaction terms $C_{i\Sigma}$ multiply the wave function while the exchange term integrates it with the kernel $C_{\rm exch}(x_1, k^\perp_{12}; x_{1'}, k^\perp_{1'2'})$. The cancellation is possible if the kernel's logarithmic dependence on m_g does not involve any change of the quark momenta. The logarithmic rise of $C_{i\Sigma}$ results from the term $\eta^\mu \eta^\nu \rho_i / p_g^{+2}$ in the sum over gluon polarizations, which blows up when the gluon x tends to 0. The blowup of $C_{i\Sigma}$ is tamed by the RGPEP vertex factor squared, $(f^t_{i,qg})^2$. In the exchange, $(f^t_{i,qg})^2$ is replaced by the product $f^t_{\bar{1},\underline{1}g} f^t_{\bar{2},\underline{2}g}$, which works in a similar way. The RGPEP vertex factors depend on the gluon mass m_g through $p_g^- = (p_g^{\perp 2} + m_g^2)/p_g^+$. Therefore, the gluon mass m_g contributes to their behavior only when $|k^\perp = p_g^\perp - p_g^+ p_{\bar{i}}^\perp / p_{\bar{i}}| \lesssim m_g$. Otherwise the gluon transverse momentum, relative to the quarks, regulates the gluon small-x behavior and prevents dependence on $m_g \to 0$. Dependence on m_g can only originate from the region of small transverse momentum of the gluon.

The self-interaction logarithmic rise for $m_g \to 0$ in Eq. (823) comes from

$$\left[\frac{C_{1\Sigma}}{p_1^+} + \frac{C_{2\Sigma}}{p_2^+}\right]_{\ln m_g} = \sum_i \frac{1}{p_i^+} \int [x k^\perp] \, \frac{\rho_i}{q_{m_g}^{+2}} j^{c+}_{iq} j^{c+}_{qi} \, (f^t_{i,qg})^2 \times \frac{1}{2p_i^+} \left\{ -2\Delta_{iqgi} + 2\left[p_i^- - p_{qG}^-\right]^{-1} \right\} \tag{826}$$

multiplying $\psi_{12}(x_1, k_{12})$. The exchange-part sensitive to $m_g \to 0$ comes from integrating the wave function,

$$\int [x_{1'} k^\perp_{1'2'}] C_{\rm exch}(x_1, k^\perp_{12}; x_{1'}, k^\perp_{1'2'}) \, \psi_{1'2'}(x_{1'}, k_{1'2'}) \tag{827}$$

$$= \frac{1}{P^+} \int [x_{1'} k^\perp_{1'2'}] \, \frac{\rho_1 + \rho_2}{2q_{m_g}^{+2}} \quad j^{c+}_{11'} \, \bar{j}^{c+}_{2'2} \, f^t_{\bar{1},\underline{1}q} f^t_{\bar{2},\underline{2}q}$$

$$\times \frac{1}{2q^+_{m_g}} \left\{ -2\Delta_{12g2'1'} + \left[P^-_{12} - P^-_3\right]^{-1} + \left[P^-_{1'2'} - P^-_3\right]^{-1} \right\} \psi_{1'2'}(x_{1'}, k_{1'2'}) \, . \tag{828}$$

Inserting some formulas for some factors, such as for the masses squared in emission of absorption of a gluon by a quark i,

$$\mathcal{M}^2_{qg} - m_i^2 = \frac{p^+_{\bar{i}}}{q^+_{m_g}} \, \rho_i = \frac{p^+_{\bar{i}}}{q^+_{m_g}} (m_g^2 + q_i^2) \, , \tag{829}$$

I arrive at the comparison

$$\tilde{\Sigma}\psi = \left[\frac{C_{1\Sigma}}{p_1^+} + \frac{C_{2\Sigma}}{p_2^+}\right]_{\ln m_g} \psi_{12}(x_1, k_{12}) = \sum_i \frac{1}{p_i^+} \int [x k^\perp] \, \frac{\rho_i}{q_{m_g}^{+2}} j^{c+}_{iq} j^{c+}_{qi} \, (f^t_{i,qg})^2 \times \left[\frac{1}{\mathcal{M}^2_{qg} - m_i^2} - \frac{1}{\mathcal{M}^2_{qG} - m_i^2}\right] \psi_{12}(x_1, k_{12}) \, , \tag{830}$$

$$\frac{1}{\mathcal{M}^2_{qg} - m_i^2} - \frac{1}{\mathcal{M}^2_{qG} - m_i^2} = \frac{q^+_{m_g}}{p_i^+} \left[\frac{1}{\rho_i} - \frac{1}{\rho_i + m_G^2 - m_g^2}\right] \tag{831}$$

$$\tilde{C}\psi = \int [x_{1'} k^\perp_{1'2'}] C_{\rm exch}(x_1, k^\perp_{12}; x_{1'}, k^\perp_{1'2'}) \, \psi_{1'2'}(x_{1'}, k_{1'2'}) \tag{832}$$

$$= \frac{1}{P^+} \int [x_{1'} k^\perp_{1'2'}] \, \frac{\rho_1 + \rho_2}{2q_{m_g}^{+2}} \quad j^{c+}_{11'} \, \bar{j}^{c+}_{2'2} \, f^t_{\bar{1},\underline{1}q} f^t_{\bar{2},\underline{2}q}$$

$$\times \frac{1}{2q^+_{m_g}} \left\{ -2\Delta_{12g2'1'} + \left[P^-_{12} - P^-_3\right]^{-1} + \left[P^-_{1'2'} - P^-_3\right]^{-1} \right\} \psi_{1'2'}(x_{1'}, k_{1'2'}) \, . \tag{833}$$

$$\frac{-2\Delta_{12g2'1'}}{2q^+_{m_g}} = \frac{\rho_1 \pi_{\bar{1}}^2 + \rho_2 \pi_{\bar{2}}^2}{\rho_1^2 \pi_{\bar{1}}^2 + \rho_2^2 \pi_{\bar{2}}^2 - (\rho_1 - \rho_2)^2 \pi_{12}^2} \, , \tag{834}$$

$$\frac{1}{2q^+_{m_g}} \left[\left(P^-_{12} - P^-_3\right)^{-1} + \left(P^-_{1'2'} - P^-_3\right)^{-1} \right] \, , \tag{835}$$

$$= -\frac{1}{2} \left[\frac{1}{\rho_1 + m_G^2 - m_g^2} + \frac{1}{\rho_2 + m_G^2 - m_g^2}\right] \, . \tag{836}$$

I need to integrate over [23]. Stop 20250105 21:40 nie San Dimas, Start 20250106 17:11 pon San Dimas, Start 20250107 13:30 wto San Dimas Now 20250107 15:01 wto San Dimas I write these two terms in abbreviated forms, using also

$M^2 = m_G^2 - m_g^2$.

$$\tilde\Sigma\psi \;=\; \sum_i \frac{1}{p_i^+} \int [xk^\perp]\; \frac{\rho_i}{q^+_{m_g} p_i^+} j^{c+}_{iq}\, j^{c+}_{qi}\; (f^t_{i,qg})^2 \; \left[\frac{1}{\rho_i} - \frac{1}{\rho_i + M^2}\right] \psi_{12}(x_1, k_{12}) \;, \tag{837}$$

$$\begin{aligned}\tilde C\psi \;&=\; \frac{1}{P^+} \int [x_{1'} k^\perp_{1'2'}]\; \frac{\rho_1+\rho_2}{2q^{+2}_{m_g}} \;\; j^{c+}_{11'}\; \bar j^{c+}_{2'2}\; f^t_{\bar 1, \underline{1} q} f^t_{\bar 2, \underline{2} q} \\ &\times \; \left[\frac{\rho_1 \pi_{\bar 1}^2 + \rho_2 \pi_{\bar 2}^2}{\rho_1^2 \pi_{\bar 1}^2 + \rho_2^2 \pi_{\bar 2}^2 - (\rho_1-\rho_2)^2 \pi_{12}^2} - \frac{1/2}{\rho_1 + M^2} - \frac{1/2}{\rho_2 + M^2}\right] \psi_{1'2'}(x_{1'}, k_{1'2'}) \;. \end{aligned} \tag{838}$$

20250109 18:02 I am certain Eqs. (837) and (838) are correct.

Now (20250109 20:43), first I pay attention to the RGPEP form factors. They exclude gluons with $x \to 0$ by the term $(p_g^{\perp 2} + m_g^2)/p_g^+$ in the exponent. So, the greater $|p_g^\perp|$ the stronger suppression of small x_g and vice versa, the smaller x_g the smaller $|p_g^\perp|$ must be for the factors not to vanish. But when $|p_g^\perp| \to 0$, gluon mass parameter m_g takes over and regulates the gluon small-x diverging factor.

Also, if the gluon transverse momentum is small, for example, proportional to the square root of the gluon x, the integration measure provides the factor x. Can the exchange cancel the self-interaction divergence for $m_g \to 0$ in the colorless states?

Since the exchange changes the relative momentum of quarks and the self-interaction does not, the question is if replacing $\psi_{1'2'}(x_{1'}, k_{1'2'})$ by $\psi_{1'2'}(x_{1'}, k_{1'2'}) - \psi_{12}(x_1, k_{12})$ takes the dependence on m_g away from the integral with the wave function. Stop 20250109 21:50 czw San Dimas, Start 20250110 19:47 pia San Dimas The currents j^+ conserve spin projections. The color factor is: C_F for the self-interaction, C_F for the gluon exchange in quarkonium, and C_A for the gluon exchange in the octet quarkonium. The only arguments of the wave functions that need to be considered are x and $k^\perp$. Using

$$\mathcal{M}^2_{qg} - m_i^2 \;=\; \frac{p^+_{\bar i}}{q^+_{m_g}}\; \rho_i \;=\; \frac{p^+_{\bar i}}{q^+_{m_g}} (m_g^2 - q_i^2) \;, \tag{839}$$

the dependence of the RGPEP form factors on $q^+_{m_g}$ is fully displayed in

$$\begin{aligned} (f^t_{i,qg})^2 \;&=\; e^{-2\left\{t\,[(p_{\underline i} + p_g)^2 - m_1^2]\, \pi_{\bar i}/p^+_{\bar i}\right\}^2} && (840)\\ &=\; e^{-2\left[t\,(m_g^2 - q_i^2)\,\pi_{\bar i}/q^+_{m_g}\right]^2} \;=\; e^{-2\left(t\,\rho_i\,\pi_{\bar i}/q^+_{m_g}\right)^2} \;, && (841)\\ f^t_{\bar 1, \underline{1} g} f^t_{\bar 2, \underline{2} g} \;&=\; e^{-\left\{t\,[(p_{\underline 1}+p_g)^2 - m_1^2]\,\pi_{\bar 1}/p^+_{\bar 1}\right\}^2} e^{-\left\{t\,[(p_{\underline 2}+p_g)^2 - m_2^2]\,\pi_{\bar 2}/p^+_{\bar 2}\right\}^2} && (842)\\ &=\; e^{-\left[t\,(m_g^2 - q_1^2)\,\pi_{\bar 1}/q^+_{m_g}\right]^2} e^{-\left[t\,(m_g^2-q_2^2)\,\pi_{\bar 2}/q^+_{m_g}\right]^2} && (843)\\ &=\; e^{-\left(t\,\rho_1\,\pi_{\bar 1}/q^+_{m_g}\right)^2} e^{-\left(t\,\rho_2\,\pi_{\bar 2}/q^+_{m_g}\right)^2} \;. && (844)\end{aligned}$$

Using $z = x_{1'} - x_1$, the key comparison concerns $-g^2\,\psi(C_F \text{ or } C_A)/[2(2\pi)^3]$ times

$$\tilde\Sigma \;=\; \sum_i \frac{1}{p_i^+} \int \frac{dx\, d^2k^\perp}{x(1-x)}\; \frac{\rho_i}{x} 4(1-x)\; (f^t_{i,qg})^2 \; \left[\frac{1}{\rho_i} - \frac{1}{\rho_i + M^2}\right] \;, \tag{845}$$

$$\begin{aligned}\tilde C \;&=\; \frac{1}{P^+} \int \frac{dx_{1'}\, d^2k^\perp_{1'2'}}{x_{1'} x_{2'}}\; \frac{\rho_1+\rho_2}{2z^2}\; (-4\sqrt{x_1 x_{1'} x_2 x_{2'}})\; f^t_{\bar 1, \underline{1} q} f^t_{\bar 2, \underline{2} q} \\ &\times \; \left[\frac{\rho_1 \pi_{\bar 1}^2 + \rho_2 \pi_{\bar 2}^2}{\rho_1^2 \pi_{\bar 1}^2 + \rho_2^2 \pi_{\bar 2}^2 - (\rho_1-\rho_2)^2 \pi_{12}^2} - \frac{1/2}{\rho_1 + M^2} - \frac{1/2}{\rho_2 + M^2}\right] \;, \end{aligned} \tag{846}$$

provided that the exchange integral,

$$\begin{aligned}\tilde C(\psi' - \psi) \;&=\; \frac{1}{P^+} \int_{-x_1}^{x_2} dz \int d^2q^\perp\; \frac{\rho_1+\rho_2}{2z^2}\; 4\sqrt{\frac{x_1 x_2}{x_{1'} x_{2'}}} \\ &\times \; \left[\frac{\rho_1 \pi_{\bar 1}^2 + \rho_2 \pi_{\bar 2}^2}{\rho_1^2 \pi_{\bar 1}^2 + \rho_2^2 \pi_{\bar 2}^2 - (\rho_1-\rho_2)^2 \pi_{12}^2} - \frac{1/2}{\rho_1 + M^2} - \frac{1/2}{\rho_2 + M^2}\right] \\ &\times \; \left[\psi(x_1 + z, k^\perp_{12} + q^\perp) - \psi(x_1, k^\perp_{12})\right]\; e^{-\left[t\,(\rho_1/z)\,(\pi_{\bar 1}/P^+)\right]^2} e^{-\left[t\,(\rho_2/z)\,(\pi_{\bar 2}/P^+)\right]^2} \;, \end{aligned} \tag{847}$$

has a finite limit for $m_g \to 0$. I need to check the above condition before I tackle the comparison. Start 20250121 13:39 wto Ekologiczna Convergence of $\tilde{C}(\psi' - \psi)$ is shown using a change of variables, rather than an explicit integration. I use

$$q^\perp \;=\; \sqrt{|z|}\, k^\perp \;, \tag{848}$$

and I obtain

$$\begin{aligned}\tilde{C}(\psi' - \psi) \;&=\; \frac{1}{P^+}\int_{-x_1}^{x_2} dz \int d^2k^\perp \;\frac{\rho_1+\rho_2}{2|z|}\; 4\sqrt{\frac{x_1 x_2}{x_{1'} x_{2'}}} \\ &\times\; \left[\frac{\rho_1\pi_{\bar 1}^2+\rho_2\pi_{\bar 2}^2}{\rho_1^2\pi_{\bar 1}^2+\rho_2^2\pi_{\bar 2}^2-(\rho_1-\rho_2)^2\pi_{12}^2} - \frac{1/2}{\rho_1+M^2} - \frac{1/2}{\rho_2+M^2}\right] \\ &\times\; \left(\psi_z z + \psi_\perp\, k^\perp\,\sqrt{|z|}\right)\; e^{-\left[t\,(\rho_1/z)\,(\pi_{\bar 1}/P^+)\right]^2} e^{-\left[t\,(\rho_2/z)\,(\pi_{\bar 2}/P^+)\right]^2} \;,\end{aligned} \tag{849}$$

where, using $p_1 = x_1 P + k$, $p_{1'} = x_{1'} P + k'$ etc., I have

$$\rho_1 \;=\; m_g^2 + \frac{m_1^2 (z/x_1)^2 + (\sqrt{|z|} k^\perp - k_{12}^\perp z/x_1)^2}{1+z/x_1} \;, \tag{850}$$

$$\rho_2 \;=\; m_g^2 + \frac{m_2^2 (z/x_2)^2 + (\sqrt{|z|} k^\perp + k_{12}^\perp z/x_2)^2}{1-z/x_2} \;. \tag{851}$$

Then,

$$\rho_1 \;=\; m_g^2 + |z| \frac{(m_1/x_1)^2|z| + (k^\perp - s_z\sqrt{|z|}\, k_{12}^\perp/x_1)^2}{1+z/x_1} \;, \tag{852}$$

$$\rho_2 \;=\; m_g^2 + |z| \frac{(m_2/x_2)^2|z| + (k^\perp + s_z\sqrt{|z|}\, k_{12}^\perp/x_2)^2}{1-z/x_2)} \;. \tag{853}$$

It is visible that ρ_1 and ρ_2 are non-negative, they can be 0 only when $m_g = 0$ and $z = 0$, and they both approach m_g^2 plus terms that tend to 0 as $|z| k^{\perp 2}$ when $z \to 0$ and $|k^\perp| > 0$ is kept fixed. For all values of momenta $k^\perp$ and fraction z,

$$\rho_1 - \rho_2 \;\sim\; a\,|z| \;, \tag{854}$$

where

$$\rho_1/|z| \;=\; \frac{m_g^2}{|z|} + \frac{m_1^2|z|/x_1^2 + (k^\perp - k_{12}^\perp s_z\sqrt{|z|}/x_1)^2}{1+z/x_1} \;, \tag{855}$$

$$\rho_2/|z| \;=\; \frac{m_g^2}{|z|} + \frac{m_2^2|z|/x_2^2 + (k^\perp + k_{12}^\perp s_z\sqrt{|z|}/x_2)^2}{1-z/x_2} \;. \tag{856}$$

It is now visible that $\rho_1/|z|$ and $\rho_2/|z|$ behave for fixed z and large $k^\perp$ as $k^{\perp 2}/(1 + z/x_1)$ and $k^{\perp 2}/(1 - z/x_2)$, respectively. In the integral

$$\begin{aligned}\tilde{C}(\psi' - \psi) \;&=\; \frac{1}{P^+}\int_{-x_1}^{x_2} dz \int d^2k^\perp \;\frac{\rho_1+\rho_2}{2|z|}\; 4\sqrt{\frac{x_1 x_2}{x_{1'} x_{2'}}} \\ &\times\; \left[\frac{\rho_1\pi_{\bar 1}^2+\rho_2\pi_{\bar 2}^2}{\rho_1^2\pi_{\bar 1}^2+\rho_2^2\pi_{\bar 2}^2-(\rho_1-\rho_2)^2\pi_{12}^2} - \frac{1/2}{\rho_1+M^2} - \frac{1/2}{\rho_2+M^2}\right] \\ &\times\; \left(\psi_z z + \psi_\perp\, k^\perp\,\sqrt{|z|}\right)\; e^{-\left[t\,(\rho_1/z)\,(\pi_{\bar 1}/P^+)\right]^2} e^{-\left[t\,(\rho_2/z)\,(\pi_{\bar 2}/P^+)\right]^2} \;,\end{aligned} \tag{857}$$

the range of integration over $k^{\perp 2}$ when $m_g^2 = 0$ is limited from above by about

$$\Lambda_k^2 \;=\; \left\{[t(\pi_{\bar 1}/P^+)(x_1/x_{1'})]^2 + [t(\pi_{\bar 2}/P^+)(x_2/x_{2'})]^2\right\}^{-1/2} \tag{858}$$

$$=\; \frac{1}{t}\left[\left(\frac{\pi_{\bar 1} x_1}{P^+ x_{1'}}\right)^2 + \left(\frac{\pi_{\bar 2} x_2}{P^+ x_{2'}}\right)^2\right]^{-1/2} \;=\; \frac{1}{t}\left[\left(x_{\bar 1}\,\frac{x_1}{x_{1'}}\right)^2 + \left(x_{\bar 2}\,\frac{x_2}{x_{2'}}\right)^2\right]^{-1/2} \;. \tag{859}$$

This is a finite upper bound. For $m_g = 0$ and $z \to 0$ one has

$$\Lambda_k^2 = \frac{1}{t\sqrt{x_1^2 + x_2^2}} . \tag{860}$$

The wave-function difference softens $1/|z|$ to $1/\sqrt{|z|}$, which is integrable over z. Start 20250122 03:30 sro Ekologiczna, Continued 20250123 from early morning czw Ekologiczna The remaining question concerning large transverse momentum $k^\perp$ is if the limit $m_g \to 0$ exists and if it is the same when one starts with $m_g > 0$. Regarding the exponential factors, their arguments are

$$\rho_i/|z| = m_g^2/|z| + \rho_{i0}/|z| , \tag{861}$$
$$(\rho_i/|z|)^2 = (m_g^2/z)^2 + 2m_g^2\rho_{i0}/z^2 + \rho_{i0}^2/z^2 , \tag{862}$$

and

$$e^{-a(\rho_i/|z|)^2} = e^{-a(m_g^2/z)^2}\, e^{-a2m_g^2\rho_{i0}/|z|}\, e^{-a(\rho_{i0}/z)^2} . \tag{863}$$

The third exponential function is the same as the one for $m_g = 0$. The first and second exponential functions provide additional suppression of the integrand for small z or large $k^\perp$. The third exponential limits z and $k^\perp$ as in the case of $m_g = 0$. The additional suppression by the first and second exponentials goes away when $m_g \to 0$. The limit of the exponential factors is the same as the result for $m_g = 0$.

The next question is what happens when $k^\perp$ is limited or tends to 0. The terms with M^2 in denominator are safe, because M^2 is positive and excludes zero in the denominator while ρ_is in the numerator have definite limits when $m_g \to 0$. The issue is how Δ_{232} behaves for small ρ_is. Given expressions for ρ_is, I can replace every ρ by $\tilde\rho = \rho/|z|$ in both the RGPEP Δ_{232} and the gluon spin factor $\rho_1 + \rho_2$ by $\tilde\rho = \rho/|z|$,

$$\tilde\rho_1 = \rho_1/|z| = \frac{m_g^2}{|z|} + \frac{m_1^2|z|/x_1^2 + (k^\perp - k_{12}^\perp s_z\sqrt{|z|}/x_1)^2}{1 + z/x_1} , \tag{864}$$
$$\tilde\rho_2 = \rho_2/|z| = \frac{m_g^2}{|z|} + \frac{m_2^2|z|/x_2^2 + (k^\perp + k_{12}^\perp s_z\sqrt{|z|}/x_2)^2}{1 - z/x_2} . \tag{865}$$

I look at

$$\Delta = \frac{(\tilde\rho_1 + \tilde\rho_2)(\tilde\rho_1\pi_1^2 + \tilde\rho_2\pi_2^2)}{\tilde\rho_1^2\pi_1^2 + \tilde\rho_2^2\pi_2^2 - (\tilde\rho_1 - \tilde\rho_2)^2\pi_{12}^2} = \frac{\tilde\rho_1^2\pi_1^2 + \tilde\rho_2^2\pi_2^2 + \tilde\rho_1\tilde\rho_2(\pi_1^2 + \pi_2^2)}{\tilde\rho_1^2\pi_1^2 + \tilde\rho_2^2\pi_2^2 - (\tilde\rho_1 - \tilde\rho_2)^2\pi_{12}^2} . \tag{866}$$

Cancellation in the denominator was thought to be eliminated by the corresponding term in part B of the eigenvalue equation. Stop 20250123 18:23 czw Ekologiczna padam Start 20250124 04:32 pia Ekologiczna In the case of $m_g = 0$ and all π_i being the same, the factor to check is

$$\Delta = \frac{(\tilde\rho_1 + \tilde\rho_2)^2}{2\tilde\rho_1\tilde\rho_2} = \frac{1}{2}\left(\frac{\tilde\rho_1}{\tilde\rho_2} + \frac{\tilde\rho_2}{\tilde\rho_1} + 2\right) . \tag{867}$$

Can only one $\tilde\rho$ vanish? This would require that z vanishes and consequently also that $k^\perp$ vanishes. Hence, both $\tilde\rho$ would vanish. If both vanish, one vanishes, too. So one vanishing is equivalent to both vanishing. The question is then if the ratio of two $\tilde\rho$s can diverge as $1/k^{\perp 2}$ or $1/\sqrt{|z|}$. Consider the example of ratio

$$\frac{\tilde\rho_1}{\tilde\rho_2} = \frac{\frac{m_g^2}{|z|} + m_1^2|z|/x_1^2 + (k^\perp - k_{12}^\perp s_z\sqrt{|z|}/x_1)^2}{\frac{m_g^2}{|z|} + m_2^2|z|/x_2^2 + (k^\perp + k_{12}^\perp s_z\sqrt{|z|}/x_2)^2}\,\frac{1 - z/x_2}{1 + z/x_1} . \tag{868}$$

Since z cannot approach $-x_1$ or x_2 too closely in the eigenvalue problem (the basis states would have too large invariant masses, which are suppressed by the RGPEP width limit on changes of invariant masses), I can drop the second factor. It is close to 1 and it cannot diverge. Dividing the numerator and denominator by $|z|$ I get

$$\frac{\tilde\rho_1}{\tilde\rho_2} \sim \frac{\frac{m_g^2}{z^2} + m_1^2/x_1^2 + (k^\perp/\sqrt{|z|} - k_{12}^\perp s_z/x_1)^2}{\frac{m_g^2}{z^2} + m_2^2/x_2^2 + (k^\perp/\sqrt{|z|} + k_{12}^\perp s_z/x_2)^2} . \tag{869}$$

Now the behavior of the ratio $R(\xi^\perp) = \tilde\rho_1/\tilde\rho_2$ is reduced to a function of one variable $\xi^\perp = k^\perp/\sqrt{|z|}$ that can range from $-\infty$ to $+\infty$,

$$R(\xi^\perp) \ \sim \ \frac{\frac{m_g^2}{z^2} + a_1 + (\xi^\perp - b_1^\perp)^2}{\frac{m_g^2}{z^2} + a_2 + (\xi^\perp - b_2^\perp)^2} \ . \tag{870}$$

with finite positive a_1, a_2, $b_1^\perp$ and $b_2^\perp$. This function can be written in terms of $\zeta^\perp = \xi^\perp - b_2^\perp$ in the form

$$\tilde R(\zeta^\perp) \ \sim \ \frac{\frac{m_g^2}{z^2} + a_1 + (\zeta^\perp + b^\perp)^2}{\frac{m_g^2}{z^2} + a_2 + \zeta^{\perp 2}} \ = \ \frac{\frac{m_g^2}{z^2} + a_1 + \zeta_\perp^2 + (\zeta_{||} + b)^2}{\frac{m_g^2}{z^2} + a_2 + \zeta_\perp^2 + \zeta_{||}^2} \ . \tag{871}$$

It is visible that for large $\zeta^\perp$ one obtains $\tilde r = 1$. One can find the minimum and maximum but it is not necessary. It suffices that the ratio $\tilde R(\zeta^\perp)$ has limited positive size for finite positive a_1, a_2 and limited $b^\perp$. This means that $1/\tilde R(\zeta^\perp)$ is also positive and limited. Therefore both $R(\xi^\perp)$ and $1/R(\xi^\perp)$ are limited. Hence, Δ is limited and thus the integral $\tilde C(\psi' - \psi)$ in Eq. (857) is finite for $m_g = 0$. When $m_g > 0$, I get $R = 1$ for $z \to 0$ and there is no divergence. But there is a range of variation around 1.

Now the question is what happens when π_i are not all equal. I can still write Δ in terms of the ratio $R(\xi^\perp) = \tilde\rho_1/\tilde\rho_2$,

$$\Delta \ = \ \frac{(R+1)(R\pi_1^2 + \pi_2^2)}{R^2\pi_1^2 + \pi_2^2 - (R-1)^2\pi_{12}^2} \ . \tag{872}$$

Since R is limited and positive, separated from 0, one could conclude that Δ is limited if the denominator could not be zero. But it can be 0. However, the complete $Q\bar Q$ eigenvalue problem includes the part B that effectively provides a factor that vanishes precisely when the denominator vanishes (Is it true? YES, IT IS TRUE, see Eqs. (752) to (762). I inspect below how the interplay between Δ, the RGPEP factors f and the parts B and C eliminates the denominator 0 and leads to the conclusion that the subtraction $\psi' - \psi$ is a legitimate step. However, for $m_g > 0$, I get for $z \to 0$ that $k^\perp$ is limited, $R = 1$ and $\Delta = 2$, independently of 0 that may occur away from $z = 0$.

1. *The denominator 0 in Δ_{LIR}*

The $Q\bar Q$ eigenvalue Eq. (751)

$$\begin{aligned} \mathcal{H}_{LR} \ &= \ \mathcal{H}_f + f_{LR} \left[\mathcal{H}_0^{(2)} + \Delta_{LIR}\, \mathcal{H}_0^{(1)}\, \mathcal{H}_0^{(1)} \right] \\ &+ \ \frac{1}{2} f_{LI} f_{IR}\, \mathcal{H}_0^{(1)} \left[(P_L^- - \mathcal{H}_f)^{-1} + (P_R^- - \mathcal{H}_f)^{-1} - 2\Delta_{LIR} \right] \mathcal{H}_0^{(1)} \ . \end{aligned} \tag{873}$$

The product $\mathcal{H}_0^{(1)}\,\mathcal{H}_0^{(1)}$ provides the factor $(\rho_1 + \rho_2)/z^2$. The divergent dependence on r occurs in the exchange with one f due to $1/z^2$. In the case of exchange with ff, the same factor $(\rho_1+\rho_2)/z^2$ appears but the factors ff suppress small z. The wave function difference turns $1/z^2$ to $1/|z|$ or to $1/|z|^{3/2}$, which is seen using $q^\perp = \sqrt{|z|} k^\perp$. The issue is that in

$$(\rho_1 + \rho_2)\,(f_{LR} - f_{LI} f_{IR})\,\Delta_{LIR} \tag{874}$$

the denominator zero in $\Delta = (\rho_1 + \rho_2)\Delta_{LIR}$, which comes from the denominator in Δ_{LIR}, is compensated by the difference of fs in the numerator. Therefore, I need to verify if dealing with divergences involving Δ_{LIR} with f_{LR} or Δ_{LIR} with $f_{LI} f_{IR}$ separately, as I did so far (20250125 05:46 sob Ekologiczna), is legitimate. In other words, are zeros in the denominator of Δ_{LIR} invalidating my analysis and computation of counterterms.

First, I use ρ_is to compare the functions f_{LR} and $f_{LI} f_{IR}$, the latter already displayed in Eq. (842). The former is $\exp\{-[t(\mathcal{M}_{12}^2 - \mathcal{M}_{1'2'}^2)\pi_{12}/P^+]^2\}$, where

$$\begin{aligned} \mathcal{M}_{12}^2 - \mathcal{M}_{1'2'}^2 \ &= \ (p_{1'}^+ + p_{2'}^+)(p_{1'}^- + p_{2'}^-) - (p_1^+ + p_2^+)(p_1^- + p_2^-) && (875) \\ &= \ P^+(p_{1'}^- + p_{2'}^- - p_1^+ - p_2^+) \ = \ s_z P^+(q_1^- - q_2^-) && (876) \\ &= \ (s_z P^+/q_1^+)(q_1^+ q_1^- - q_1^{\perp 2} - q_2^+ q_2^- + q_2^{\perp 2}) && (877) \\ &= \ (1/z)(q_1^2 - q_2^2) \ = \ (\rho_1 - \rho_2)/z \ . && (878) \end{aligned}$$

Therefore,

$$
\begin{aligned}
f_{LR} &= e^{-\left[t(\rho_1/z-\rho_2/z)\pi_{12}/P^+\right]^2} \ , && (879)\\
f_{LI}f_{IR} &= e^{-\left[t\,(\rho_1/z)\,\pi_{\bar 1}/P^+\right]^2} e^{-\left[t\,(\rho_2/z)\,\pi_{\bar 2}/P^+\right]^2} \ . && (880)
\end{aligned}
$$

In the case of my boost-ivariant version,

$$
\begin{aligned}
f_{LR} &= e^{-[t(\rho_1-\rho_2)/z]^2} \ , && (881)\\
f_{LI}f_{IR} &= e^{-[t\,\rho_1\,x_{\bar 1}/z]^2} e^{-[t\,\rho_2\,x_{\bar 2}/z]^2} \ . && (882)
\end{aligned}
$$

Thus, I can change variables using $q^\perp = \sqrt{|z|}k^\perp$ that reduces the singularity for small $|z|$ to the integrable $1/\sqrt{|z|}$ while I use

$$
\begin{aligned}
\tilde\rho_1 &= \rho_1/|z| = \frac{m_1^2|z|/x_1^2 + (k^\perp - k_{12}^\perp s_z\sqrt{|z|}/x_1)^2}{1+z/x_1} \ , && (883)\\
\tilde\rho_2 &= \rho_2/|z| = \frac{m_2^2|z|/x_2^2 + (k^\perp + k_{12}^\perp s_z\sqrt{|z|}/x_2)^2}{1-z/x_2} \ . && (884)
\end{aligned}
$$

in

$$
\Delta = \frac{(\tilde\rho_1+\tilde\rho_2)(\tilde\rho_1\pi_{\bar 1}^2+\tilde\rho_2\pi_{\bar 2}^2)}{\tilde\rho_1^2\pi_{\bar 1}^2+\tilde\rho_2^2\pi_{\bar 2}^2-(\tilde\rho_1-\tilde\rho_2)^2\pi_{12}^2} \qquad (885)
$$

with

$$
\begin{aligned}
f_{LR} &= e^{-[t(\tilde\rho_1-\tilde\rho_2)]^2} \ , && (886)\\
f_{LI}f_{IR} &= e^{-[t\,\tilde\rho_1\,x_{\bar 1}]^2} e^{-[t\,\tilde\rho_2\,x_{\bar 2}]^2} \ . && (887)
\end{aligned}
$$

I can use the argument that $k^\perp$ is limited with $f - ff$ as well as I had done it for f and for ff separately.

Second, I note that in the self-interactions and mass counterterms, the factor Δ_{LIR} is not sensitive to π_is. It only provides the denominator that is the same as in the old-fashioned Hamiltonian perturbation theory, no matter what π_is I choose. In the self-interaction terms, there is no issue with denominator 0 in Δ because the invariant mass of quark and gluon is always greater than that of the quark, except the case when the gluon momentum is zero, which lies on the edge of the integration region. The singularity from that region is accounted for and a counterterm is introduced.

In the exchange terms, one ought to consider

$$
(f_{LR} - f_{LI}f_{IR})\Delta_{LIR}\,\mathcal{H}_0^{(1)}\,\mathcal{H}_0^{(1)} \ , \qquad (888)
$$

where the bracket factor cancels the denominator of Δ_{LIR} when the latter tends to 0. My analysis of the exchange was so far (20250125 11:14 sob Ekologiczna) carried out for the two terms separately, the term with the factor $f_{LR}\Delta_{LIR}$, which is singular for finite s because of the gluon small-x singularity, and the term with the factor $f_{LI}f_{IR}\Delta_{LIR}$ that for finite s excludes small-x singularities but depends on the gluon mass m_g. The question is how is each of these analyses altered when the factors f_{LR} or $f_{LI}f_{IR}$ are replaced by their difference. In the case of part B, see Eq. (763), repeated here,

$$
B = f_{LR}\left[H_0^{(2)} + \Delta_{L3R}\,H_0^{(1)}\,H_0^{(1)}\right] \ , \qquad (889)
$$

I go back to Eq. (763) and check what changes in the subsequent reasoning and equations for the exchange terms when I replace f_{LR} by $f_{LR} - f_{LI}f_{IR}$. The seagull analysis is not changed at all because it does not involve Δ . The self-interactions are free from the denominator 0 in Δ as already explained in a paragraph above; the denominator 0 is absent. I need to identify the change in the computation of the ccounterterm to the diverging dependence on r when instead of $f_{LR}\Delta_{L3R}\,H_0^{(1)}\,H_0^{(1)}$ I consider the dependence on r of the exchange

$$
(f_{LR} - f_{LI}f_{IR})\Delta_{LIR}\,\mathcal{H}_0^{(1)}\,\mathcal{H}_0^{(1)} \ . \qquad (890)
$$

The dependence on r comes only from the first term. One can set r to 0 in the second term; the second term has a finite limit when $r \to 0$. The counterterm to the first term contains $\delta(z)$. For $z = 0$, the second term vanishes because of the factor $f_{LI} f_{IR}$ that vanishes for $m_g^2/z \to \infty$. Start 20250126 06:50 nie Ekologiczna I go back to Eqs. (850) and (851), repeated here

$$\rho_1 = m_g^2 + \frac{m_1^2(z/x_1)^2 + (q^\perp + k_{12}^\perp z/x_1)^2}{1 + z/x_1} , \tag{891}$$

$$\rho_2 = m_g^2 + \frac{m_2^2(z/x_2)^2 + (q^\perp + k_{12}^\perp z/x_2)^2}{1 - z/x_2} . \tag{892}$$

It is clear that for $q^\perp$ limited by fs and $z \to 0$ I have $\rho_i = m_g^2 + q^{\perp 2}$ and there is no zero in the denominator of $\Delta = 2$.

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Higher orders
Extrapolation $g \to g_s$
Beyond perturbative expansion for H_s
Successive approximations
Increasing $m_g \to m_G$
No need for artificial potential and scaling of oscillator
Mass for confinement instead of potential
Gluons needed above the oscillator

Appendix A: Gluon self-interaction computation

$$\hat{A}_f^\mu = \sum_{c=1}^{8} \sum_{\sigma=1}^{2} \int [p] \left[\varepsilon_{p\sigma}^\mu T^c \hat{a}_{p\sigma c} e^{-ipx} + \varepsilon_{p\sigma}^{\mu *} T^c \hat{a}_{p\sigma c}^\dagger e^{ipx}\right]_{x^+=0} , \tag{A1}$$

$$\hat{\phi} = \sum_{c=1}^{8} \int [p] \left[-iT^c \hat{a}_{p3c} e^{-ipx} + iT^c \hat{a}_{p3c}^\dagger e^{ipx}\right]_{x^+=0} . \tag{A2}$$

1. Computation of H_{+1}

$$J_A^{a+} = -g[i\partial^+ A^\perp, A^\perp]^a , \tag{A3}$$

$$\mathcal{H}_{+1} = \left(\partial^\perp A_f^{a\perp} + m_g \phi^a\right) \frac{1}{\partial^+} J_A^{a+} \equiv \left[\frac{-1}{i\partial^+}\left(i\partial^\perp A_f^{a\perp} + i m_g \phi^a\right)\right] J_A^{a+} . \tag{A4}$$

Common factors.

$$\frac{\Sigma_g^{(2)}}{p^+} = \frac{g^2}{p^+} \int \frac{dk^+ d^2k^\perp}{2(2\pi)^3 k^+(p^+ - k^+)} \frac{|N|^2}{P^- - Q^- - p_1^- - p_2^-} . \tag{A5}$$

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$$J_A^{a+} \to -gif^{12a} p_1^+ \left(\varepsilon_1^\perp a_1 - \varepsilon_1^{\perp *} a_1^\dagger\right)\left(\varepsilon_2^\perp a_2 + \varepsilon_2^{\perp *} a_2^\dagger\right) \tag{A6}$$

$$\begin{aligned} &= -gif^{12a}\, p_1^+\, \varepsilon_1^\perp a_1\, \varepsilon_2^\perp a_2 - gif^{12a}\, p_1^+\, \varepsilon_1^\perp a_1\, \varepsilon_2^{\perp *} a_2^\dagger \\ &+ gif^{12a}\, p_1^+\, \varepsilon_1^{\perp *} a_1^\dagger\, \varepsilon_2^\perp a_2 + gif^{12a}\, p_1^+\, \varepsilon_1^{\perp *} a_1^\dagger\, \varepsilon_2^{\perp *} a_2^\dagger . \end{aligned} \tag{A7}$$

Normal order and arrangement.

$$\begin{aligned} J_A^{3+} \to & -gif^{123}\, p_1^+\, \varepsilon_1^\perp \varepsilon_2^\perp\, a_1 a_2 - gif^{123}\, p_1^+\, \varepsilon_2^{\perp *} \varepsilon_1^\perp\, a_2^\dagger a_1 \\ & - gif^{123}\, p_2^+\, \varepsilon_2^{\perp *} \varepsilon_1^\perp\, a_2^\dagger a_1 + gif^{123}\, p_1^+\, \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\, a_1^\dagger a_2^\dagger . \end{aligned} \tag{A8}$$

$$\frac{-1}{i\partial^+}\left(i\partial^\perp A_f^{3\perp} + i m_g \phi^3\right) \;\rightarrow\; -\left[\left(\frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} a_3 + \frac{p_3^\perp \varepsilon_3^{\perp *}}{p_3^+} a_3^\dagger\right) + \frac{m_g}{p_3^+}\left(\alpha_3 + \alpha_3^\dagger\right)\right] \; . \tag{A9}$$

Simplify.

$$J_A^{3+} \;\rightarrow\; -gif^{123} \;\left[p_1^+ \;\varepsilon_1^\perp \varepsilon_2^\perp \; a_1 a_2 + (p_1^+ + p_2^+)\; \varepsilon_2^{\perp *}\varepsilon_1^\perp \; a_2^\dagger a_1 - p_1^+ \; \varepsilon_1^{\perp *}\varepsilon_2^{\perp *} \; a_1^\dagger a_2^\dagger\right] \; . \tag{A10}$$

$$\frac{-1}{i\partial^+}\left(i\partial^\perp A_f^{3\perp} + i m_g \phi^3\right) \;\rightarrow\; -\left[\left(\frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} a_3 + \frac{p_3^\perp \varepsilon_3^{\perp *}}{p_3^+} a_3^\dagger\right) + \frac{m_g}{p_3^+}\left(\alpha_3 + \alpha_3^\dagger\right)\right] \; . \tag{A11}$$

Multiplication.

$$H_{+1} \;=\; \int_F \frac{-1}{i\partial^+}\left(i\partial^\perp A_f^{3\perp} + i m_g \phi^3\right) J_A^{3+} \tag{A12}$$

$$\begin{aligned} &\rightarrow\; -\sum_{123}\int [123]\;\tilde{\delta}_{c.a}\left(\frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} a_3 + \frac{p_3^\perp \varepsilon_3^{\perp *}}{p_3^+} a_3^\dagger + \frac{m_g}{p_3^+}\alpha_3 + \frac{m_g}{p_3^+}\alpha_3^\dagger\right) \\ &\times\; -gif^{123}\;\left[p_1^+ \;\varepsilon_1^\perp \varepsilon_2^\perp \; a_1 a_2 + (p_1^+ + p_2^+)\; \varepsilon_2^{\perp *}\varepsilon_1^\perp \; a_2^\dagger a_1 - p_1^+ \; \varepsilon_1^{\perp *}\varepsilon_2^{\perp *} \; a_1^\dagger a_2^\dagger\right] \end{aligned} \tag{A13}$$

Simplify.

$$\begin{aligned} H_{+1} \;&\rightarrow\; \sum_{123}\int [123]\; gif^{123}\;\tilde{\delta}_{c.a}\left(\frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} a_3 + \frac{p_3^\perp \varepsilon_3^{\perp *}}{p_3^+} a_3^\dagger + \frac{m_g}{p_3^+}\alpha_3 + \frac{m_g}{p_3^+}\alpha_3^\dagger\right) \\ &\times\;\;\left[p_1^+ \;\varepsilon_1^\perp \varepsilon_2^\perp \; a_1 a_2 + (p_1^+ + p_2^+)\; \varepsilon_2^{\perp *}\varepsilon_1^\perp \; a_2^\dagger a_1 - p_1^+ \; \varepsilon_1^{\perp *}\varepsilon_2^{\perp *} \; a_1^\dagger a_2^\dagger\right] \end{aligned} \tag{A14}$$

$$= \; \sum_{123}\int [123]\; gif^{123}\;\tilde{\delta}_{c.a}\; ig\; S \; . \tag{A15}$$

There are 12 terms to simplify.

$$\begin{aligned} S \;=\;& f^{123}\frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} a_3 \left[(p_1^+ + p_2^+)\; \varepsilon_2^{\perp *}\varepsilon_1^\perp \; a_2^\dagger a_1 - p_1^+ \; \varepsilon_1^{\perp *}\varepsilon_2^{\perp *} \; a_1^\dagger a_2^\dagger\right] \\ +\;& f^{123}\frac{p_3^\perp \varepsilon_3^{\perp *}}{p_3^+} a_3^\dagger \left[p_1^+ \;\varepsilon_1^\perp \varepsilon_2^\perp \; a_1 a_2 + (p_1^+ + p_2^+)\; \varepsilon_2^{\perp *}\varepsilon_1^\perp \; a_2^\dagger a_1\right] \\ +\;& f^{123}\frac{m_g}{p_3^+}\alpha_3 \left[(p_1^+ + p_2^+)\; \varepsilon_2^{\perp *}\varepsilon_1^\perp \; a_2^\dagger a_1 - p_1^+ \; \varepsilon_1^{\perp *}\varepsilon_2^{\perp *} \; a_1^\dagger a_2^\dagger\right] \\ +\;& f^{123}\frac{m_g}{p_3^+}\alpha_3^\dagger \left[p_1^+ \;\varepsilon_1^\perp \varepsilon_2^\perp \; a_1 a_2 + (p_1^+ + p_2^+)\; \varepsilon_2^{\perp *}\varepsilon_1^\perp \; a_2^\dagger a_1\right] \; . \end{aligned} \tag{A16}$$

Move.

$$\begin{aligned} S \;=\;& f^{123}\frac{p_3^\perp \varepsilon_3^\perp}{p_3^+}(p_1^+ + p_2^+)\; \varepsilon_2^{\perp *}\varepsilon_1^\perp \; a_2^\dagger a_3 a_1 \qquad 2 \leftrightarrow 3 \\ -\;& f^{123}\frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} p_1^+ \; \varepsilon_1^{\perp *}\varepsilon_2^{\perp *} \; a_1^\dagger a_2^\dagger a_3 \\ +\;& f^{123}\frac{p_3^\perp \varepsilon_3^{\perp *}}{p_3^+} p_1^+ \; \varepsilon_1^\perp \varepsilon_2^\perp \; a_3^\dagger a_2 a_1 \\ +\;& f^{123}\frac{p_3^\perp \varepsilon_3^{\perp *}}{p_3^+}(p_1^+ + p_2^+)\; \varepsilon_2^{\perp *}\varepsilon_1^\perp \; a_3^\dagger a_2^\dagger a_1 \qquad 1 \leftrightarrow 3 \\ +\;& f^{123}\frac{m_g}{p_3^+}(p_1^+ + p_2^+)\; \varepsilon_2^{\perp *}\varepsilon_1^\perp \; a_2^\dagger \alpha_3 a_1 \qquad 2 \leftrightarrow 3 \\ -\;& f^{123}\frac{m_g}{p_3^+} p_1^+ \; \varepsilon_1^{\perp *}\varepsilon_2^{\perp *} \; a_1^\dagger a_2^\dagger \alpha_3 \\ +\;& f^{123}\frac{m_g}{p_3^+} p_1^+ \; \varepsilon_1^\perp \varepsilon_2^\perp \; \alpha_3^\dagger a_2 a_1 \\ +\;& f^{123}\frac{m_g}{p_3^+}(p_1^+ + p_2^+)\; \varepsilon_2^{\perp *}\varepsilon_1^\perp \; \alpha_3^\dagger a_2^\dagger a_1 \qquad 1 \leftrightarrow 3 \; . \end{aligned} \tag{A17}$$

Changes.

$$
\begin{aligned}
S &= f^{123} S' \ , && \text{(A18)}\\
S' &= -\frac{p_2^\perp \varepsilon_2^\perp}{p_2^+}(p_1^+ + p_3^+)\ \varepsilon_3^{\perp *}\varepsilon_1^\perp\ a_3^\dagger a_2 a_1 \\
&- \frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} p_1^+\ \varepsilon_1^{\perp *}\varepsilon_2^{\perp *}\ a_1^\dagger a_2^\dagger a_3 \\
&+ \frac{p_3^\perp \varepsilon_3^{\perp *}}{p_3^+} p_1^+\ \varepsilon_1^\perp \varepsilon_2^\perp\ a_3^\dagger a_2 a_1 \\
&- \frac{p_1^\perp \varepsilon_1^{\perp *}}{p_1^+}(p_3^+ + p_2^+)\ \varepsilon_2^{\perp *}\varepsilon_3^\perp\ a_1^\dagger a_2^\dagger a_3 \\
&- \frac{m_g}{p_2^+}(p_1^+ + p_3^+)\ \varepsilon_3^{\perp *}\varepsilon_1^\perp\ a_3^\dagger \alpha_2 a_1 \\
&- \frac{m_g}{p_3^+} p_1^+\ \varepsilon_1^{\perp *}\varepsilon_2^{\perp *}\ a_1^\dagger a_2^\dagger \alpha_3 \\
&+ \frac{m_g}{p_3^+} p_1^+\ \varepsilon_1^\perp \varepsilon_2^\perp\ \alpha_3^\dagger a_2 a_1 \\
&- \frac{m_g}{p_1^+}(p_3^+ + p_2^+)\ \varepsilon_2^{\perp *}\varepsilon_3^\perp\ \alpha_1^\dagger a_2^\dagger a_3 \ . && \text{(A19)}
\end{aligned}
$$

Collection.

$$
\begin{aligned}
S &= f^{123} S' \ , && \text{(A20)}\\
S' &= -\left[\frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} p_1^+\ \varepsilon_1^{\perp *}\varepsilon_2^{\perp *} + \frac{p_1^\perp \varepsilon_1^{\perp *}}{p_1^+}(p_3^+ + p_2^+)\ \varepsilon_2^{\perp *}\varepsilon_3^\perp\right]\ a_1^\dagger a_2^\dagger a_3 \\
&+ \left[\frac{p_3^\perp \varepsilon_3^{\perp *}}{p_3^+} p_1^+\ \varepsilon_1^\perp \varepsilon_2^\perp + \frac{p_1^\perp \varepsilon_1^\perp}{p_1^+}(p_3^+ + p_2^+)\ \varepsilon_2^\perp \varepsilon_3^{\perp *}\right]\ a_3^\dagger a_2 a_1 \\
&- \frac{m_g}{p_2^+}(p_3^+ + p_1^+)\ \varepsilon_1^\perp \varepsilon_3^{\perp *}\ a_3^\dagger \alpha_2 a_1 + \frac{m_g}{p_2^+}(p_3^+ + p_1^+)\ \varepsilon_1^{\perp *}\varepsilon_3^\perp\ a_1^\dagger \alpha_2^\dagger a_3 \\
&- \frac{m_g}{p_3^+} p_1^+\ \varepsilon_1^{\perp *}\varepsilon_2^{\perp *}\ a_1^\dagger a_2^\dagger \alpha_3 + \frac{m_g}{p_3^+} p_1^+\ \varepsilon_1^\perp \varepsilon_2^\perp\ \alpha_3^\dagger a_2 a_1 \ . && \text{(A21)}
\end{aligned}
$$

Result.

$$
H_{+1} = g \sum_{123} \int [123]\ \tilde{\delta}_{c.a}\ f^r_{12.3}\ X_{+123} + h.c.\ . \qquad \text{(A22)}
$$

$$
\begin{aligned}
X_{+123} &= i f^{c_1 c_2 c_3} \left[-\frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} p_1^+\ \varepsilon_1^{\perp *}\varepsilon_2^{\perp *} - \frac{p_1^\perp \varepsilon_1^{\perp *}}{p_1^+}(p_3^+ + p_2^+)\ \varepsilon_2^{\perp *}\varepsilon_3^\perp\right]\ a_1^\dagger a_2^\dagger a_3 \\
&- i f^{c_1 c_2 c_3} \frac{m_g}{p_2^+}(p_3^+ + p_1^+)\ \varepsilon_1^\perp \varepsilon_3^{\perp *}\ a_3^\dagger \alpha_2 a_1 \\
&- i f^{c_1 c_2 c_3} \frac{m_g}{p_3^+} p_1^+\ \varepsilon_1^{\perp *}\varepsilon_2^{\perp *}\ a_1^\dagger a_2^\dagger \alpha_3 \ . && \text{(A23)}
\end{aligned}
$$

2. Computation of $H_{\perp 1}$

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The complementary transverse gluon interaction density term, the 3rd term on the right-hand side in Eq. (67), is

$$
(J^{aj}_{\psi f} + J^{aj}_{Af}) A^{aj}_f \ , \qquad \text{(A24)}
$$

of which the purely transverse gluon part reads

$$\mathcal{H}_{\perp 1} \;=\; A_f^{aj} J_{Af}^{aj} \;. \tag{A25}$$

$$J_{Af}^{aj} \;=\; -g[i\partial^j A^\perp, A^\perp]^a \;, \tag{A26}$$

$$J_{Af}^{aj} \;\to\; -gif^{12a} p_1^j \left(\varepsilon_1^\perp a_1 - \varepsilon_1^{\perp *} a_1^\dagger\right)\left(\varepsilon_2^\perp a_2 + \varepsilon_2^{\perp *} a_2^\dagger\right) \tag{A27}$$

$$\begin{aligned} &= -gif^{12a}\; p_1^j\; \varepsilon_1^\perp a_1\; \varepsilon_2^\perp a_2 - gif^{12a}\; p_1^j\; \varepsilon_1^\perp a_1\; \varepsilon_2^{\perp *} a_2^\dagger \\ &+ \; gif^{12a}\; p_1^j\; \varepsilon_1^{\perp *} a_1^\dagger\; \varepsilon_2^\perp a_2 + gif^{12a}\; p_1^j\; \varepsilon_1^{\perp *} a_1^\dagger\; \varepsilon_2^{\perp *} a_2^\dagger \;. \end{aligned} \tag{A28}$$

Normal order and arrangement.

$$\begin{aligned} J_A^{3j} \;&\to\; -gif^{123}\; p_1^j\; \varepsilon_1^\perp \varepsilon_2^\perp\; a_1 a_2 - gif^{123}\; p_1^j\; \varepsilon_2^{\perp *} \varepsilon_1^\perp\; a_2^\dagger a_1 \\ &- \; gif^{123}\; p_2^j\; \varepsilon_2^{\perp *} \varepsilon_1^\perp\; a_2^\dagger a_1 + gif^{123}\; p_1^j\; \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\; a_1^\dagger a_2^\dagger \;, \end{aligned} \tag{A29}$$

$$A_f^{3j} \;\to\; \left(\varepsilon_3^j a_3 + \varepsilon_3^{j*} a_3^\dagger\right) \;. \tag{A30}$$

Simplify.

$$J_A^{3j} \;\to\; -gif^{123}\; \left[p_1^j\; \varepsilon_1^\perp \varepsilon_2^\perp\; a_1 a_2 + (p_1^j + p_2^j)\; \varepsilon_2^{\perp *} \varepsilon_1^\perp\; a_2^\dagger a_1 - p_1^j\; \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\; a_1^\dagger a_2^\dagger\right] \;. \tag{A31}$$

Multiplication.

$$H_{\perp 1} \;=\; \int_F A_f^{3j} J_A^{3+} \;\;\to\;\; \sum_{123} \int [123]\; \tilde{\delta}_{c.a}\; (-ig)\; S \;, \tag{A32}$$

$$S \;=\; f^{123}(\varepsilon_3^j a_3 + \varepsilon_3^{j*} a_3^\dagger)[p_1^j\; \varepsilon_1^\perp \varepsilon_2^\perp\; a_1 a_2 + (p_1^j + p_2^j)\; \varepsilon_2^{\perp *} \varepsilon_1^\perp\; a_2^\dagger a_1 - p_1^j\; \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\; a_1^\dagger a_2^\dagger]. \tag{A33}$$

There are 4 terms to simplify.

$$\begin{aligned} S \;&=\; f^{123} \varepsilon_3^j a_3 [(p_1^j + p_2^j)\; \varepsilon_2^{\perp *} \varepsilon_1^\perp\; a_2^\dagger a_1 - p_1^j\; \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\; a_1^\dagger a_2^\dagger] \\ &+ \; f^{123} \varepsilon_3^{j*} a_3^\dagger [p_1^j\; \varepsilon_1^\perp \varepsilon_2^\perp\; a_1 a_2 + (p_1^j + p_2^j)\; \varepsilon_2^{\perp *} \varepsilon_1^\perp\; a_2^\dagger a_1] \;. \end{aligned} \tag{A34}$$

Move.

$$\begin{aligned} S \;&=\; f^{123} \varepsilon_3^j (p_1^j + p_2^j)\; \varepsilon_2^{\perp *} \varepsilon_1^\perp\; a_2^\dagger a_3 a_1 \quad 2 \leftrightarrow 3 \\ &- \; f^{123} \varepsilon_3^j p_1^j\; \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\; a_1^\dagger a_2^\dagger a_3 \\ &+ \; f^{123} \varepsilon_3^{j*} p_1^j\; \varepsilon_1^\perp \varepsilon_2^\perp\; a_3^\dagger a_2 a_1 \\ &+ \; f^{123} \varepsilon_3^{j*} (p_1^j + p_2^j)\; \varepsilon_2^{\perp *} \varepsilon_1^\perp\; a_3^\dagger a_2^\dagger a_1 \quad 1 \leftrightarrow 3 \;. \end{aligned} \tag{A35}$$

Changes.

$$\begin{aligned} S \;&=\; -f^{123} \varepsilon_2^j (p_1^j + p_3^j)\; \varepsilon_3^{\perp *} \varepsilon_1^\perp\; a_3^\dagger a_2 a_1 \\ &+ \; f^{123} \varepsilon_3^{j*} p_1^j\; \varepsilon_1^\perp \varepsilon_2^\perp\; a_3^\dagger a_2 a_1 \\ &- \; f^{123} \varepsilon_3^j p_1^j\; \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\; a_1^\dagger a_2^\dagger a_3 \\ &- \; f^{123} \varepsilon_1^{j*} (p_3^j + p_2^j)\; \varepsilon_2^{\perp *} \varepsilon_3^\perp\; a_1^\dagger a_2^\dagger a_3 \;. \end{aligned} \tag{A36}$$

Collection.

$$\begin{aligned} S \;&=\; f^{123} \left[\varepsilon_1^j (p_3^j + p_2^j)\; \varepsilon_2^\perp \varepsilon_3^{\perp *} + \varepsilon_3^{j*} p_1^j\; \varepsilon_1^\perp \varepsilon_2^\perp\right]\; a_3^\dagger a_2 a_1 \\ &- \; f^{123} \left[\varepsilon_1^{j*} (p_3^j + p_2^j)\; \varepsilon_2^{\perp *} \varepsilon_3^\perp + \varepsilon_3^j p_1^j\; \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\right]\; a_1^\dagger a_2^\dagger a_3 \;. \end{aligned} \tag{A37}$$

Result.

$$H_{\perp 1} \;=\; g \sum_{123} \int [123]\; \tilde{\delta}_{c.a}\; f_{12.3}^r\; X_{\perp 123} + h.c. \;. \tag{A38}$$

$$X_{\perp 123} \;=\; if^{123} \left[\varepsilon_1^{j*} (p_3^j + p_2^j)\; \varepsilon_2^{\perp *} \varepsilon_3^\perp + \varepsilon_3^j p_1^j\; \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\right]\; a_1^\dagger a_2^\dagger a_3 \;. \tag{A39}$$

3. Summary of H_{+1} and $H_{\perp 1}$

$$H_{+1} = g \sum_{123} \int [123]\ \tilde{\delta}_{c.a}\ f^r_{12.3}\ X_{+123} + h.c.\ , \tag{A40}$$

$$\begin{aligned} X_{+123} &= if^{c_1c_2c_3} \left[-\frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} p_1^+\ \varepsilon_1^{\perp *} \varepsilon_2^{\perp *} - \frac{p_1^\perp \varepsilon_1^{\perp *}}{p_1^+} (p_3^+ + p_2^+)\ \varepsilon_2^{\perp *} \varepsilon_3^\perp \right]\ a_1^\dagger a_2^\dagger a_3 \\ &- if^{c_1c_2c_3} \frac{m_g}{p_2^+} (p_3^+ + p_1^+)\ \varepsilon_1^\perp \varepsilon_3^{\perp *}\ a_3^\dagger \alpha_2 a_1 - if^{c_1c_2c_3} \frac{m_g}{p_3^+} p_1^+\ \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\ a_1^\dagger a_2^\dagger \alpha_3\ , \end{aligned} \tag{A41}$$

$$H_{\perp 1} = g \sum_{123} \int [123]\ \tilde{\delta}_{c.a}\ f^r_{12.3}\ X_{\perp 123} + h.c.\ , \tag{A42}$$

$$X_{\perp 123} = if^{c_1c_2c_3} \left[\varepsilon_1^{j*} (p_3^j + p_2^j)\ \varepsilon_2^{\perp *} \varepsilon_3^\perp + \varepsilon_3^j p_1^j\ \varepsilon_1^{\perp *} \varepsilon_2^{\perp *} \right]\ a_1^\dagger a_2^\dagger a_3\ . \tag{A43}$$

The combined result is

$$\begin{aligned} H_{+1} + H_{\perp 1} &= g \sum_{123} \int [123]\ \tilde{\delta}_{c.a}\ f^r_{12.3}\ if^{123}\ \tilde{Y}_{123}\ a_1^\dagger a_2^\dagger a_3 + h.c. \\ &+ g \sum_{123} \int [123]\ \tilde{\delta}_{c.a}\ f^r_{12.3}\ if^{123} \left(\tilde{Y}_{A123}\ a_3^\dagger \alpha_2 a_1 + \tilde{Y}_{\phi 123}\ a_1^\dagger a_2^\dagger \alpha_3 \right) + h.c.\ , \end{aligned} \tag{A44}$$

$$\begin{aligned} \tilde{Y}_{123} &= -\frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} p_1^+\ \varepsilon_1^{\perp *} \varepsilon_2^{\perp *} - \frac{p_1^\perp \varepsilon_1^{\perp *}}{p_1^+} (p_3^+ + p_2^+)\ \varepsilon_2^{\perp *} \varepsilon_3^\perp \\ &+ \varepsilon_1^{j*} (p_3^j + p_2^j)\ \varepsilon_2^{\perp *} \varepsilon_3^\perp + \varepsilon_3^j p_1^j\ \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\ , \end{aligned} \tag{A45}$$

$$\tilde{Y}_{A123} = -\frac{m_g}{p_2^+} (p_3^+ + p_1^+)\ \varepsilon_1^\perp \varepsilon_3^{\perp *}\ , \tag{A46}$$

$$\tilde{Y}_{\phi 123} = -\frac{m_g}{p_3^+} p_1^+\ \varepsilon_1^{\perp *} \varepsilon_2^{\perp *}\ . \tag{A47}$$

4. Crosscheck of $\tilde{Y}_{123}$

$$\begin{aligned} \tilde{Y}_{123} &= \varepsilon_3^\perp p_1^\perp \cdot \varepsilon_1^{\perp *} \varepsilon_2^{\perp *} - \frac{p_3^\perp \varepsilon_3^\perp}{p_3^+} p_1^+ \cdot \varepsilon_1^{\perp *} \varepsilon_2^{\perp *} - \frac{p_1^\perp \varepsilon_1^{\perp *}}{p_1^+} (p_3^+ + p_2^+) \cdot \varepsilon_2^{\perp *} \varepsilon_3^\perp \\ &+ \varepsilon_1^{\perp *} (p_3^\perp + p_2^\perp) \cdot \varepsilon_2^{\perp *} \varepsilon_3^\perp\ . \end{aligned} \tag{A48}$$

The first two terms are symmetric in spins 1 and 2.

$$\tilde{Y}_{123} = \varepsilon_1^{\perp *} \varepsilon_2^{\perp *} \cdot \varepsilon_3^\perp \left(p_1^\perp - x_1 p_3^\perp \right) - \frac{p_1^\perp \varepsilon_1^{\perp *}}{p_1^+} (p_3^+ + p_2^+) \cdot \varepsilon_2^{\perp *} \varepsilon_3^\perp + \varepsilon_1^{\perp *} (p_3^\perp + p_2^\perp) \cdot \varepsilon_2^{\perp *} \varepsilon_3^\perp \tag{A49}$$

$$= \varepsilon_1^{\perp *} \varepsilon_2^{\perp *} \cdot \varepsilon_3^\perp k_{12}^\perp + \left[-\frac{p_1^\perp}{p_1^+} (p_3^+ + p_2^+) + (p_3^\perp + p_2^\perp) \right] \varepsilon_1^{\perp *} \cdot \varepsilon_2^{\perp *} \varepsilon_3^\perp\ . \tag{A50}$$

The square bracket can be written using $p_3 = p_1 + p_2$ for the $+$ and $\perp$ components.

$$-\frac{p_1^\perp}{p_1^+} (p_1^+ + 2p_2^+) + (p_1^\perp + 2p_2^\perp) = -p_1^\perp - \frac{2p_2^+}{p_1^+} p_1^\perp + p_1^\perp + 2p_2^\perp \tag{A51}$$

$$= -\frac{2p_2^+}{p_1^+} p_1^\perp + 2p_2^\perp = \frac{2}{p_1^+} (-p_2^+ p_1^\perp + p_1^+ p_2^\perp) \tag{A52}$$

$$= \frac{2}{x_1} (-x_2 p_1^\perp + x_1 p_2^\perp) = \frac{-2k_{12}^\perp}{x_1}\ , \tag{A53}$$

So,

$$if^{123}\tilde{Y}_{123} \;=\; if^{123}\varepsilon_1^{\perp *}\varepsilon_2^{\perp *}\cdot\varepsilon_3^{\perp}k_{12}^{\perp} + S_{12}\;, \tag{A54}$$

where

$$S_{12} \;=\; -if^{123}\frac{2k_{12}^{\perp}}{x_1}\varepsilon_1^{\perp *}\cdot\varepsilon_2^{\perp *}\varepsilon_3^{\perp}\;. \tag{A55}$$

One can use

$$S = \frac{1}{2}\left(S_{12}+S_{21}\right) \tag{A56}$$

instead of S_{12} in Eq. (A44), because one sums over 1 and 2 there.

$$2S \;=\; -if^{123}\frac{2k_{12}^{\perp}}{x_1}\varepsilon_1^{\perp *}\cdot\varepsilon_2^{\perp *}\varepsilon_3^{\perp} - if^{213}\frac{2k_{21}^{\perp}}{x_2}\varepsilon_2^{\perp *}\cdot\varepsilon_1^{\perp *}\varepsilon_3^{\perp}\;, \tag{A57}$$

Using $f^{213}=-f^{123}$ and $k_{21}=-k_{12}$ I obtain

$$if^{123}\tilde{Y}_{123} \;=\; if^{123}\left[\varepsilon_1^{\perp *}\varepsilon_2^{\perp *}\cdot\varepsilon_3^{\perp}k_{12}^{\perp} - \frac{k_{12}^{\perp}\varepsilon_1^{\perp *}}{x_1}\cdot\varepsilon_2^{\perp *}\varepsilon_3^{\perp} - \frac{k_{21}^{\perp}\varepsilon_2^{\perp *}}{x_2}\cdot\varepsilon_1^{\perp *}\varepsilon_3^{\perp}\right]\;, \tag{A58}$$

which matches my old result

$$Y_{123} \;=\; if^{c_1c_2c_3}\left[\varepsilon_1^*\varepsilon_2^*\,\varepsilon_3 k_{12} - \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2 - \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1\right]\; a_1^\dagger a_2^\dagger a_3\;. \tag{A59}$$

5. Evaluation of the gluon 2nd-order self-interaction

I use $x_1 = p_1^+/p_3^+$, $x_2 = p_2^+/p_3^+$ and $k_{12}^{\perp} = x_2 p_1^{\perp} - x_1 p_2^{\perp}$, so that $p_1^{\perp} = x_1 p_3^{\perp} + k_{12}^{\perp}$, $p_2^{\perp} = x_2 p_3^{\perp} - k_{12}^{\perp}$, 1 = transverse gluon and 2 = longitudinal gluon. The Hamiltonian interaction term is given in Eq. (A44),

$$\begin{aligned} H_{+1}+H_{\perp 1} \;&=\; g\sum_{123}\int[123]\;\tilde{\delta}_{c.a}\;f_{12.3}^{r}\;Y_{123}\;a_1^\dagger a_2^\dagger a_3 + h.c. \\ &+\; g\sum_{123}\int[123]\;\tilde{\delta}_{c.a}\;f_{12.3}^{r}\;\left(Y_{A123}\;a_3^\dagger\alpha_2 a_1 + Y_{\phi 123}\;a_1^\dagger a_2^\dagger\alpha_3\right) + h.c.\;, \end{aligned} \tag{A60}$$

$$Y_{123} \;=\; if^{c_1c_2c_3}\left[\varepsilon_1^*\varepsilon_2^*\,\varepsilon_3 k_{12} - \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2 - \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1\right]\; a_1^\dagger a_2^\dagger a_3\;, \tag{A61}$$

$$Y_{A123} \;=\; -if^{123}\frac{m_g}{p_2^+}(p_3^+ + p_1^+)\;\varepsilon_1^{\perp}\varepsilon_3^{\perp *}\;, \tag{A62}$$

$$Y_{\phi 123} \;=\; -if^{123}\frac{m_g}{p_3^+}p_1^+\;\varepsilon_1^{\perp *}\varepsilon_2^{\perp *}\;. \tag{A63}$$

Start 20240929 11:35 -¿ 11:54 nie Cienega Ave. So, I have

$$\langle p'|H_1^2|p\rangle_{\perp g} \;\to\; \langle 0|Y_{\tilde{1}\tilde{2}}^*\;a_{\tilde{2}}a_{\tilde{1}}\;Y_{\tilde{1}\tilde{2}}\;a_{\tilde{1}}^\dagger a_{\tilde{1}}^\dagger|0\rangle \tag{A64}$$

$$\to\; Y_{\tilde{1}\tilde{2}}^* Y_{\tilde{1}\tilde{2}} + Y_{\tilde{1}\tilde{2}}^* Y_{\tilde{2}\tilde{1}}\;. \tag{A65}$$

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$$|N|^2 \;=\; \sum_{\tilde{1}\tilde{2}}\left(Y_{\tilde{1}\tilde{2}}^* Y_{\tilde{1}\tilde{2}} + Y_{\tilde{1}\tilde{2}}^* Y_{\tilde{2}\tilde{1}}\right) \;=\; \sum_{12}\left(Y_{12}^* Y_{12} + Y_{12}^* Y_{21}\right) \tag{A66}$$

$$\begin{aligned} &=\; \sum_{12} -if^{123}\left[\varepsilon_1\varepsilon_2\,\varepsilon_3^* k_{12} - \varepsilon_1\varepsilon_3^*\,\varepsilon_2 k_{12}/x_2 - \varepsilon_2\varepsilon_3^*\,\varepsilon_1 k_{12}/x_1\right] \\ &\times\; if^{123}\left[\varepsilon_1^*\varepsilon_2^*\,\varepsilon_3 k_{12} - \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2 - \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1\right] \\ &+\; \sum_{12} -if^{123}\left[\varepsilon_1\varepsilon_2\,\varepsilon_3^* k_{12} - \varepsilon_1\varepsilon_3^*\,\varepsilon_2 k_{12}/x_2 - \varepsilon_2\varepsilon_3^*\,\varepsilon_1 k_{12}/x_1\right] \\ &\times\; if^{123}\left[\varepsilon_2^*\varepsilon_1^*\,\varepsilon_3 k_{12} - \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1 - \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2\right]\;, \end{aligned} \tag{A67}$$

where the last line results from $f^{213} = -f^{123}$ combined with $k_{21} = -k_{12}$. Simplify.

$$\begin{aligned} |N|^2 \;&=\; 2N_c \sum_{12} [\varepsilon_1\varepsilon_2\,\varepsilon_3^* k_{12} - \varepsilon_1\varepsilon_3^*\,\varepsilon_2 k_{12}/x_2 - \varepsilon_2\varepsilon_3^*\,\varepsilon_1 k_{12}/x_1] \\ &\times\; [\varepsilon_1^*\varepsilon_2^*\,\varepsilon_3 k_{12} - \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2 - \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1] \ . \end{aligned} \tag{A68}$$

The factor 2 comes from symmetry $Y_{12} = Y_{21}$.

****

Here starts my summing over polarizations of intermediate gluons that is written in red because I now doubt that it is correct. 20240930 19:00 pon Cienega Ave.

****

Also, integration over $k_{12}^{\perp}$ is symmetric with respect to change of transverse directions $(x,y) \leftrightarrow (y,x)$. Therefore, $k_{12}^i k_{12}^j \equiv k_{12}^{\perp 2}/2$. So,

$$\begin{aligned} |N|^2 \;&=\; \frac{2N_c k_{12}^{\perp 2}}{2} \sum_{12\ k} \left[\varepsilon_1\varepsilon_2\,\varepsilon_3^{*k} - \varepsilon_1\varepsilon_3^*\,\varepsilon_2^k/x_2 - \varepsilon_2\varepsilon_3^*\,\varepsilon_1^k/x_1\right] \\ &\times\; \left[\varepsilon_1^*\varepsilon_2^*\,\varepsilon_3^k - \varepsilon_1^*\varepsilon_3\,\varepsilon_2^{*k}/x_2 - \varepsilon_2^*\varepsilon_3\,\varepsilon_1^{*k}/x_1\right] \ . \end{aligned} \tag{A69}$$

9 terms.

$$|N|^2 \;=\; N_c k_{12}^{\perp 2} S \ , \tag{A70}$$

$$\begin{aligned} S \;&=\; \sum_{12} \varepsilon_1\varepsilon_2\ [\varepsilon_1^*\varepsilon_2^*\ \varepsilon_3^*\varepsilon_3 - \varepsilon_1^*\varepsilon_3\ \varepsilon_3^*\varepsilon_2^*/x_2 - \varepsilon_2^*\varepsilon_3\ \varepsilon_3^*\varepsilon_1^*/x_1] \\ &-\; \varepsilon_1\varepsilon_3^*\,[\varepsilon_1^*\varepsilon_2^*\ \varepsilon_2\varepsilon_3 - \varepsilon_1^*\varepsilon_3\ \varepsilon_2\varepsilon_2^*/x_2 - \varepsilon_2^*\varepsilon_3\ \varepsilon_2\varepsilon_1^*/x_1]\,/x_2 \\ &-\; \varepsilon_2\varepsilon_3^*\,[\varepsilon_1^*\varepsilon_2^*\ \varepsilon_1\varepsilon_3 - \varepsilon_1^*\varepsilon_3\ \varepsilon_1\varepsilon_2^*/x_2 - \varepsilon_2^*\varepsilon_3\ \varepsilon_1\varepsilon_1^*/x_1]\,/x_1 \ . \end{aligned} \tag{A71}$$

Factors in.

$$\begin{aligned} S \;&=\; \sum_{12} [\varepsilon_1\varepsilon_2\ \varepsilon_1^*\varepsilon_2^*\ \varepsilon_3^*\varepsilon_3 - \varepsilon_1\varepsilon_2\ \varepsilon_1^*\varepsilon_3\ \varepsilon_3^*\varepsilon_2^*/x_2 - \varepsilon_1\varepsilon_2\ \varepsilon_2^*\varepsilon_3\ \varepsilon_3^*\varepsilon_1^*/x_1] \\ &-\; [\varepsilon_1\varepsilon_3^*\ \varepsilon_1^*\varepsilon_2^*\ \varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_3^*\ \varepsilon_1^*\varepsilon_3\ \varepsilon_2\varepsilon_2^*/x_2 - \varepsilon_1\varepsilon_3^*\ \varepsilon_2^*\varepsilon_3\ \varepsilon_2\varepsilon_1^*/x_1]\,/x_2 \\ &-\; [\varepsilon_2\varepsilon_3^*\ \varepsilon_1^*\varepsilon_2^*\ \varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_3^*\ \varepsilon_1^*\varepsilon_3\ \varepsilon_1\varepsilon_2^*/x_2 - \varepsilon_2\varepsilon_3^*\ \varepsilon_2^*\varepsilon_3\ \varepsilon_1\varepsilon_1^*/x_1]\,/x_1 \ . \end{aligned} \tag{A72}$$

Sum over 2.

$$\begin{aligned} S \;&=\; \sum_{1} [\varepsilon_1\varepsilon_1^*\ \varepsilon_3^*\varepsilon_3 - \varepsilon_1^*\varepsilon_3\ \varepsilon_3^*\varepsilon_1/x_2 - \varepsilon_1\varepsilon_3\ \varepsilon_3^*\varepsilon_1^*/x_1] \\ &-\; [\varepsilon_1\varepsilon_3^*\ \varepsilon_1^*\varepsilon_3 - \varepsilon_1\varepsilon_3^*\ \varepsilon_1^*\varepsilon_3/x_2 - \varepsilon_1\varepsilon_3^*\ \varepsilon_1^*\varepsilon_3/x_1]\,/x_2 \\ &-\; [\varepsilon_3^*\ \varepsilon_1^*\ \varepsilon_1\varepsilon_3 - \varepsilon_1^*\varepsilon_3\ \varepsilon_1\varepsilon_3^*/x_2 - \varepsilon_3^*\varepsilon_3\ \varepsilon_1\varepsilon_1^*/x_1]\,/x_1 \ . \end{aligned} \tag{A73}$$

Sum over 1.

$$\begin{aligned} S \;&=\; \varepsilon_3^*\varepsilon_3\,[1 - 1/x_2 - 1/x_1] \\ &-\; \varepsilon_3^*\varepsilon_3\,[1 - 1/x_2 - 1/x_1]\,/x_2 \\ &-\; \varepsilon_3^*\varepsilon_3\,[1 - 1/x_2 - 1/x_1]\,/x_1 \ . \end{aligned} \tag{A74}$$

Result for S.

$$S \;=\; \varepsilon_3^*\varepsilon_3\,[1 - 1/x_1 - 1/x_2]^2 \ . \tag{A75}$$

Result. WRONG

$$|N|^2 \;=\; N_c k_{12}^{\perp 2}\ [1 - 1/x_1 - 1/x_2]^2\ \varepsilon_3^*\varepsilon_3 \ . \tag{A76}$$

Here in green is my corrected sum over polarizations, 20240930 19:08 pon Cienega Ave..

From Eq. (A68) I take the formula

$$|N|^2 \;=\; 2N_c S \ , \tag{A77}$$

$$\begin{aligned} S \;&=\; \sum_{12} [\varepsilon_1\varepsilon_2\,\varepsilon_3^* k_{12} - \varepsilon_1\varepsilon_3^*\,\varepsilon_2 k_{12}/x_2 - \varepsilon_2\varepsilon_3^*\,\varepsilon_1 k_{12}/x_1] \\ &\times\; [\varepsilon_1^*\varepsilon_2^*\,\varepsilon_3 k_{12} - \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2 - \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1] \ . \end{aligned} \tag{A78}$$

9 terms.

$$S = \sum_{12} \varepsilon_1\varepsilon_2\,\varepsilon_3^* k_{12}\left[\varepsilon_1^*\varepsilon_2^*\,\varepsilon_3 k_{12} - \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2 - \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1\right]$$
$$- \varepsilon_1\varepsilon_3^*\,\varepsilon_2 k_{12}\left[\varepsilon_1^*\varepsilon_2^*\,\varepsilon_3 k_{12} - \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2 - \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1\right]/x_2$$
$$- \varepsilon_2\varepsilon_3^*\,\varepsilon_1 k_{12}\left[\varepsilon_1^*\varepsilon_2^*\,\varepsilon_3 k_{12} - \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2 - \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1\right]/x_1\ . \tag{A79}$$

Factors in.

$$S = \sum_{12}\left[\varepsilon_1\varepsilon_2\,\varepsilon_3^* k_{12}\ \varepsilon_1^*\varepsilon_2^*\,\varepsilon_3 k_{12} - \varepsilon_1\varepsilon_2\,\varepsilon_3^* k_{12}\ \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2 - \varepsilon_1\varepsilon_2\,\varepsilon_3^* k_{12}\ \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1\right]$$
$$- \left[\varepsilon_1\varepsilon_3^*\,\varepsilon_2 k_{12}\ \varepsilon_1^*\varepsilon_2^*\,\varepsilon_3 k_{12} - \varepsilon_1\varepsilon_3^*\,\varepsilon_2 k_{12}\ \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2 - \varepsilon_1\varepsilon_3^*\,\varepsilon_2 k_{12}\ \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1\right]/x_2$$
$$- \left[\varepsilon_2\varepsilon_3^*\,\varepsilon_1 k_{12}\ \varepsilon_1^*\varepsilon_2^*\,\varepsilon_3 k_{12} - \varepsilon_2\varepsilon_3^*\,\varepsilon_1 k_{12}\ \varepsilon_1^*\varepsilon_3\,\varepsilon_2^* k_{12}/x_2 - \varepsilon_2\varepsilon_3^*\,\varepsilon_1 k_{12}\ \varepsilon_2^*\varepsilon_3\,\varepsilon_1^* k_{12}/x_1\right]/x_1. \tag{A80}$$

Sum over 2.

$$S = \sum_{1}\left[\varepsilon_3^* k_{12}\ \varepsilon_1^*\varepsilon_1\ \varepsilon_3 k_{12} - \varepsilon_3^* k_{12}\ \varepsilon_1^*\varepsilon_3\ \varepsilon_1 k_{12}/x_2 - \varepsilon_3^* k_{12}\ \varepsilon_1\varepsilon_3\ \varepsilon_1^* k_{12}/x_1\right]$$
$$- \left[\varepsilon_1\varepsilon_3^*\ \varepsilon_1^* k_{12}\ \varepsilon_3 k_{12} - \varepsilon_1\varepsilon_3^*\ \varepsilon_1^*\varepsilon_3\ k_{12}^2/x_2 - \varepsilon_1\varepsilon_3^*\ \varepsilon_3 k_{12}\ \varepsilon_1^* k_{12}/x_1\right]/x_2$$
$$- \left[\varepsilon_1^*\varepsilon_3^*\ \varepsilon_1 k_{12}\ \varepsilon_3 k_{12} - \varepsilon_1 k_{12}\ \varepsilon_1^*\varepsilon_3\ \varepsilon_3^* k_{12}/x_2 - \varepsilon_1 k_{12}\ \varepsilon_3^*\varepsilon_3\ \varepsilon_1^* k_{12}/x_1\right]/x_1. \tag{A81}$$

Sum over 1.

$$S = \varepsilon_3^* k_{12}\ 2\ \varepsilon_3 k_{12} - \varepsilon_3^* k_{12}\ \varepsilon_3 k_{12}/x_2 - \varepsilon_3^* k_{12}\ \varepsilon_3 k_{12}/x_1$$
$$- \left[\varepsilon_3^* k_{12}\ \varepsilon_3 k_{12} - \varepsilon_3^*\varepsilon_3\ k_{12}^2/x_2 - \varepsilon_3 k_{12}\ \varepsilon_3^* k_{12}/x_1\right]/x_2$$
$$- \left[\varepsilon_3^* k_{12}\ \varepsilon_3 k_{12} - \varepsilon_3 k_{12}\ \varepsilon_3^* k_{12}/x_2 - \varepsilon_3^*\varepsilon_3\ k_{12}^2/x_1\right]/x_1. \tag{A82}$$

Remove brackets.

$$S = \varepsilon_3^* k_{12}\ 2\ \varepsilon_3 k_{12} - \varepsilon_3^* k_{12}\ \varepsilon_3 k_{12}/x_2 - \varepsilon_3^* k_{12}\ \varepsilon_3 k_{12}/x_1$$
$$- \varepsilon_3^* k_{12}\ \varepsilon_3 k_{12}/x_2 + \varepsilon_3^*\varepsilon_3\ k_{12}^2/x_2^2 + \varepsilon_3 k_{12}\ \varepsilon_3^* k_{12}/(x_1 x_2)$$
$$- \varepsilon_3^* k_{12}\ \varepsilon_3 k_{12}/x_1 + \varepsilon_3 k_{12}\ \varepsilon_3^* k_{12}/(x_2 x_1) + \varepsilon_3^*\varepsilon_3\ k_{12}^2/x_1^2\ . \tag{A83}$$

Common factors.

$$S = \varepsilon_3^* k_{12}\ \varepsilon_3 k_{12}\left[2 - 2/x_2 - 2/x_1 + 2/(x_1 x_2)\right]$$
$$+ \varepsilon_3^*\varepsilon_3\ k_{12}^2\left(1/x_2^2 + 1/x_1^2\right)\ . \tag{A84}$$

Simplify.

$$S = 2\varepsilon_3^* k_{12}\ \varepsilon_3 k_{12} + \varepsilon_3^*\varepsilon_3\ k_{12}^2\left(1/x_2^2 + 1/x_1^2\right)\ . \tag{A85}$$

Azimuthal symmetry of integral over $k_{12}^\perp$.

$$S = \varepsilon_3^*\varepsilon_3\ k_{12}^{\perp 2} + \varepsilon_3^*\varepsilon_3\ k_{12}^{\perp 2}\left(1/x_1^2 + 1/x_2^2\right)\ . \tag{A86}$$

So, instead of the wrong result in Eq. (A76), I now have the fixed result

$$|N|^2 = 2N_c\varepsilon_3^*\varepsilon_3\ k_{12}^{\perp 2}\left[1 + 1/x_1^2 + 1/x_2^2\right]\ . \tag{A87}$$

Below I use the fixed result. Since $\varepsilon_3^*\varepsilon_3$ belongs to the normalization, or equals 1, coming back to Eq. (A5),

$$\frac{\Sigma_{\perp g}^{(2)}}{p^+} = \frac{g^2 N}{p^+}\int \frac{dk^+ d^2k^\perp}{2(2\pi)^3 k^+(p^+ - k^+)}\ \frac{2k^{\perp 2}\ \left[1 + 1/x^2 + 1/(1-x)^2\right]}{P^- - Q^- - p_1^- - p_2^-}\ . \tag{A88}$$

From Eq. (A40), dissociation of $\perp$ gluon into one $\perp$ and one $+$ gluon proceeds via

$$g\sum_{123}\int [123]\ \tilde\delta_{c.a}\ f_{12.3}^r\ if^{123}\ \tilde Y_{A123}\ a_3^\dagger \alpha_2 a_1 + h.c. \tag{A89}$$

where from Eq. (A46)

$$\tilde{Y}_{A123} \;=\; -\frac{m_g}{p_2^+}(p_3^+ + p_1^+)\;\varepsilon_1^\perp \varepsilon_3^{\perp *} \;, \tag{A90}$$

or from Eq. (A62)

$$Y_{A123} \;=\; -if^{123}\frac{m_g}{p_2^+}(p_3^+ + p_1^+)\;\varepsilon_1^\perp \varepsilon_3^{\perp *} \;. \tag{A91}$$

So,

$$\langle p'|H_{+1}^2|p\rangle_{+g} \;\to\; \langle 0|Y_{A123}\alpha_2 a_1\; a_1^\dagger \alpha_2^\dagger Y_{A123}^*|0\rangle \tag{A92}$$

$$\;\to\; Y_{A\tilde{1}\tilde{2}}Y_{A\tilde{1}\tilde{2}}^* \;. \tag{A93}$$

Numerator factor.

$$|N_+|^2 \;=\; Y_{A\tilde{1}\tilde{2}}Y_{A\tilde{1}\tilde{2}}^* \tag{A94}$$

$$=\; f^{123'}f^{123}\;\frac{m_g^2}{p_2^{+2}}\;(p_{3'}^+ + p_1^+)(p_3^+ + p_1^+)\;\varepsilon_{3'}^{\perp *}\varepsilon_1^\perp\;\varepsilon_1^\perp\varepsilon_3^{\perp *} \;. \tag{A95}$$

Color factors yield N_c. The sum over spin 1 of the transverse gluon yields $\varepsilon_{3'}^{\perp *}\varepsilon_3^{\perp *}$, which goes into normalization. I note there is no transverse momentum, but there is a denominator. In terms of $x = p_1^+/p_3^+$, with 1 corresponding to the transverse intermedaie gluon, I have

$$|N_+|^2 \;\to\; N\;\frac{m_g^2}{(1-x)^2}\;(1+x)^2 \;. \tag{A96}$$

Thus,

$$\frac{\Sigma_{+g}^{(2)}}{p^+} \;=\; \frac{g^2N}{p^+}\int\frac{dk^+d^2k^\perp}{2(2\pi)^3k^+(p^+-k^+)}\;\frac{m_g^2(1+x)^2/(1-x)^2}{P^- - Q^- - p_1^- - p_2^-} \;, \tag{A97}$$

and the combined gluon self-interaction due to the transverse and longitudinal intermediate gluons is

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Corrected gluon self-interaction is

$$\frac{\Sigma_{\perp g}^{(2)} + \Sigma_{+g}^{(2)}}{p^+} \;=\; \frac{g^2N}{p^+}\int\frac{dk^+d^2k^\perp}{2(2\pi)^3k^+(p^+-k^+)}\;\frac{2k^{\perp 2}\,\left[1+\frac{1}{x^2}+\frac{1}{(1-x)^2}\right] + m_g^2\left(\frac{1+x}{1-x}\right)^2}{P^- - Q^- - p_1^- - p_2^-} \;. \tag{A98}$$

I can write it in the form

$$\frac{\Sigma_{A+\phi}^{(2)}}{p^+} \;=\; \frac{\Sigma_{\perp g}^{(2)} + \Sigma_{+g}^{(2)}}{p^+} \;, \tag{A99}$$

$$\frac{\Sigma_{A+\phi}^{(2)}}{p^+} \;=\; \frac{g^2N}{p^+}\int\frac{dxd^2k^\perp}{2(2\pi)^3}\;\frac{2k^{\perp 2}\,\left[1+\frac{1}{x^2}+\frac{1}{(1-x)^2}\right] + m_g^2\left(\frac{1+x}{1-x}\right)^2}{x(1-x)p^+(P^- - Q^-) - k^{\perp 2} - m_g^2} \;. \tag{A100}$$

The mass term can be symmetrized using $f(x) \equiv [f(x) + f(1-x)]/2$ since the rest of the integrand is symmetric with respect to exchange $x \leftrightarrow 1-x$. I have

$$\frac{1+x}{1-x} \;=\; \frac{1+1-(1-x)}{1-x} \;=\; \frac{2}{1-x} - 1 \;, \tag{A101}$$

$$\left(\frac{1+x}{1-x}\right)^2 \;\equiv\; \frac{1}{2}\left(\frac{2}{1-x} - 1\right)^2 + \frac{1}{2}\left(\frac{2}{x} - 1\right)^2 \tag{A102}$$

$$=\; \frac{1}{2}\left[4 + \frac{4}{x^2} + \frac{4}{(1-x)^2} - \frac{4}{x(1-x)} - 2\right] \tag{A103}$$

$$=\; 2\left[1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}\right] - \left[\frac{2}{x(1-x)} + 1\right] \;. \tag{A104}$$

Consequently,

$$\frac{\Sigma^{(2)}_{A+\phi}}{p^+} = \frac{g^2 N}{p^+}\int \frac{dx d^2 k^\perp}{2(2\pi)^3}\,\frac{2(k^{\perp 2}+m_g^2)\,\left[1+\frac{1}{x^2}+\frac{1}{(1-x)^2}\right]-m_g^2\left[\frac{2}{x(1-x)}+1\right]}{x(1-x)p^+(P^- - Q^-)-k^{\perp 2}-m_g^2}\ . \tag{A105}$$

Appendix B: Change of variables for homogeneous counterterm computation

To compute the counterterm coefficients $\mathcal{U}_{122'1'}$ that eliminate dependence on $r\to 0$ from matrix elements in Eq. (**??**), we change the integration variables $x_1, k^\perp_{12}, x_{1'}, k^\perp_{1'2'}$, to x, y, $k^\perp$ and $l^\perp$,

$$x = (x_1+x_1')/2\ ,\ y = (x_{1'}-x_1)/2 = z/2\ , \tag{B1}$$

$$x_1 = x-y\ ,\ x_{1'} = x+y\ , \tag{B2}$$

$$x_2 = 1-x+y\ ,\ x_{2'} = 1-x-y\ , \tag{B3}$$

$$dx_1\,dx_{1'} = 2\,dx\,dy\ , \tag{B4}$$

$$\int_0^1 dx_1\int_0^1 dx_{1'} = 2\left[\int_0^{1/2}dx\int_{-x}^{x}dy+\int_{1/2}^1 dx\int_{x-1}^{1-x}dy\right]\ , \tag{B5}$$

$$k^\perp = (k^\perp_{12}+k^\perp_{1'2'})/2\ ,\ l^\perp = (k^\perp_{1'2'}-k^\perp_{12})/2\ , \tag{B6}$$

$$k^\perp_{12} = k^\perp - l^\perp\ ,\ k^\perp_{1'2'} = k^\perp + l^\perp\ , \tag{B7}$$

$$d^2k^\perp_{12}\,d^2k^\perp_{1'2'} = 4 d^2k^\perp\,d^2l^\perp\ , \tag{B8}$$

$$\sqrt{x_1x_2x_{1'}x_{2'}} = \sqrt{(x-y)(1-x+y)(x+y)(1-x-y)} \tag{B9}$$

$$= \sqrt{(x^2-y^2)[(1-x)^2-y^2]}\ . \tag{B10}$$

$$\int[12] = \int[P]\int[xk]\ . \tag{B11}$$

For equal masses, $\mathcal{M}=2E$,

$$\int[xk] = \int\frac{dx\,d^2k^\perp}{2(2\pi)^3x(1-x)} = \int\frac{d^3k}{(2\pi)^3E}\ , \tag{B12}$$

For different masses, like a quark and a gluon in the quark self-interaction, , $\mathcal{M}=E_1+E_2$ and

$$\int[xk] = \int\frac{dx\,d^2k^\perp}{2(2\pi)^3x(1-x)} = \int\frac{d^3k}{2(2\pi)^3}\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\ . \tag{B13}$$

$$\begin{aligned}\int[x_1k^\perp_{12}]\int[x_{1'}k^\perp_{1'2'}] &= \frac{2^3}{[2(2\pi)^3]^2}\int d^2k^\perp\int d^2l^\perp\\ &\times\left[\int_0^{1/2}dx\int_{-x}^{x}dy+\int_{1/2}^1dx\int_{x-1}^{1-x}dy\right]\\ &\times\left[\frac{1}{x_1x_2x_{1'}x_{2'}} = \frac{1}{(x^2-y^2)[(1-x)^2-y^2]}\right]\ .\end{aligned} \tag{B14}$$

Appendix C: Formulas for tree-level counterterm computations

The invariant masses are

$$\mathcal{M}^2_{12} = \frac{k^{\perp 2}_{12}+m_1^2}{x_1}+\frac{k^{\perp 2}_{12}+m_2^2}{x_2} = \frac{(k-l)^{\perp 2}+m_1^2}{x-y}+\frac{(k-l)^{\perp 2}+m_2^2}{1-x+y}\ , \tag{C1}$$

$$\mathcal{M}^2_{1'2'} = \frac{k^{\perp 2}_{1'2'}+m_1^2}{x_{1'}}+\frac{k^{\perp 2}_{1'2'}+m_2^2}{x_{2'}} = \frac{(k+l)^{\perp 2}+m_1^2}{x+y}+\frac{(k+l)^{\perp 2}+m_2^2}{1-x-y}\ . \tag{C2}$$

The exchanged-gluon fourmomentum has the kinematic three-momentum components $q_3^{+,\perp} = p_{\bar{1}}^{+,\perp} - p_{\underline{1}}^{+,\perp}$ and $q_3^- = (q_3^{\perp 2} + m_g^2)/q_3^+$.

$$\frac{(p_{\underline{1}}+q_3)^2 - m_1^2}{x_{\bar{1}}} = \mathrm{sgn}(y)\left[\frac{(k-l)^{\perp 2}+m_1^2}{x-y} + \frac{4l^{\perp 2}+m_g^2}{2y} - \frac{(k+l)^{\perp 2}+m_1^2}{x+y}\right] , \tag{C3}$$

$$\frac{(p_{\underline{2}}+q_3)^2 - m_2^2}{x_{\bar{2}}} = \mathrm{sgn}(y)\left[\frac{(k+l)^{\perp 2}+m_2^2}{1-x-y} + \frac{4l^{\perp 2}+m_g^2}{2y} - \frac{(k-l)^{\perp 2}+m_2^2}{1-x+y}\right] , \tag{C4}$$

where $s_y = \mathrm{sgn}(y)$. Also,

$$\frac{(p_{\underline{1}}+q_3)^2 - m_1^2}{x_{\bar{1}}} = \frac{4l^{\perp 2}+m_g^2}{2|y|} + \frac{(k-s_y l)^{\perp 2}+m_1^2}{x-|y|} - \frac{(k+s_y l)^{\perp 2}+m_1^2}{x+|y|} , \tag{C5}$$

$$\frac{(p_{\underline{2}}+q_3)^2 - m_2^2}{x_{\bar{2}}} = \frac{4l^{\perp 2}+m_g^2}{2|y|} + \frac{(k+s_y l)^{\perp 2}+m_2^2}{1-x-|y|} - \frac{(k-s_y l)^{\perp 2}+m_2^2}{1-x+|y|} . \tag{C6}$$

So, with $c = 4l^{\perp 2} + m_g^2$,

$$\frac{(p_{\underline{1}}+q_3)^2 - m_1^2}{x_{\bar{1}}} = \frac{c}{2|y|} + a_1 \ , \ a_1 = \frac{(k-s_y l)^{\perp 2}+m_1^2}{x-|y|} - \frac{(k+s_y l)^{\perp 2}+m_1^2}{x+|y|} , \tag{C7}$$

$$\frac{(p_{\underline{2}}+q_3)^2 - m_2^2}{x_{\bar{2}}} = \frac{c}{2|y|} + a_2 \ , \ a_2 = \frac{(k+s_y l)^{\perp 2}+m_2^2}{1-x-|y|} - \frac{(k-s_y l)^{\perp 2}+m_2^2}{1-x+|y|} , \tag{C8}$$

with the original exressions

$$a_1 = \mathrm{sgn}(y)\left[\frac{(k-l)^{\perp 2}+m_1^2}{x-y} - \frac{(k+l)^{\perp 2}+m_1^2}{x+y}\right] , \tag{C9}$$

$$a_2 = \mathrm{sgn}(y)\left[\frac{(k+l)^{\perp 2}+m_2^2}{1-x-y} - \frac{(k-l)^{\perp 2}+m_2^2}{1-x+y}\right] , \tag{C10}$$

and

$$f^r_{\bar{1},\underline{1}3} = \exp\left\{-r^2\left[\frac{c}{2|y|} + a_1\right]^2 (\pi_{\bar{1}}/P_{12}^+)^2\right\} , \tag{C11}$$

$$f^r_{\bar{2},\underline{2}3} = \exp\left\{-r^2\left[\frac{c}{2|y|} + a_2\right]^2 (\pi_{\bar{2}}/P_{12}^+)^2\right\} . \tag{C12}$$

I have to eventually take into account that $c = 4l^{\perp 2} + m_g^2 \to m_g^2 \to 0$. I note that the product

$$x_1 x_2 x_{1'} x_{2'} = (x^2 - y^2)[(1-x)^2 - y^2] \tag{C13}$$

is a function of y^2 and for small y differs from $x^2(1-x)^2$ by terms on the order of y^2.

Appendix D: Source of homogeneous divergence in integral over $y \sim 0$

The test states used in evaluation of the Hamiltonian matrix elements involve significant contributions only from quarks with x_1 and x_2 in a limited range. I denote that range by $(x_0, 1-x_0)$, it does not depend on $k^\perp$. The test wave functions are assumed to fall off quickly for x_1 or x_2 smaller than x_0. The same is assumed about the range of $x_{1'}$ and $x_{2'}$. Hence the range of significant contribution to integrals over x and y is limited, see Fig. 6. In that figure, the two near-diagonal parallel lines are described by relations $x_{1'} = x_1 \pm y_0$. Consequently, the integration over $x_{1'}$ and x_1 can be limited according to the replacement illustrated in Figs. 6 and 7,

$$\int_0^1 dx_1 \int_0^1 dx_{1'} \ \to \ \int_{xy} , \tag{D1}$$

FIG. 6: Schematic view of the ranges of integration over $x_{1'}$ and x_1 in the matrix elements in Eq. (**??**), *cf.* Eqs. (B5) and (D2). The large square indicates the ranges $(0,1)$. The crossed diagonal lines correspond to $y = (x_{1'} - x_1)/2 = 0$ and $x = (x_{1'} + x_1)/2 = 1/2$. The two parallel lines near a diagonal correspond to $y = \pm y_0$ with $y_0 \ll x_0$. The test functions in the matrix elements of Eq. (**??**) are negligible near the edge of and outside the small square.

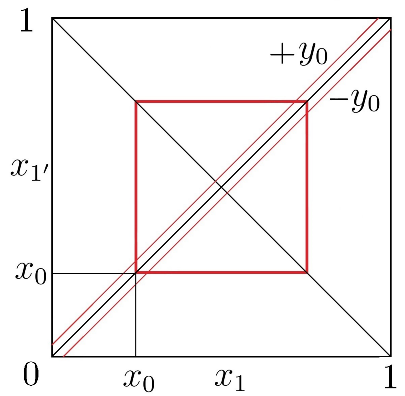
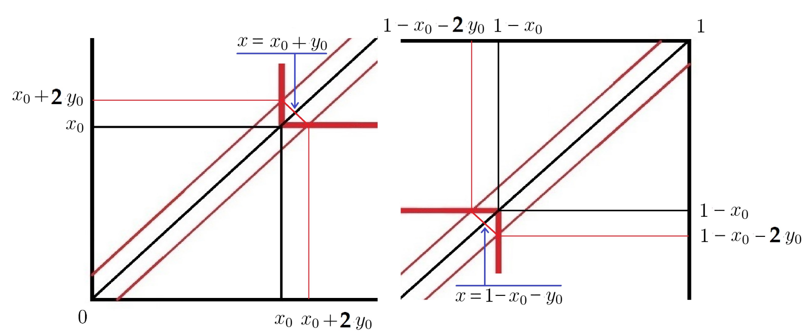


FIG. 7: Enlargements of the lower-left and upper-right corners of Fig. 6, which explain the meaning of parameter y_0. By construction, the results of integration do not depend on y_0.



where

$$
\begin{aligned}
\int_{xy} &= 2\int_{x_0+y_0}^{1/2} dx \left[\int_{x_0-x}^{-y_0} + \int_{-y_0}^{y_0} + \int_{y_0}^{x-x_0}\right] dy \\
&+ 2\int_{1/2}^{1-x_0-y_0} dx \left[\int_{x_0-1+x}^{-y_0} + \int_{-y_0}^{y_0} + \int_{y_0}^{1-x_0-x}\right] dy \ . \qquad \text{(D2)}
\end{aligned}
$$

The only source of dependence on $r \to 0$ is the region of y close to 0. Therefore, the integral $\int_{xy}$ is split up into two parts: a convergent one (c) and a divergent one (d);

$$
\int_{xy} = \int_c + \int_d \ , \quad \int_c = \int_{xy} \theta(|y| - y_0) \ , \quad \int_d = \int_{xy} \theta(y_0 - |y|) \ . \qquad \text{(D3)}
$$

The Jacobian factor 2 is included in $2^3 = 8$ in front of a final formula for the matrix elements. We also use the shorthand notation

$$
\int_\perp = \frac{8}{[2(2\pi)^3]^2} \int d^2k^\perp \int d^2l^\perp \ . \qquad \text{(D4)}
$$

Review of the homogeneous CT

After the change of variables, the matrix element of Eq. (??) with the complete integration factors from Eq. (B14), reads

$$X_{nn'122'1'} = \int F(z) \left[\frac{f^r_{\bar{1},\underline{1}q} f^r_{\bar{2},\underline{2}q}}{z^2} + \mathcal{U}_{122'1'} \right] , \tag{D5}$$

$$\int = \frac{2^3}{[2(2\pi)^3]^2} \int d^2k^\perp \int d^2l^\perp \left(\int_0^{1/2} dx \int_{-x}^{x} dy + \int_{1/2}^{1} dx \int_{x-1}^{1-x} dy \right) , \tag{D6}$$

where, after change of variables, $z = 2y$, including all factors,

$$\begin{aligned} F(z) &= f^t_{\tilde{c},\bar{a}} \frac{4\sqrt{x_1 x_{1'} x_2 x_{2'}}}{x_1 x_2 x_{1'} x_{2'}} \, \delta_{\sigma_1 \sigma_{1'}} \delta_{\sigma_2 \sigma_{2'}} (-T^{\tilde{c}}_{11'} T^{\tilde{c}}_{22'}) \, \delta_{f_1 f_{1'}} \delta_{f_2 f_{2'}} \\ &\times \; \psi^{*n}_{12}(x_1, k^\perp_{12}) \, \psi^{n'}_{1'2'}(x_{1'}, k^\perp_{1'2'}) , \end{aligned} \tag{D7}$$

is a smooth function of z around $z = 0$. Now I use the formula with prime meaning d/dz,

$$F(z) = F(0) + F'(0)z + [F(z) - F(0) - F'(0)z] , \tag{D8}$$

of which only the first two terms contribute to divergence when $r \to 0$. The term in the square bracket yields an integral that has a finite limit when $r \to 0$. The part of the matrix element that determines the counterterm is

$$X^{\rm div}_{nn'122'1'} = \int \left\{ [F(0) + F'(0)z] \, \frac{f^r_{\bar{1},\underline{1}q} f^r_{\bar{2},\underline{2}q}}{z^2} + F(z)\,\mathcal{U}_{122'1'} \right\} . \tag{D9}$$

The divergence is solely due to integration over y near $z = 2y = 0$. I write

$$X^{\rm div}_{nn'122'1'} = \int \left[F(0) \, \frac{f^r_{\bar{1},\underline{1}q} f^r_{\bar{2},\underline{2}q}}{z^2} + F'(0) \, \frac{f^r_{\bar{1},\underline{1}q} f^r_{\bar{2},\underline{2}q}}{z} + F(z)\,\mathcal{U}_{122'1'} \right] , \tag{D10}$$

to see that I need the counterterm kernel to be

$$\lim_{r \to 0} \mathcal{U}_{122'1'} = -\delta(z)\,\mathcal{V}_{122'1'} + \delta'(z)\,\mathcal{W}_{122'1'} , \tag{D11}$$

where

$$\mathcal{V}_{122'1'} = \lim_{r\to 0} \left[\theta_{1/2-x} \int_{-x}^{x} dy + \theta_{x-1/2} \int_{x-1}^{1-x} dy \right] dy \, \frac{f^r_{\bar{1},\underline{1}q} f^r_{\bar{2},\underline{2}q}}{4y^2} , \tag{D12}$$

$$\mathcal{W}_{122'1'} = \lim_{r\to 0} \left[\theta_{1/2-x} \int_{-x}^{x} dy + \theta_{x-1/2} \int_{x-1}^{1-x} dy \right] dy \, \frac{f^r_{\bar{1},\underline{1}q} f^r_{\bar{2},\underline{2}q}}{2y} . \tag{D13}$$

By comparison with the section below, I get in the review the following results.

a. *Review of the canonical case of $\pi_{\bar{i}}/P^+_{12} = 1$*

$$4\mathcal{V}_{122'1'} = \theta_{1/2-x} \left[\frac{\sqrt{\pi/A}}{r} - \frac{2}{x} + O(r) \right] + \theta_{x-1/2} \left[\frac{\sqrt{\pi/A}}{r} - \frac{2}{1-x} + O(r) \right] . \tag{D14}$$

This result can be written as

$$4\mathcal{V}_{122'1'} = \frac{\sqrt{\pi/A}}{r} - \frac{2}{\min(x, 1-x)} + O(r) , \tag{D15}$$

$$\mathcal{W}_{122'1'} = O(r) \, , \; A = \frac{1}{2}\,(4l^{\perp 2} + m_g^2)^2 . \tag{D16}$$

In summary,

$$4\mathcal{V}_{122'1'} = \frac{\sqrt{2\pi}}{r\,(4l^{\perp 2} + m_g^2)} - \frac{2}{\min(x, 1-x)} + O(r) . \tag{D17}$$

b. Review of the boost-invariant case of $\pi_{\bar{i}} = p^+_{\bar{i}}$

The result can be written as

$$4\mathcal{V}_{122'1'} = \frac{\sqrt{\pi/A}}{r} - \frac{2}{\min(x,1-x)} - \frac{2}{x^2+(1-x)^2} + O(r) \ , \tag{D18}$$

$$\mathcal{W}_{122'1'} = O(r) \ , \ A = \frac{1}{2}(4l^{\perp 2}+m_g^2)^2[x^2+(1-x)^2]/2 \ . \tag{D19}$$

In summary,

$$4\mathcal{V}_{122'1'} = \frac{\sqrt{2\pi a}}{r\,(4l^{\perp 2}+m_g^2)} - \frac{2}{\min(x,1-x)} - a + O(r) \ , \tag{D20}$$

$$a = \frac{2}{x^2+(1-x)^2} \ . \tag{D21}$$

1. Computation of the homogenous counterterms

The counterterm operator $\mathcal{U}^{q\bar{q}}_{0\,\mathrm{seagull}}$ in Eq. (**??**) is set to have the structure given in Eq. (**??**) in order that the matrix elements of 2nd-order $\mathcal{U}^{q\bar{q}}_{t\,\mathrm{seagull}} = f^t_{\bar{c},\bar{a}}\mathcal{U}^{q\bar{q}}_{0\,\mathrm{seagull}}$ can counter the small-r divergences identified in matrix elements $X_{nn'122'1'}$ of Eq. (D5). The divergences originate from the integration over y irrespective of the integration over variables x, $k^\perp$ and $l^\perp$. **I see in the review that instead of integrating from $-y_0$ to y_0 I need to integrate from $-x$ to x or from $x-1$ to $1-x$, which means the same formulas but with y_0 replaced with x or $1-x$. So, I check if the integration goes through and all I need to do is to make such replacements in the previously calculated integrals. The previous calculation follows.**

$$\mathcal{V}_{122'1'} = \lim_{r\to 0}\int_{-y_0}^{y_0} \frac{f^r_{\bar{1},\underline{1}q} f^r_{\bar{2},\underline{2}q}}{y^2} \ , \tag{D22}$$

$$\mathcal{W}_{122'1'} = \lim_{r\to 0}\int_{-y_0}^{y_0} \frac{f^r_{\bar{1},\underline{1}q} f^r_{\bar{2},\underline{2}q}}{y} = \lim_{r\to 0}\int_{-y_0}^{y_0} \frac{f^r_{\bar{1},\underline{1}q} f^r_{\bar{2},\underline{2}q}}{y^2}\ y \ . \tag{D23}$$

Computation of the tree-level counterterm amounts to evaluation of the functions $\mathcal{V}_{122'1'}$ and $\mathcal{W}_{122'1'}$. Using Appendix Eqs. (C11) and (C12), we have

$$f^r_{\bar{1},\underline{1}3} = \exp\left\{-r^2\left[\frac{c}{2|y|}+a_1\right]^2 (\pi_{\bar{1}}/P^+_{12})^2\right\} \ , \tag{D24}$$

$$f^r_{\bar{2},\underline{2}3} = \exp\left\{-r^2\left[\frac{c}{2|y|}+a_2\right]^2 (\pi_{\bar{2}}/P^+_{12})^2\right\} \ , \tag{D25}$$

where

$$c = 4l^{\perp 2}+m_g^2 \ , \tag{D26}$$

$$a_1 = \frac{-4k^\perp l^\perp}{x}\,\mathrm{sgn}(y) + \frac{k^2+l^2+m_1^2}{x^2}\,2|y| + O(y^2) \ , \tag{D27}$$

$$a_2 = \frac{4k^\perp l^\perp}{1-x}\,\mathrm{sgn}(y) + \frac{k^2+l^2+m_2^2}{(1-x)^2}\,2|y| + O(y^2) \ . \tag{D28}$$

Start 20240824 17:50 sob Leszno Since all variables in the integration are limited, the sensitivity to $r \to 0$ comes solely from $y \lesssim r \ll y_0$, where y_0 is an arbitrarily small but fixed number.

a. The canonical case of $\pi_{\bar{i}}/P^+_{12} = 1$

The exponent in factors $f^r_{\bar{1},\underline{1}q} f^r_{\bar{2},\underline{2}q}$ reads

$$E = E_1^2 + E_2^2 = -r^2[c/(2|y|)+a_1]^2 - r^2[c/(2|y|)+a_2]^2 \ . \tag{D29}$$

When y increases from $y<0$ to $y>0$ I have a_1 jumping from $a_{1L} = \frac{(k+l)^{\perp 2}+m_1^2}{x} - \frac{(k-l)^{\perp 2}+m_1^2}{x}$ to $a_{1R}=-a_{1L}$ and a_2 jumping from $a_{2L} = \frac{(k-l)^{\perp 2}+m_2^2}{1-x} - \frac{(k+l)^{\perp 2}+m_2^2}{1-x}$ to $a_{2R}=-a_{2L}$.

$$a_{1L} \;=\; 4k^\perp l^\perp/x \;,\; a_{1R} \;=\; -4k^\perp l^\perp/x \;, \tag{D30}$$
$$a_{2L} \;=\; -4k^\perp l^\perp/(1-x) \;,\; a_{2R} \;=\; 4k^\perp l^\perp/(1-x) \;. \tag{D31}$$

Stop 20240121 23:30 nie Ekologiczna Start 20240122 09:59 pon Ekologiczna I have

$$a_1 \;=\; \frac{-4k^\perp l^\perp}{x}\,\mathrm{sgn}(y) + \frac{k^2+l^2+m_1^2}{x^2}\,2|y| + O(y^2) \;, \tag{D32}$$
$$a_2 \;=\; \frac{4k^\perp l^\perp}{1-x}\,\mathrm{sgn}(y) + \frac{k^2+l^2+m_2^2}{(1-x)^2}\,2|y| + O(y^2) \;. \tag{D33}$$

Using $s_y = \mathrm{sgn}(y)$, the exponent is obtained in the form

$$\begin{aligned} E/(-r^2) \;=\; & \left[\frac{c}{2|y|} - \frac{4k^\perp l^\perp}{x}\,s_y + \frac{k^2+l^2+m_1^2}{x^2}\,2|y| + O(y^2)\right]^2 \\ + & \left[\frac{c}{2|y|} + \frac{4k^\perp l^\perp}{1-x}\,s_y + \frac{k^2+l^2+m_2^2}{(1-x)^2}\,2|y| + O(y^2)\right]^2 \;. \end{aligned} \tag{D34}$$

The squares yield

$$E/(-r^2) \;=\; \frac{c^2}{2y^2} + \frac{8c\,k^\perp l^\perp(x-1/2)}{x(1-x)\,y} + O(y^0) \tag{D35}$$
$$\;=\; \frac{(4l^{\perp 2}+m_g^2)^2}{2y^2} + \frac{8(4l^{\perp 2}+m_g^2)\,k^\perp l^\perp(x-1/2)}{x(1-x)\,y} + O(y^0) \;. \tag{D36}$$

The sign of y is canceled out in terms that grow when $y\to 0$. The exponent thus has the form

$$E \;=\; -r^2\left(\frac{A}{y^2}+\frac{B}{y}+C\right) \;, \tag{D37}$$
$$A \;=\; \frac{1}{2}\,(4l^{\perp 2}+m_g^2)^2 \;,\quad B \;=\; \frac{8(4l^{\perp 2}+m_g^2)\,k^\perp l^\perp(x-1/2)}{x(1-x)} \;. \tag{D38}$$

A is positive, sign of B varies with x and C has a finite limit when $y\to 0$. The diverging integrals read

$$\mathcal{V}_{122'1'} \;=\; \lim_{r\to 0}\int_{-y_0}^{y_0}\frac{dy}{y^2}\; e^{-r^2\left(A/y^2+B/y+C\right)} \;, \tag{D39}$$
$$\mathcal{W}_{122'1'} \;=\; \lim_{r\to 0}\int_{-y_0}^{y_0}\frac{dy}{y^2}\; e^{-r^2\left(A/y^2+B/y+C\right)}\; y \;. \tag{D40}$$

The integrals extend over positive and negative y. The integrals to tackle are

$$[\mathcal{Z}] \;=\; \left[\int_{-y_0}^{0}\frac{dy}{y^2} + \int_0^{y_0}\frac{dy}{y^2}\right]\; e^{-r^2\left(A/y^2+B/y+C\right)}\;[1,y] \;. \tag{D41}$$

Stop 20240824 20:22 sob Leszno I am not going to cheat myself that I can do much tonight. The exponent diverges as A/y^2 and B/y. B may be negative. First I eliminate C from consideration.

$$e^{-r^2(A/y^2+B/y+C)} \;=\; e^{-r^2(A/y^2+B/y+C)} - e^{-r^2(A/y^2+B/y)} + e^{-r^2(A/y^2+B/y)} \tag{D42}$$
$$\;=\; e^{-r^2(A/y^2+B/y)}\left(e^{-r^2C}-1\right) + e^{-r^2(A/y^2+B/y)} \;. \tag{D43}$$

The round bracket in the limit $r\to 0$ is on the order of $-r^2C$ with a limited C, while the divergence is at most on the order of $1/r$. Therefore the first term, the one with the round bracket, tends to 0 when r tends to 0. The diverging and finite dependence of the integrals $\mathcal{Z}$ on r originates entirely in $[\mathcal{Z}]$,

$$[\mathcal{Z}] \;=\; \left[\int_{-y_0}^{0}\frac{dy}{y^2} + \int_0^{y_0}\frac{dy}{y^2}\right]\; e^{-r^2\left(A/y^2+B/y\right)}\;[1,y] \;, \tag{D44}$$
$$A \;=\; \frac{1}{2}\,(4l^{\perp 2}+m_g^2)^2 \;,\quad B \;=\; \frac{8(4l^{\perp 2}+m_g^2)\,k^\perp l^\perp(x-1/2)}{x(1-x)} \tag{D45}$$

By changing integration variable y to $-y$ in the integral over negative y, I get

$$[\mathcal{Z}] = [\mathcal{Z}_1, \mathcal{Z}_y] \ , \tag{D46}$$

$$\mathcal{Z}_1 = \int_0^{y_0} \frac{dy}{y^2} \left[e^{-r^2\left(A/y^2+B/y\right)} + e^{-r^2\left(A/y^2-B/y\right)} \right] \ , \tag{D47}$$

$$\mathcal{Z}_y = \int_0^{y_0} \frac{dy}{y^2} \left[e^{-r^2\left(A/y^2+B/y\right)} - e^{-r^2\left(A/y^2-B/y\right)} \right] y \ . \tag{D48}$$

Stop 2024826 14:30 pon Pasteura Start 2024827 18:10 wto Ekologiczna I change the integration variable to $u = 1/y$ and obtain

$$\mathcal{Z}_1 = \int_{1/y_0}^{\infty} du \ e^{-r^2 A u^2} \left(e^{-r^2 B u} + e^{r^2 B u} \right) \ , \tag{D49}$$

$$\mathcal{Z}_y = \int_{1/y_0}^{\infty} \frac{du}{u} \ e^{-r^2 A u^2} \left(e^{-r^2 B u} - e^{r^2 B u} \right) \ . \tag{D50}$$

Next I change the variable u to $v = ru$.

$$\mathcal{Z}_1 = \frac{1}{r} \int_{r/y_0}^{\infty} dv \ e^{-Av^2} \left(e^{-rBv} + e^{rBv} \right) \ , \tag{D51}$$

$$\mathcal{Z}_y = \int_{r/y_0}^{\infty} \frac{dv}{v} \ e^{-Av^2} \left(e^{-rBv} - e^{rBv} \right) \ . \tag{D52}$$

The Gaussian integrand prevents v from being much larger than $1/\sqrt{A}$, which is always finite for finite m_g. Using the expansion

$$e^{\pm rBv} = 1 \pm rBv + O(v^2 r^2) \ , \tag{D53}$$

for $r \to 0$, I obtain

$$\mathcal{Z}_1 = \frac{1}{r} \int_{r/y_0}^{\infty} dv \ e^{-Av^2} \left[2 + O(v^2 r^2) \right] \ , \tag{D54}$$

$$\mathcal{Z}_y = \int_{r/y_0}^{\infty} dv \ e^{-Av^2} \left[-2rB + O(v^2 r^3) \right] \ . \tag{D55}$$

In the limit $r \to 0$ I get the leading terms

$$\mathcal{Z}_1 \to \frac{2}{r\sqrt{A}} \int_{r\sqrt{A}/y_0}^{\infty} dz \ e^{-z^2} = \frac{2}{r\sqrt{A}} \left[\frac{\sqrt{\pi}}{2} - r\sqrt{A}/y_0 + O(r^2) \right] \ , \tag{D56}$$

$$\mathcal{Z}_y \to O(r) \ . \tag{D57}$$

Hence, in Eqs. (D39) and (D40)

$$\mathcal{V}_{122'1'} = \frac{\sqrt{\pi/A}}{r} - 2/y_0 + O(r) \ , \tag{D58}$$

$$\mathcal{W}_{122'1'} = O(r) \ . \tag{D59}$$

Inclusion of $2/y_0$ secures independence of the complete result from the chosen value of y_0 when one includes the complementary part $\int_c$ in Eq. (D5) of the integral over y. However, since I used an arbitrary y_0 and all integrals with different values of y_0 exhibit the same dependence on r in the limit $r \to 0$, the dependence on r that needs to be removed is given by the term that does not depend on y_0. Consequently, after dropping y_0, the counterterm in Eq. (D11) must be

$$\lim_{r\to 0} 4\mathcal{U}_{122'1'} = -\delta(y) \, \frac{\sqrt{\pi/A}}{r} = \frac{-\delta(y)\sqrt{2\pi}}{r(4l^{\perp 2} + m_g^2)} \ . \tag{D60}$$

Stop 20240827 22:57 wto Ekologiczna Start 20240828 21:51 sro Ekologiczna Using Eq. (D61), I obtain the counterterm

$$\begin{aligned} \mathcal{U}^{q\bar{q}}_{0\,\mathrm{seagull}} &= \sum_{122'1'} \int [122'1'] \, \frac{4\sqrt{p_1^+ p_{1'}^+ p_2^+ p_{2'}^+}}{P_{12}^{+2}} \, \tilde{\delta}_{c.a} \, \delta_{sf} \, (-T^a_{11'} T^a_{2'2}) \\ &\times \frac{-\delta(y)\,\sqrt{2\pi}}{4r\,(4l^{\perp 2} + m_g^2)} \, b_1^\dagger d_2^\dagger d_{2'} b_{1'} \ , \end{aligned} \tag{D61}$$

where

$$y = (x_{1'} - x_1)/2 \ , \tag{D62}$$

$$l^\perp = (k^\perp_{1'2'} - k^\perp_{12})/2 \ , \tag{D63}$$

and

$$x_1 = p_1^+/P_{12}^+ \ , \ x_{1'} = p_{1'}^+/P_{1'2'}^+ \ , \tag{D64}$$

$$x_2 = 1 - x_1 \ , \ x_{2'} = 1 - x_{1'} \ , \tag{D65}$$

$$k^\perp_{12} = x_2 p_1^\perp - x_1 p_2^\perp \ , \ k^\perp_{1'2'} = x_{2'} p_{1'}^\perp - x_{1'} p_{2'}^\perp \ . \tag{D66}$$

Stop 20240828 24:00 sro Ekologiczna Start 20240829 19:42 czw Ekologiczna

b. The boost-invariant case of $\pi_{\bar{i}} = p^+_{\bar{i}}$

Using Eqs. (**??**) and (**??**) together with (B2) and (B3), I have

$$x_{\bar{1}} = x + |y| \ , \ x_{\bar{2}} = 1 - x + |y| \ . \tag{D67}$$

and the exponent arguments are

$$E = -r^2[c/(2|y|) + a_1]^2 (x + |y|)^2 - r^2[c/(2|y|) + a_2]^2 (1 - x + |y|)^2 \ , \tag{D68}$$

$$a_1 = \mathrm{sgn}(y) \left[\frac{(k-l)^{\perp 2} + m_1^2}{x - y} - \frac{(k+l)^{\perp 2} + m_1^2}{x + y} \right] \ , \tag{D69}$$

$$a_2 = \mathrm{sgn}(y) \left[\frac{(k+l)^{\perp 2} + m_2^2}{1 - x - y} - \frac{(k-l)^{\perp 2} + m_2^2}{1 - x + y} \right] \ . \tag{D70}$$

Analysis of a_1 and a_2 proceeds in the same way as in the simplest case, so that

$$\begin{aligned} E/(-r^2) &= \left\{ \left[\frac{c}{2|y|} - \frac{4k^\perp l^\perp}{x} \, s_y + \frac{k^2 + l^2 + m_1^2}{x^2} \, 2|y| + O(y^2) \right] (x + |y|) \right\}^2 \\ &+ \left\{ \left[\frac{c}{2|y|} + \frac{4k^\perp l^\perp}{1 - x} \, s_y + \frac{k^2 + l^2 + m_2^2}{(1-x)^2} \, 2|y| + O(y^2) \right] (1 - x + |y|) \right\}^2 \ . \end{aligned} \tag{D71}$$

Multiplication before squaring produces

$$\begin{aligned} E/(-r^2) &= \left[\frac{cx}{2|y|} + \frac{c}{2} - 4k^\perp l^\perp \, s_y - \frac{4k^\perp l^\perp \, y}{x} + \frac{k^2 + l^2 + m_1^2}{x} \, 2|y| + O(y^2) \right]^2 \\ &+ \left[\frac{c(1-x)}{2|y|} + \frac{c}{2} + 4k^\perp l^\perp s_y + \frac{4k^\perp l^\perp}{1 - x} \, y + \frac{k^2 + l^2 + m_2^2}{1 - x} \, 2|y| + O(y^2) \right]^2 \ . \end{aligned} \tag{D72}$$

Evaluating the squares I get

$$\begin{aligned} E/(-r^2) &= \frac{c^2}{4y^2} [x^2 + (1-x)^2] \\ &+ \frac{cx}{|y|} \left(\frac{c}{2} - 4k^\perp l^\perp \, s_y \right) + \frac{c(1-x)}{|y|} \left(\frac{c}{2} + 4k^\perp l^\perp s_y \right) + O(y^0) \ , \end{aligned} \tag{D73}$$

which simplifies to

$$E/(-r^2) \;=\; \frac{c^2}{4y^2}[x^2+(1-x)^2] + \frac{c^2}{2|y|} - \frac{8ck^\perp l^\perp (x-1/2)}{y} + O(y^0)\;, \tag{D74}$$

or

$$E/(-r^2) \;=\; A/y^2 + B_1/y + B_2/|y| + C\;, \tag{D75}$$

where

$$A \;=\; \frac{c^2}{4}[x^2+(1-x)^2]\;, \tag{D76}$$
$$B_1 \;=\; -8ck^\perp l^\perp (x-1/2)\;, \tag{D77}$$
$$B_2 \;=\; \frac{c^2}{2}\;. \tag{D78}$$

The new element is appearance of $O(1/y)$ as well as $O(1/|y|)$. As in Eq. (D41), the divergence originates in the region of $|y| < y_0$.

$$[\mathcal{Z}] \;=\; \left[\int_{-y_0}^{0} \frac{dy}{y^2} + \int_0^{y_0} \frac{dy}{y^2}\right]\; e^{-r^2\left(A/y^2+B_1/y+B_2/|y|+C\right)}\;[1,y]\;. \tag{D79}$$

Eliminating C as before, I proceed to changing integration variable y to $-y$ in the integral over negative y, which leads to the source of divergent dependence on $r \to 0$ in

$$[\mathcal{Z}] \;=\; [\mathcal{Z}_1, \mathcal{Z}_y]\;, \tag{D80}$$
$$\mathcal{Z}_1 \;=\; \int_0^{y_0} \frac{dy}{y^2}\;\left\{e^{-r^2\left[A/y^2+(B_2-B_1)/y\right]} + e^{-r^2\left[A/y^2+(B_2+B_1)/y\right]}\right\}\;, \tag{D81}$$
$$\mathcal{Z}_y \;=\; \int_0^{y_0} \frac{dy}{y^2}\;\left\{e^{-r^2\left[A/y^2+(B_2+B_1)/y\right]} - e^{-r^2\left[A/y^2+(B_2-B_1)/y\right]}\right\} y\;. \tag{D82}$$

Change of integration variable from y to $u = 1/y$ yields

$$\mathcal{Z}_1 \;=\; \int_{1/y_0}^{\infty} du\;\left\{e^{-r^2\left[Au^2+(B_2-B_1)u\right]} + e^{-r^2\left[Au^2+(B_2+B_1)u\right]}\right\}\;, \tag{D83}$$
$$\mathcal{Z}_y \;=\; \int_{1/y_0}^{\infty} \frac{du}{u}\;\left\{e^{-r^2\left[Au^2+(B_2+B_1)u\right]} - e^{-r^2\left[Au^2+(B_2-B_1)u\right]}\right\}\;, \tag{D84}$$

where

$$A \;=\; \frac{1}{2}\,(4l^{\perp 2}+m_g^2)^2[x^2+(1-x)^2]/2\;, \tag{D85}$$
$$B_1 \;=\; -8(4l^{\perp 2}+m_g^2)k^\perp l^\perp (x-1/2)\;, \tag{D86}$$
$$B_2 \;=\; \frac{1}{2}\,(4l^{\perp 2}+m_g^2)^2\;. \tag{D87}$$

Subsequently, the change of integration variable to $v = ru$ yields

$$\mathcal{Z}_1 \;=\; \frac{1}{r}\int_{r/y_0}^{\infty} dv\; e^{-Av^2}\left[e^{-r(B_2-B_1)v} + e^{-r(B_2+B_1)v}\right]\;, \tag{D88}$$
$$\mathcal{Z}_y \;=\; \int_{r/y_0}^{\infty} \frac{dv}{v}\; e^{-Av^2}\left[e^{-r(B_2+B_1)v} - e^{-r(B_2-B_1)v}\right]\;. \tag{D89}$$

As in the canonical case before, the coefficient A is limited from below by $m_g^4/8$. Therefore, v is limited from above by $\sim 1/\sqrt{A} \le 2\sqrt{2}/m_g^2$. In the limit $r \to 0$, I can expand $e^{-r(B_2 \mp B_1)u}$ in a series of powers of $r(B_2 \mp B_1)u$.

$$\mathcal{Z}_1 \;=\; \frac{1}{r}\int_{r/y_0}^{\infty} dv\; e^{-Av^2}\left[2 - r(B_2-B_1)v - r(B_2+B_1)v + O(v^2r^2)\right]\;, \tag{D90}$$
$$\mathcal{Z}_y \;=\; \int_{r/y_0}^{\infty} dv\; e^{-Av^2}\left[-r(B_2+B_1) + r(B_2-B_1) + O(vr^2)\right]\;. \tag{D91}$$

Neglecting terms $O(r)$ whose contribution vanishes when $r \to 0$, I get

$$\mathcal{Z}_1 = \frac{2}{r} \int_{r/y_0}^{\infty} dv\, e^{-Av^2} \left[1 - rB_2 v - O(v^2 r^2)\right] , \tag{D92}$$

$$\mathcal{Z}_y = \int_{r/y_0}^{\infty} dv\, e^{-Av^2} \left[-2rB_1 + O(vr^2)\right] . \tag{D93}$$

Evaluation yields

$$\mathcal{Z}_1 \to \frac{\sqrt{\pi/A}}{r} - 2/y_0 - B_2/A , \tag{D94}$$

$$\mathcal{Z}_y \to 0 , \tag{D95}$$

with

$$A = \frac{1}{2}\,(4l^{\perp 2} + m_g^2)^2 [x^2 + (1-x)^2]/2 , \tag{D96}$$

$$B_1 = -8(4l^{\perp 2} + m_g^2) k^{\perp} l^{\perp} (x - 1/2) , \tag{D97}$$

$$B_2 = \frac{1}{2}\,(4l^{\perp 2} + m_g^2)^2 . \tag{D98}$$

Hence, as in the canonical case, in Eqs. (D39) and (D40)

$$\mathcal{V}_{122'1'} = \frac{\sqrt{\pi/A}}{r} - 2/y_0 - \frac{2}{x^2 + (1-x)^2} + O(r) , \tag{D99}$$

$$\mathcal{W}_{122'1'} = O(r) . \tag{D100}$$

Consequently, the counterterm in Eq. (**??**) must be

$$\lim_{r\to 0} 4\mathcal{U}_{122'1'} = -\delta(y)\,\frac{\sqrt{\pi/A}}{r} = \frac{-\delta(y)\,\sqrt{2\pi}}{r(4l^{\perp 2} + m_g^2)} \sqrt{\frac{2}{x^2 + (1-x)^2}} , \tag{D101}$$

which differs from the canonical case by the square-root factor. According to Eq. (**??**), the counterterm is

$$\mathcal{U}^{q\bar{q}}_{0\,\mathrm{seagull}} = \sum_{122'1'} \int [122'1']\, \frac{4\sqrt{p_1^+ p_{1'}^+ p_2^+ p_{2'}^+}}{P_{12}^{+2}}\, \tilde{\delta}_{c.a}\, \delta_{sf}\, (-T^a_{11'} T^a_{2'2})$$
$$\times\, \frac{-\delta(y)\,\sqrt{2\pi}}{4r\,(4l^{\perp 2} + m_g^2)} \sqrt{\frac{2}{x^2 + (1-x)^2}}\; b_1^\dagger d_2^\dagger d_{2'} b_{1'} , \tag{D102}$$

where

$$y = (x_{1'} - x_1)/2 , \tag{D103}$$

$$l^{\perp} = (k^{\perp}_{1'2'} - k^{\perp}_{12})/2 , \tag{D104}$$

$$x_1 = p_1^+/P_{12}^+ \;,\; x_{1'} = p_{1'}^+/P_{1'2'}^+ , \tag{D105}$$

$$x_2 = 1 - x_1 \;,\; x_{2'} = 1 - x_{1'} , \tag{D106}$$

$$k^{\perp}_{12} = x_2 p_1^{\perp} - x_1 p_2^{\perp} \;,\; k^{\perp}_{1'2'} = x_{2'} p_{1'}^{\perp} - x_{1'} p_{2'}^{\perp} . \tag{D107}$$

c. *Counterterm computation summary: extension to H_0*

Start 20240907 13:41 sob Ekologiczna
In Eq. (D61) the canonical counterterm is

$$\mathcal{U}^{q\bar{q}}_{0\,\mathrm{seagull}}/F^r = \sum_{122'1'} \int [122'1']\, \frac{4\sqrt{p_1^+ p_{1'}^+ p_2^+ p_{2'}^+}}{P_{12}^{+2}}\, \tilde{\delta}_{c.a}\, \delta_{sf}\, (-T^a_{11'} T^a_{2'2})$$
$$\times\, \frac{-\delta(y)\,\sqrt{2\pi}}{4r\,(4l^{\perp 2} + m_g^2)}\; b_1^\dagger d_2^\dagger d_{2'} b_{1'} , \tag{D108}$$

and in Eq. (D102) the boost invariant counterterm is

$$\mathcal{U}^{q\bar{q}}_{0\,\mathrm{seagull}}/F^r = \sum_{122'1'} \int [122'1'] \, \frac{4\sqrt{p_1^+ p_{1'}^+ p_2^+ p_{2'}^+}}{P_{12}^{+2}} \, \tilde{\delta}_{c.a} \, \delta_{sf} \, (-T^a_{11'} T^a_{2'2}) \\ \times \frac{-\delta(y)\,\sqrt{2\pi}}{4r\,(4l^{\perp 2} + m_g^2)} \sqrt{\frac{2}{x^2 + (1-x)^2}} \; b_1^\dagger d_2^\dagger d_{2'} b_{1'} \ , \tag{D109}$$

where F^r denotes a suitable regularization factor and the variables are

$$y = (x_{1'} - x_1)/2 \ , \ l^\perp = (k^\perp_{1'2'} - k^\perp_{12})/2 \ , \tag{D110}$$
$$x_1 = p_1^+/P_{12}^+ \ , \ x_2 = 1 - x_1 \ , \ x_{1'} = p_{1'}^+/P_{1'2'}^+ \ , \ x_{2'} = 1 - x_{1'} \ , \tag{D111}$$
$$k^\perp_{12} = x_2 p_1^\perp - x_1 p_2^\perp \ , \ k^\perp_{1'2'} = x_{2'} p_{1'}^\perp - x_{1'} p_{2'}^\perp \ . \tag{D112}$$

The regularization factor F^r for $\mathcal{U}^{q\bar{q}}_{0\,\mathrm{seagull}}$ is set in the same way as in the third term in Eq. (62) that shows how the canonical interaction term in Eq. (61) is regulated. Namely,

$$F^r = f^r_{12,1+2} \; f^r_{1'+2',1'2'} \ , \tag{D113}$$

where the mass parameter for the *gedanken* quanta of kinematic momenta $P_{12} = p_1 + p_2$ and $P_{1'2'} = p_{1'} + p_{2'}$ is set to any finite, fixed value that is smaller than $m_1 + m_2$. In the limit $r \to 0$, the choice of the mass parameter in F^r does not matter because it is nullified by r anyways.

Appendix E: Review of the RGPEP single-quark term

Stop 20241115 23:42 pia Ekologiczna The whole evaluation of single-quark eigenvalue condition needed precise verification!

Start 20241116 14:10 sob Ekologiczna **I review the expressions, especially their signs, in the single-quark case that suddenly appears so illuminating.**.

The RGPEP for 2nd-order terms, Eq. (106)

$$\mathcal{H}_I^{(2)'} = \left[\left[\mathcal{H}_f, \mathcal{H}_I^{\pi(2)}\right], \mathcal{H}_f\right] + \left[\left[\mathcal{H}_f, \mathcal{H}_I^{\pi(1)}\right], \mathcal{H}_I^{(1)}\right]_I \ . \tag{E1}$$

Written in terms of p_q^-, for which the derivative of $p^{\perp 2} + m^2$ vanishes, taking into account that there is no homogeneous term, I drop the sum over quark spins,

$$\int [p_q] \; (p_q^-)' \; b^\dagger_{p_q} b_{p_q} = 2(p_q^- - p_{qg}^-)\pi_q^2 \, e^{-2t(p_q^- - p_{qg}^-)^2 \pi_q^2} \, [H_1^r H_1^r]_\Sigma \ . \tag{E2}$$

The momentum dependent factors are actually operators that insert functions in the integrand d In p_q^-, there is a free term that does not depend on s,

$$p_q^- = \frac{p^{\perp 2} + m^2}{p^+} + \frac{\delta m_s^2}{p^+} \ . \tag{E3}$$

Therefore

$$(p_q^-)' = \frac{(\delta m_s^2)'}{p^+} \ . \tag{E4}$$

Eq. (E2) can be integrated explicitly.

$$\int_0^t d\tau \int [p_q] \, \frac{(\delta m_\tau^2)'}{p^+} \, b^\dagger_{p_q} b_{p_q} = 2(p_q^- - p_{qg}^-)\pi_q^2 \int_0^t d\tau \; e^{-2\tau(p_q^- - p_{qg}^-)^2 \pi_q^2} \, [H_1^r H_1^r]_\Sigma \ , \tag{E5}$$
$$\int [p_q] \frac{\delta m_s^2 - \delta m_0^2}{p^+} = \frac{2(p_q^- - p_{qg}^-)\pi_q^2}{-2(p_q^- - p_{qg}^-)^2 \pi_q^2} \left[e^{-2t(p_q^- - p_{qg}^-)^2 \pi_q^2} - 1\right] \; [H_1^r H_1^r]_\Sigma \ , \tag{E6}$$
$$\int [p_q] \frac{\delta m_s^2}{p^+} = \int [p_q] \frac{\delta m_0^2}{p^+} + \left[1 - e^{-2t(p_q^- - p_{qg}^-)^2 \pi_q^2}\right] \frac{[H_1^r H_1^r]_\Sigma}{p_q^- - p_{qg}^-} \ . \tag{E7}$$

The divergences show up only in the term 1 in the square bracket; the exponential kills divergences. The counterterm in δm_0^2 is designed to remove the gluon severe small-x divergences from

$$\frac{[H_1^r H_1^r]_{\Sigma\ p\sigma cf}}{p_q^- - p_{qg}^-} \ . \tag{E8}$$

The divergence is computed in Σ_q in Eq. (**??**), which is the same as δm_s^2 considered here. **This is false statement because in the scattering p^- is replaced by $P^- - Q^-$, where P^- is the shell value and Q^- are the spectators of the self-interaction.**

20250627: See correction in scattering in Sec. E 0 a

But besides the severely diverging terms there are also logarithmic terms, not included in the previous discussion of the most severe terms in scattering and bound states. Nevertheless, the severe divergences suggest the definite counterterm to themselves. Using Eq. (**??**),

$$\mathcal{H}_\Sigma^{\rm div} \ = \ \left[\Delta_{LIR}\,\mathcal{H}_0^{(1)}\,\mathcal{H}_0^{(1)}\right]_\Sigma \ \equiv \ \frac{[H_1^r H_1^r]_\Sigma}{p_q^- - p_{qg}^-} \ , \tag{E9}$$

From $\mathcal{H}_\Sigma^{\rm div}$, I had obtained the full result for $\Sigma_q^{(2)}$ in Eq. (**??**), which reads

$$\frac{\Sigma_q^{(2)}}{p^+} \ = \ \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\, \frac{N}{D}\, f^{r2}_{p_q p_g, p} \ . \tag{E10}$$

I also obtained the counterterm integrand to the severe small-x divergence, see Eq. (**??**), and I discovered that the ultraviolet quadratic transverse divergence can be also removed if the counterterm integrand is changed to $1/x^2 + 1/[2(1-x)]$. But then the discovery is that the numerator becomes very simple. Start 20241117 15:05 nie Ekologiczna I check it now using p^- and taking advantage of the quark current conservation.

$$\begin{aligned}
\mathcal{H}^{\rm div}_{\Sigma\ p\sigma} \ &= \ \frac{[H_1^r H_1^r]_{\Sigma\ p\sigma}}{p_q^- - p_{qg}^-} && \text{(E11)}\\
&= \ \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\, \frac{\bar u_{p\sigma}\gamma_\mu(\not p_q + m)\gamma_\nu u_{p\sigma}}{(1-x)p_g^+(p^- - p_q^- - p_g^-)} \left[-g^{\mu\nu} + \eta^\mu\eta^\nu(p_g^- + p_q^- - p^-)/p_g^+\right] f^{r2}_{p_q p_g, p} \\
&= \ \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\, \bar u_p \left\{ \frac{2\not p_q - 4m}{(1-x)p_g^+(p^- - p_q^- - p_g^-)} - \frac{2p_q^+\gamma^+}{(1-x)p_g^{+2}} \right\} u_p f^{r2}_{p_q p_g, p} && \text{(E12)}\\
&= \ \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3} \left[\frac{4p_q p - 8m^2}{(1-x)p_g^+(p^- - p_q^- - p_g^-)} - \frac{4p^+p_q^+}{(1-x)p_g^{+2}} \right] f^{r2}_{p_q p_g, p} && \text{(E13)}
\end{aligned}$$

The small-x severe divergence results from the last term, which means, from the current non-conservation in direction of η. The intermediate fermion four-momentum is

$$p_q \ = \ (1-x)p - k + \frac{1}{2}\,\eta[p_q^- - (1-x)p^-] \ , \tag{E14}$$

where $k^+ = k^- = 0$. So, the integral of pk gives 0 and

$$\begin{aligned}
\mathcal{H}^{\rm div}_{\Sigma\ p\sigma cf} \ &= \ \frac{4g^2 C_F}{p^+} \int \frac{dx\, d^2k^\perp}{2(2\pi)^3} \left[\frac{\tilde N}{D} - \frac{1}{x^2}\right] f^{r2}_{p_q p_g, p} \ , && \text{(E15)}\\
\tilde N \ &= \ (1-x)m^2 + p^\perp k^\perp + \frac{1}{2}\,p^+[p_q^- - (1-x)p^-] - 2m^2 \ , && \text{(E16)}\\
D \ &= \ x(1-x)p^+(p^- - p_q^- - p_g^-) \ , && \text{(E17)}\\
4\left(\frac{\tilde N}{D} - \frac{1}{x^2}\right) \ &= \ \frac{N}{D} \ \text{ in Eq. (??)} \ , && \text{(E18)}\\
N \ &= \ 4[k^{\perp 2} + (1-x)m_g^2]/x^2 + 2(k^{\perp 2} + x^2 m_i^2)/(1-x) \ , && \text{(E19)}\\
D \ &= \ x(1-x)p^+(p^- - p_q^- - p_g^-) \ = \ x(1-x)(m_i^2 - \mathcal{M}_{qg}^2) \ . && \text{(E20)}
\end{aligned}$$

Now I verify what is obtained by adding the modified counterterm integrand $4\sigma_q^{(2)} = 4/x^2 + 2/(1-x)$ to N/D.

$$\frac{N}{D} + 4/x^2 + 2/(1-x) \;=\; \left(N + 4D\left\{1/x^2 + 1/[2(1-x)]\right\}\right)/D \;, \tag{E21}$$

$$N + D\left[4/x^2 + 2/(1-x)\right] \tag{E22}$$

$$\begin{aligned}
&= 4[k^{\perp 2} + (1-x)m_g^2]/x^2 + 2(k^{\perp 2} + x^2 m_i^2)/(1-x) \\
&- x(1-x)(\mathcal{M}_{qg}^2 - m_i^2)\left[4/x^2 + 2/(1-x)\right]
\end{aligned} \tag{E23}$$

$$\begin{aligned}
&= 4[k^{\perp 2} + (1-x)m_g^2]/x^2 + 2(k^{\perp 2} + x^2 m_i^2)/(1-x) \\
&- x(1-x)\left[(k^{\perp 2} + m_g^2)/x + (k^{\perp 2} + m_i^2)/(1-x) - m_i^2\right]\left[4/x^2 + 2/(1-x)\right]
\end{aligned} \tag{E24}$$

$$\begin{aligned}
&= 4[k^{\perp 2} + (1-x)m_g^2]/x^2 + 2(k^{\perp 2} + x^2 m_i^2)/(1-x) \\
&- \left[k^{\perp 2} + (1-x)m_g^2 + x^2 m_i^2\right]\left[4/x^2 + 2/(1-x)\right]
\end{aligned} \tag{E25}$$

$$\begin{aligned}
&= 4[(1-x)m_g^2]/x^2 + 2x^2 m_i^2/(1-x) \\
&- (1-x)m_g^2 4/x^2 - (1-x)m_g^2 2/(1-x) - x^2 m_i^2 4/x^2 - x^2 m_i^2 2/(1-x)
\end{aligned} \tag{E26}$$

$$\begin{aligned}
&= 4(1-x)m_g^2/x^2 + 2x^2 m_i^2/(1-x) \\
&- 4(1-x)m_g^2/x^2 - 2m_g^2 - 4m_i^2 - m_i^2 2x^2/(1-x)
\end{aligned} \tag{E27}$$

$$= -(4m_i^2 + 2m_g^2) \;. \tag{E28}$$

This confirms my original result. **End of review 20241117 22:32 nie Ekologiczna**

a. 20250627: Correction due to $p^- \to P^- - Q^-$ in self-interaction in scattering

$$\mathcal{H}^{\text{div scattering}}_{\Sigma\; p\sigma} \;=\; \frac{[H_1^r H_1^r]_{\Sigma\; p\sigma}}{P^- - Q^- - p^-_{qg}} \tag{E29}$$

$$= \frac{g^2 C_F}{p^+}\int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\, \frac{\bar u_{p\sigma}\gamma_\mu(\not p_q + m)\gamma_\nu u_{p\sigma}}{(1-x)p_g^+(\Delta^- + p^- - p_q^- - p_g^-)}\left[-g^{\mu\nu} + \eta^\mu\eta^\nu(p_g^- + p_q^- - p^-)/p_g^+\right] f^{r2}_{p_q p_g, p}$$

$$= \frac{g^2 C_F}{p^+}\int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\, \bar u_p\left\{\frac{2\not p_q - 4m}{(1-x)p_g^+(\Delta^- + d^-)} - \frac{2p_q^+\gamma^+}{(1-x)p_g^{+2}}\,\frac{d^-}{\Delta^- + d^-}\right\} u_p f^{r2}_{p_q p_g, p}, \tag{E30}$$

where

$$\Delta^- \;=\; P^- - Q^- - p^- \;,\; d^- \;=\; p^- - p_q^- - p_g^- \;. \tag{E31}$$

Therefore,

$$\mathcal{H}^{\text{div scattering}}_{\Sigma\; p\sigma} \;=\; \frac{[H_1^r H_1^r]_{\Sigma\; p\sigma}}{\Delta^- - p^-_{qg}} \tag{E32}$$

$$= \frac{g^2 C_F}{p^+}\int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\left[\frac{4p_q p - 8m^2}{(1-x)p_g^+(\Delta^- + d^-)} - \frac{4p^+ p_q^+}{(1-x)p_g^{+2}}\,\frac{d^-}{\Delta^- + d^-}\right] f^{r2}_{p_q p_g, p} \tag{E33}$$

$$= \frac{g^2 C_F}{p^+}\int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\left[\frac{4p_q p - 8m^2}{(1-x)p_g^+ d^-} - \frac{4p^+ p_q^+}{(1-x)p_g^{+2}}\right]\frac{d^-}{\Delta^- + d^-} f^{r2}_{p_q p_g, p} \tag{E34}$$

$$= \frac{g^2 C_F}{p^+}\int \frac{dx\, d^2k^\perp}{2(2\pi)^3}\left[\frac{4p_q p - 8m^2}{(1-x)p_g^+ d^-} - \frac{4p^+ p_q^+}{(1-x)p_g^{+2}}\right]\left(1 - \frac{\Delta^-}{\Delta^- + d^-}\right) f^{r2}_{p_q p_g, p} \tag{E35}$$

The square bracket is as in the self-interaction in the RGPEP. The term with 1 in the round bracket can be treated by the same counterterm as before in my flawed formula for scattering off-shell. The term with

$$\frac{\Delta^-}{\Delta^- + d^-} \;=\; \frac{1}{1 + d^-/\Delta^-} \tag{E36}$$

tends to 0 like Δ^-/d^-, which is inverse of the gluon p_g^-. Such factor eliminates $1/x^2$ as well as quadratic transverse divergence. I can return to the text on scattering and insert a comment about the off-shell result. **20250627: end of correction to quark Σ due to off-shellness.**

b. Evaluation of m_E^2

The quark self-interaction and its counterterm in Eq. (**??**) enter the quarkonium eigenvalue problem for H_{LR}. The counterterm cancels the severe small-x divergence in Eq. (**??**)

$$\sigma_q^{(2)} \;=\; 1/x^2 + 1/[2(1-x)] \;, \tag{E37}$$

$$\mathrm{Num} \;=\; 4[k^{\perp 2} + (1-x)m_g^2]/x^2 + 2(k^{\perp 2} + x^2 m_i^2)/(1-x) \;, \tag{E38}$$

$$\mathrm{Den} \;=\; x(1-x)p^+(p^- - p_q^- - p_g^-) \;=\; x(1-x)(m_i^2 - \mathcal{M}_{qg}^2) \tag{E39}$$

$$\tilde{\Sigma} \;=\; \frac{\mathrm{Num}}{\mathrm{Den}} + 4/x^2 + 2/(1-x) \;=\; \frac{\mathrm{Num} + [4/x^2 + 2/(1-x)]\mathrm{Den}}{\mathrm{Den}} \;. \tag{E40}$$

$$R \;=\; \mathrm{Num} + [4/x^2 + 2/(1-x)]\mathrm{Den} \tag{E41}$$

$$\begin{aligned} &= 4k^{\perp 2}/x^2 + 2k^{\perp 2}/(1-x) + 4(1-x)m_g^2/x^2 + 2x^2 m_i^2/(1-x) \\ &+ [4/x^2 + 2/(1-x)]x(1-x)\left\{m_i^2 - k^{\perp 2}/[x(1-x)] - m_i^2/(1-x) - m_g^2/x\right\} \;. \end{aligned} \tag{E42}$$

$$\begin{aligned} R \;&=\; 4k^{\perp 2}/x^2 + 2k^{\perp 2}/(1-x) + 4(1-x)m_g^2/x^2 + 2x^2 m_i^2/(1-x) \\ &+ [4/x^2 + 2/(1-x)]\left[-k^{\perp 2} - m_i^2 x^2 - m_g^2(1-x)\right] \;. \end{aligned} \tag{E43}$$

$$\begin{aligned} R \;&=\; 4(1-x)m_g^2/x^2 + 2x^2 m_i^2/(1-x) \\ &- [4/x^2 + 2/(1-x)]\left[m_i^2 x^2 + m_g^2(1-x)\right] \;. \end{aligned} \tag{E44}$$

$$\begin{aligned} R \;&=\; 4(1-x)m_g^2/x^2 + 2x^2 m_i^2/(1-x) \\ &- (4/x^2)\left[m_i^2 x^2 + m_g^2(1-x)\right] - [2/(1-x)]\left[m_i^2 x^2 + m_g^2(1-x)\right] \end{aligned} \tag{E45}$$

$$\begin{aligned} &= 4(1-x)m_g^2/x^2 + 2x^2 m_i^2/(1-x) \\ &- 4m_i^2 - 4m_g^2(1-x)/x^2 - 2m_i^2 x^2/(1-x) - 2m_g^2 \end{aligned} \tag{E46}$$

$$\begin{aligned} &= m_i^2\left[2x^2/(1-x) - 4 - 2x^2/(1-x)\right] \\ &+ m_g^2\left[4(1-x)/x^2 - 4(1-x)/x^2 - 2\right] \;. \end{aligned} \tag{E47}$$

$$R \;=\; -4m_i^2 - 2m_g^2 \;. \tag{E48}$$

$$\tilde{\Sigma} \;=\; \frac{\mathrm{Num}}{\mathrm{Den}} + 4/x^2 + 2/(1-x) \;=\; \frac{4m_i^2 + 2m_g^2}{x(1-x)(\mathcal{M}_{qg}^2 - m_i^2)} \tag{E49}$$

$$= \frac{4m_i^2 + 2m_g^2}{x(1-x)\left\{k^{\perp 2}/[x(1-x)] + m_i^2/(1-x) + m_g^2/x - m_i^2\right\}} \tag{E50}$$

$$= \frac{4m_i^2 + 2m_g^2}{x(1-x)\left\{k^{\perp 2}/[x(1-x)] + m_i^2 x/(1-x) + m_g^2/x\right\}} \tag{E51}$$

$$= \frac{4m_i^2 + 2m_g^2}{k^{\perp 2} + m_i^2 x^2 + m_g^2(1-x)} \;. \tag{E52}$$

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Appendix F: Formulas for TDW with m_G not cancelling RGPEP

When the numerator factors are included,

$$\begin{aligned} & f^t \mathcal{H}(\ )\, f^t \mathcal{H} && \text{(F1)} \\ =\ & \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2 k^\perp}{2(2\pi)^3} \frac{p_G^- - p_g^-}{p_G^- - p_g^- + p_g^- + p_q^- - p^-} \frac{N}{x(1-x)p^+(p_g^- + p_q^- - p^-)} f^{t2}_{p_q p_g, p} \ , && \text{(F2)} \\ N\ =\ & 4[k^{\perp 2} + (1-x)m_g^2]/x^2 + 2(k^{\perp 2} + x^2 m_i^2)/(1-x) \ . && \text{(F3)} \end{aligned}$$

I have $M^2 = m_G^2 - m_g^2$ and

$$\begin{aligned} & f^t \mathcal{H}(\ )\, f^t \mathcal{H} && \text{(F4)} \\ =\ & \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2 k^\perp}{2(2\pi)^3} \frac{M^2}{M^2 + p_g^+(p_g^- + p_q^- - p^-)} \frac{N\, f^{t2}_{p_q p_g, p}}{(1-x)p_g^+(p_g^- + p_q^- - p^-)} \ . && \text{(F5)} \end{aligned}$$

This is the same equation as Eq. (44) in [K. Serafin et al., Phys. Rev. D 109, 016017 (2024)]. Stop 20241119 04:44 wto Ekologiczna Start 20241119 22:11 -¿ 23:13 wto Ekologiczna Before the subtraction is evaluated as above here, I have

$$\begin{aligned} & f^t \mathcal{H}(\ )\, f^t \mathcal{H} && \text{(F6)} \\ =\ & \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2 k^\perp}{2(2\pi)^3 x(1-x)p^+} \left(\frac{N\, f^{t2}_{p_q p_g, p}}{p^- - p_q^- - p_g^- - A/p^+} - \frac{N\, f^{t2}_{p_q p_g, p}}{p^- - p_q^- - p_g^-} \right) \ , && \text{(F7)} \\ A\ =\ & (m_G^2 - m_g^2)/x \ , && \text{(F8)} \\ N\ =\ & 4[k^{\perp 2} + (1-x)m_g^2]/x^2 + 2(k^{\perp 2} + x^2 m_i^2)/(1-x) \ , && \text{(F9)} \\ D\ =\ & x(1-x)p^+(p^- - p_q^- - p_g^-) \ =\ x(1-x)(m_i^2 - \mathcal{M}_{qg}^2) \ , && \text{(F10)} \end{aligned}$$

according to Eq. (**??**). Then

$$\begin{aligned} f^t \mathcal{H}(\ )\, f^t \mathcal{H}\ &=\ \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2 k^\perp}{2(2\pi)^3} \left[\frac{N}{D-B} - \frac{N}{D} \right] f^{t2}_{p_q p_g, p} \ , && \text{(F11)} \\ B\ &=\ (1-x)(m_G^2 - m_g^2) \ . && \text{(F12)} \end{aligned}$$

Now I can add and subtract the previously discovered integrand $\sigma_q^{(2)}$ that provided a counterterm to both $1/x^2$ and $k^{\perp 2}$ divergences, given in Eq. (**??**) as $\sigma_q^{(2)} = 1/x^2 + 1/[2(1-x)]$.

$$\begin{aligned} f^t \mathcal{H}(\ )\, f^t \mathcal{H}\ &=\ \frac{g^2 C_F}{p^+} \int \frac{dx\, d^2 k^\perp}{2(2\pi)^3} \left[\frac{N}{D-B} + 4\sigma_q^{(2)} - \left(\frac{N}{D} + 4\sigma_q^{(2)} \right) \right] f^{t2}_{p_q p_g, p} \ , && \text{(F13)} \\ B\ &=\ (1-x)(m_G^2 - m_g^2) \ . && \text{(F14)} \end{aligned}$$

Nothing has changed due to this adding and subtracting the same term. I have the result from Eqs. (**??**), (**??**), and (E52),

$$\begin{aligned} \frac{N}{D} + 4\sigma_q^{(2)}\ &=\ \frac{N + 4\sigma_q^{(2)} D}{D}\ =\ \frac{4m_i^2 + 2m_g^2}{-D} \ , && \text{(F15)} \\ \frac{N}{D-B} + 4\sigma_q^{(2)}\ &=\ \frac{N + 4\sigma_q^{(2)}(D-B)}{D-B}\ =\ \frac{4m_i^2 + 2m_g^2 + 4\sigma_q^{(2)} B}{-D+B} \ . && \text{(F16)} \end{aligned}$$

Now I substitute D.

$$\begin{aligned} \frac{N}{D} + 4\sigma_q^{(2)}\ &=\ \frac{4m_i^2 + 2m_g^2}{k^{\perp 2} + m_i^2 x^2 + m_g^2(1-x)} \ , && \text{(F17)} \\ \frac{N}{D-B} + 4\sigma_q^{(2)}\ &=\ \frac{4m_i^2 + 2m_g^2 + 4\sigma_q^{(2)} B}{k^{\perp 2} + m_i^2 x^2 + m_g^2(1-x) + B} \ , && \text{(F18)} \end{aligned}$$

and then I substitute B, using

$$4\sigma_q^{(2)}B \;=\; 4\sigma_q^{(2)}(1-x)(m_G^2-m_g^2) \;=\; \frac{4(1-x)}{x^2}(m_G^2-m_q^2)+2m_G^2-2m_g^2\;, \tag{F19}$$

so that

$$\frac{N}{D}+4\sigma_q^{(2)} \;=\; \frac{4m_i^2+2m_g^2}{k^{\perp 2}+m_i^2x^2+m_g^2(1-x)}\;, \tag{F20}$$

$$\frac{N}{D-B}+4\sigma_q^{(2)} \;=\; \frac{4m_i^2+2m_G^2+4(1-x)(m_G^2-m_g^2)/x^2}{k^{\perp 2}+m_i^2x^2+m_G^2(1-x)}\;, \tag{F21}$$

The difference Dif = [ ] in Eq. (F13) is

$$\mathrm{Dif} \;=\; \frac{N}{D-B}+4\sigma_q^{(2)}-\left(\frac{N}{D}+4\sigma_q^{(2)}\right) \tag{F22}$$

$$=\; \frac{4m_i^2+2m_g^2}{k^{\perp 2}+m_i^2x^2+m_g^2(1-x)}-\frac{4m_i^2+2m_G^2+4(1-x)(m_G^2-m_g^2)/x^2}{k^{\perp 2}+m_i^2x^2+m_g^2(1-x)+(m_G^2-m_g^2)(1-x)}\;, \tag{F23}$$

or

$$\mathrm{Dif} \;=\; \frac{N}{D-B}+4\sigma_q^{(2)}-\left(\frac{N}{D}+4\sigma_q^{(2)}\right) \tag{F24}$$

$$=\; \frac{4m_i^2+2m_g^2}{x(1-x)(\mathcal{M}^2-m_i^2)}-\frac{4m_i^2+2m_G^2+4(1-x)(m_G^2-m_g^2)/x^2}{x(1-x)(\mathcal{M}^2-m_i^2)+(m_G^2-m_g^2)(1-x)x/x}\;. \tag{F25}$$

Calculational trick yields

$$x(1-x)\mathrm{Dif} \;=\; x(1-x)\left[\frac{N}{D-B}+4\sigma_q^{(2)}-\left(\frac{N}{D}+4\sigma_q^{(2)}\right)\right] \tag{F26}$$

$$=\; \frac{4m_i^2+2m_g^2}{\mathcal{M}^2-m_i^2}-\frac{4m_i^2+2m_g^2+(m_G^2-m_g^2)[2+4(1-x)/x^2]}{\mathcal{M}^2-m_i^2+(m_G^2-m_g^2)/x}\;. \tag{F27}$$

I tried some other ways of writing the same thing,

$$x(1-x)\mathrm{Dif} \;=\; (m_G^2-m_g^2)\,[\quad]\;, \tag{F28}$$

$$[\quad] \;=\; \frac{4m_i^2+2m_g^2}{x(\mathcal{M}^2-m_i^2)[\mathcal{M}^2-m_i^2+(m_G^2-m_g^2)/x]}-\frac{2+4(1-x)/x^2}{\mathcal{M}^2-m_i^2+(m_G^2-m_g^2)/x}\;, \tag{F29}$$

or

$$x(1-x)\mathrm{Dif} \;=\; (m_G^2-m_g^2)\,[\quad]\;, \tag{F30}$$

$$[\quad] \;=\; \frac{4m_i^2+2m_g^2}{x(\mathcal{M}_g^2-m_i^2)(\mathcal{M}_G^2-m_i^2)}-\frac{2+4(1-x)/x^2}{\mathcal{M}_G^2-m_i^2}\;. \tag{F31}$$

The result to think about seems to be

$$f^t\mathcal{H}\,(\;)\,f^t\mathcal{H} \;=\; \frac{g^2C_F}{p^+}\int\frac{dx\,d^2k^\perp}{2(2\pi)^3x(1-x)}\;x(1-x)\mathrm{Dif}\;f^{t2}_{p_qp_g,p}\;, \tag{F32}$$

$$x(1-x)\mathrm{Dif} \;=\; (m_G^2-m_g^2)\,[\quad]\;, \tag{F33}$$

$$[\quad] \;=\; \frac{4m_i^2+2m_g^2}{x(\mathcal{M}^2-m_i^2)[\mathcal{M}^2-m_i^2+(m_G^2-m_g^2)/x]}-\frac{2+4(1-x)/x^2}{\mathcal{M}^2-m_i^2+(m_G^2-m_g^2)/x}\;, \tag{F34}$$

combined with Eq. (**??**),

$$\int[xk] \;=\; \int\frac{dx\,d^2k^\perp}{2(2\pi)^3x(1-x)} \;=\; \int\frac{d^3k}{2(2\pi)^3}\left(\frac{1}{E_g}+\frac{1}{E_q}\right) \;=\; \int\frac{d\Omega\,k\,dE}{2(2\pi)^3}\;, \tag{F35}$$

and

$$x(1-x)\mathrm{Dif} \;=\; (m_G^2 - m_g^2)\,[\quad] \tag{F36}$$

$$= \frac{(m_G^2 - m_g^2)(4m_i^2 + 2m_g^2)}{[(E_q+E_g)^2 - m_i^2]\{x[(E_q+E_g)^2 - m_i^2] + (m_G^2 - m_g^2)\}}$$

$$- \frac{(m_G^2 - m_g^2)[2x + 4(1-x)/x]}{x[(E_q+E_g)^2 - m_i^2] + m_G^2 - m_g^2} \tag{F37}$$

$$= \frac{m_G^2 - m_g^2}{x[(E_q+E_g)^2 - m_i^2] + m_G^2 - m_g^2} \left[\frac{4m_i^2 + 2m_g^2}{(E_q+E_g)^2 - m_i^2} + 4 - 2x - 4/x\right] . \tag{F38}$$

In terms of E,

$$f^t\mathcal{H}(\quad)\, f^t\mathcal{H} \;=\; \frac{g^2 C_F}{p^+} \int \frac{d\Omega\, k\, dE}{2(2\pi)^3}\, A(E,x)\, e^{-2t^2(E^2-m_i^2)^2(\pi^+/p^+)^2} , \tag{F39}$$

$$A(E,x) \;=\; \frac{m_G^2 - m_g^2}{x(E^2 - m_i^2) + m_G^2 - m_g^2} \left[\frac{4m_i^2 + 2m_g^2}{E^2 - m_i^2} + 4 - 2x - 4/x\right] . \tag{F40}$$

Appendix G: RGPEP form factors in gluon exchange terms

From Eqs. (**??**) and (**??**), in which r is replaced by s, or from Eq. (699), I obtain

$$f_1 \;=\; f^t_{\bar{1},\underline{1}q} \;=\; e^{-t^2\,[(p_{\underline{1}}+q)^2 - m_1^2]^2\,(\pi_{\bar{1}}/p^+_{\bar{1}})^2} , \tag{G1}$$

$$f_2 \;=\; f^t_{\bar{2},\underline{2}q} \;=\; e^{-t^2\,[(p_{\underline{2}}+q)^2 - m_2^2]^2\,(\pi_{\bar{2}}/p^+_{\bar{2}})^2} . \tag{G2}$$

From Eqs. (C7) and (C8), with $z = 2y$ and

$$c \;=\; 4l^{\perp 2} + m_g^2 \;,\; x_{\bar{1}} \;=\; x + |y| \;,\; x_{\bar{2}} \;=\; 1 - x + |y| \;,\; s_y \;=\; \mathrm{sgn}(y) \;, \tag{G3}$$

I have

$$(p_{\underline{1}} + q)^2 - m_1^2 \;=\; x_{\bar{1}}(c/|z| + a_1) \;,\; a_1 \;=\; s_y\left[\frac{(k-l)^{\perp 2} + m_1^2}{x-y} - \frac{(k+l)^{\perp 2} + m_1^2}{x+y}\right], \tag{G4}$$

$$(p_{\underline{2}} + q)^2 - m_2^2 \;=\; x_{\bar{2}}(c/|z| + a_2) \;,\; a_2 \;=\; s_y\left[\frac{(k+l)^{\perp 2} + m_2^2}{1-x-y} - \frac{(k-l)^{\perp 2} + m_2^2}{1-x+y}\right]. \tag{G5}$$

****

a. *Scheme for evaluating self-interactions in gauge theories*

First there is the Hamiltonian term

$$H \;=\; \int_F -g_{\mu\nu} J^{a\mu} A^{a\nu}\; f \;, \tag{G6}$$

$$J^{a\mu} \;=\; -g\bar{\psi} T^a \gamma^\mu \psi \;, \tag{G7}$$

and the field operators

$$\psi \;=\; \sum_{c=1}^{3}\sum_{\sigma=1}^{2} \int [p] \left[u_{p\sigma}\chi_c b_{p\sigma c} e^{-ipx} + v_{p\sigma}\chi_c d^\dagger_{p\sigma c} e^{ipx}\right]_{x^+=0} , \tag{G8}$$

$$A^\mu \;=\; \sum_{c=1}^{8}\sum_{\sigma=1}^{2} \int [p] \left[\varepsilon^\mu_{p\sigma} T^c a_{p\sigma c} e^{-ipx} + \varepsilon^{\mu *}_{p\sigma} T^c a^\dagger_{p\sigma c} e^{ipx}\right]_{x^+=0} , \tag{G9}$$

$$\phi \;=\; \sum_{c=1}^{8} \int [p] \left[-iT^c a_{p3c} e^{-ipx} + iT^c a^\dagger_{p3c} e^{ipx}\right]_{x^+=0} . \tag{G10}$$

Hence,

$$H = -g_{\mu\nu}(-g)\int [123]\tilde{\delta}_{c.a} f \left[u_1\chi_1 b_1 + v_1\chi_1 d_1^\dagger\right]^\dagger \gamma^0\gamma^\mu \left[\varepsilon_3^\nu T^3 a_3 + \varepsilon_3^{\nu *} T^3 a_3^\dagger\right]\left[u_2\chi_2 b_2 + v_2\chi_2 d_2^\dagger\right] . \tag{G11}$$

Self-interaction means no pairs and no exchange.

$$H = -g_{\mu\nu}\int [123]\ \tilde{\delta}_{c.a}\ \Sigma^{\mu\nu}\ f\ , \tag{G12}$$

$$\Sigma^{\mu\nu} = (-g)\left[\bar{u}_1\chi_1^\dagger\gamma^\mu T^3 u_2\chi_2\ b_1^\dagger b_2 - \bar{v}_1\chi_1^\dagger\gamma^\mu T^3 v_2\chi_2\ d_2^\dagger d_1\right]\left[\varepsilon_3^\nu a_3 + \varepsilon_3^{\nu *} a_3^\dagger\right] . \tag{G13}$$

Rewritten,

$$\Sigma^{\mu\nu} = \left[-g\bar{u}_1\gamma^\mu u_2\ \chi_1^\dagger T^3\chi_2\ b_1^\dagger b_2 + g\bar{v}_1\gamma^\mu v_2\ \chi_1^\dagger T^3\chi_2\ d_2^\dagger d_1\right]\left[\varepsilon_3^\nu a_3 + \varepsilon_3^{\nu *} a_3^\dagger\right] \tag{G14}$$

$$= \left[j_{12}^{3\mu}\ b_1^\dagger b_2 + \bar{j}_{12}^{3\mu}\ d_2^\dagger d_1\right]\left[\varepsilon_3^\nu a_3 + \varepsilon_3^{\nu *} a_3^\dagger\right] , \tag{G15}$$

with

$$j_{12}^{3\mu} = -g\bar{u}_1\gamma^\mu u_2\ \chi_1^\dagger T^3\chi_2\ ,\ \bar{j}_{12}^{3\mu} = g\bar{v}_1\gamma^\mu v_2\ \chi_1^\dagger T^3\chi_2\ . \tag{G16}$$

Stop 20241220 20:30 pia San Dimas Start 20241221 10:38 sob San Dimas I simplify H.

$$H = -\int [123]\ \tilde{\delta}_{c.a} f\left[j_{12}^3\varepsilon_3\ b_1^\dagger b_2 a_3 + j_{12}^3\varepsilon_3^*\ a_3^\dagger b_1^\dagger b_2 + \bar{j}_{12}^3\varepsilon_3\ d_2^\dagger d_1 a_3 + \bar{j}_{12}^3\varepsilon_3^*\ a_3^\dagger d_2^\dagger d_1\right] . \tag{G17}$$

The self-interaction occurs in the square of H. Limiting H^2 to the self-interaction of fermions and anti-fermions, I have

$$\begin{aligned}
H^2 &= \int [1'2'3']\ \tilde{\delta}_{c'.a'} f \\
&\times \left[j_{1'2'}^{3'}\varepsilon_{3'}\ b_{1'}^\dagger b_{2'} a_{3'} + j_{1'2'}^{3'}\varepsilon_{3'}^*\ a_{3'}^\dagger b_{1'}^\dagger b_{2'} + \bar{j}_{1'2'}^{3'}\varepsilon_{3'}\ d_{2'}^\dagger d_{1'} a_{3'} + \bar{j}_{1'2'}^{3'}\varepsilon_{3'}^*\ a_{3'}^\dagger d_{2'}^\dagger d_{1'}\right] \\
&\times \int [1''2''3'']\ \tilde{\delta}_{c''.a''} f \\
&\times \left[j_{1''2''}^{3''}\varepsilon_{3''}\ b_{1''}^\dagger b_{2''} a_{3''} + j_{1''2''}^{3''}\varepsilon_{3''}^*\ a_{3''}^\dagger b_{1''}^\dagger b_{2''} + \bar{j}_{1''2''}^{3''}\varepsilon_{3''}\ d_{2''}^\dagger d_{1''} a_{3''} + \bar{j}_{1''2''}^{3''}\varepsilon_{3''}^*\ a_{3''}^\dagger d_{2''}^\dagger d_{1''}\right] \\
&\to \int [1'2'3']\ \tilde{\delta}_{1'.2'3'}\ \left[f j_{1'2'}^{3'}\varepsilon_{3'}\ b_{1'}^\dagger b_{2'} a_{3'}\right] \int [1''2''3'']\ \tilde{\delta}_{3''1''.2''}\ \left[f j_{1''2''}^{3''}\varepsilon_{3''}^*\ a_{3''}^\dagger b_{1''}^\dagger b_{2''}\right] \\
&+ \int [1'2'3']\ \tilde{\delta}_{2'.1'3'}\ \left[f\bar{j}_{1'2'}^{3'}\varepsilon_{3'}\ d_{2'}^\dagger d_{1'} a_{3'}\right] \int [1''2''3'']\ \tilde{\delta}_{3''2''.1''}\ \left[f\bar{j}_{1''2''}^{3''}\varepsilon_{3''}^*\ a_{3''}^\dagger d_{2''}^\dagger d_{1''}\right] .
\end{aligned} \tag{G18}$$

Contraction of $3'$ with $3''$ yields

$$\begin{aligned}
H^2 &\to \int [1'2'3']\ \tilde{\delta}_{1'.2'3'}\ \left[f j_{1'2'}^{3'}\varepsilon_{3'}\ b_{1'}^\dagger b_{2'}\right] \int [1''2'']\ \tilde{\delta}_{3'1''.2''}\ \left[f j_{1''2''}^{3'}\varepsilon_{3'}^*\ b_{1''}^\dagger b_{2''}\right] \\
&+ \int [1'2'3']\ \tilde{\delta}_{2'.1'3'}\ \left[f\bar{j}_{1'2'}^{3'}\varepsilon_{3'}\ d_{2'}^\dagger d_{1'}\right] \int [1''2'']\ \tilde{\delta}_{3'2''.1''}\ \left[f\bar{j}_{1''2''}^{3'}\varepsilon_{3'}^*\ d_{2''}^\dagger d_{1''}\right] .
\end{aligned} \tag{G19}$$

Contraction of fermions yields Continuing 20241227 13:16 pia San Dimas

$$\begin{aligned}
H^2 &\to \int [1'2'3']\ \tilde{\delta}_{1'.2'3'}\ \left[f j_{1'2'}^{3'}\varepsilon_{3'}\ b_{1'}^\dagger\right] \int [2'']\ \tilde{\delta}_{3'2'.2''}\ \left[f j_{2'2''}^{3'}\varepsilon_{3'}^*\ b_{2''}\right] \\
&+ \int [1'2'3']\ \tilde{\delta}_{2'.1'3'}\ \left[f\bar{j}_{1'2'}^{3'}\varepsilon_{3'}\ d_{2'}^\dagger\right] \int [1'']\ \tilde{\delta}_{3'1'.1''}\ \left[f\bar{j}_{1''1'}^{3'}\varepsilon_{3'}^*\ d_{1''}\right] .
\end{aligned} \tag{G20}$$

Integration over $2''$ and $1''$ gives

$$\begin{aligned}
H^2 &\to \int [1'2'3']\ \tilde{\delta}_{1'.2'3'}\ \left[f j_{1'2'}^{3'}\varepsilon_{3'}\ b_{1'}^\dagger\right]\ \frac{1}{p_{3'}^+ + p_{2'}^+}\ \left[f j_{2'3'+2'}^{3'}\varepsilon_{3'}^*\ b_{3'+2'}\right] \\
&+ \int [1'2'3']\ \tilde{\delta}_{2'.1'3'}\ \left[f\bar{j}_{1'2'}^{3'}\varepsilon_{3'}\ d_{2'}^\dagger\right]\ \frac{1}{p_{3'}^+ + p_{1'}^+}\ \left[f\bar{j}_{3'+1'1'}^{3'}\varepsilon_{3'}^*\ d_{3'+1'}\right] .
\end{aligned} \tag{G21}$$

Change of notation.

$$\begin{aligned} H^2 \;\to\; & \int [1] \frac{1}{p_1^+} \int [23]\ \tilde{\delta}_{1.23}\ f_1 f_2 \sum_{c\sigma} j_{12}^{c\mu} \varepsilon_{\sigma\mu}\ j_{21}^{c\nu} \varepsilon_{\sigma\nu}^*\ b_1^\dagger b_1 \\ + & \int [2] \frac{1}{p_2^+} \int [13]\ \tilde{\delta}_{2.13}\ f_1 f_2 \sum_{c\sigma} \bar{j}_{12}^{c\mu} \varepsilon_{\sigma\mu}\ \bar{j}_{21}^{c\nu} \varepsilon_{\sigma\nu}^*\ d_2^\dagger d_2\ . \end{aligned} \tag{G22}$$

Sum over the gluon polarizations and colors.

$$\begin{aligned} H^2 \;\to\; & \int [1] \frac{1}{p_1^+} \int [23]\ \tilde{\delta}_{1.23}\ d_{\mu\nu}\ f_1 f_2\ j_{12}^{c\mu} j_{21}^{c\nu}\ b_1^\dagger b_1 \\ + & \int [2] \frac{1}{p_2^+} \int [13]\ \tilde{\delta}_{2.13}\ d_{\mu\nu}\ f_1 f_2\ \bar{j}_{12}^{c\mu} \bar{j}_{21}^{c\nu}\ d_2^\dagger d_2\ . \end{aligned} \tag{G23}$$

Knowing that the Dirac fermion currents,

$$j_{12}^{c\mu} \;=\; -g \bar{u}_1 \gamma^\mu u_2\ \chi_1^\dagger T^c \chi_2\ ,\ \bar{j}_{12}^{c\mu} \;=\; g \bar{v}_1 \gamma^\mu v_2\ \chi_1^\dagger T^c \chi_2\ , \tag{G24}$$

are conserved and using FF rules of kinematic momentum conservation,

$$q_1 \;=\; p_{\overline{1}} - p_{\underline{1}} \;=\; q_2 \;=\; p_{\overline{2}} - p_{\underline{2}}\ , \tag{G25}$$

$$d^{\mu\nu} \;=\; -g^{\mu\nu} + \frac{1}{2q_{m_g}^+} \eta^\mu \eta^\nu \left(\frac{\mathcal{M}_{g\underline{1}}^2 - m_1^2}{p_{\overline{1}}^+} + \frac{\mathcal{M}_{g\underline{2}}^2 - m_2^2}{p_{\overline{2}}^+} \right) \tag{G26}$$

$$\;=\; -g^{\mu\nu} + \frac{1}{2q_{m_g}^{+2}} \eta^\mu \eta^\nu \left(m_g^2 - q_1^2 + m_g^2 - q_2^2 \right)\ . \tag{G27}$$

Evaluation of matrix elements

$$\sigma_{23} \;=\; d_{\mu\nu}\ f_1 f_2\ j_{12}^{c\mu} j_{21}^{c\nu}\ , \tag{G28}$$

$$\langle \tilde{1}\tilde{2} | \int [1] \int [23]\ \tilde{\delta}_{1.23}\ \frac{\sigma_{23}}{p_1^+}\ b_1^\dagger b_1 | \psi_E \rangle \tag{G29}$$

$$= \langle \tilde{1}\tilde{2} | \int [1] \int [23]\ \tilde{\delta}_{1.23}\ \sigma_{23}\ b_1^\dagger b_1 \int [56]\ P^+ \tilde{\delta}_{P.56}\ \psi(x_5, k_{56}^\perp) | 56 \rangle \tag{G30}$$

$$\begin{aligned} = & \int [1] \int [23]\ \tilde{\delta}_{1.23}\ \sigma_{23} \int [56]\ P^+ \tilde{\delta}_{P.56}\ \psi(x_5, k_{56}^\perp) \\ \times & \left[\langle \tilde{1}\tilde{2} |\ b_1^\dagger b_1\ | 56 \rangle = \langle 0 | d_{\tilde{2}} b_{\tilde{1}}\ b_1^\dagger b_1\ b_5^\dagger d_6^\dagger | 0 \rangle \right] \end{aligned} \tag{G31}$$

$$= \int [1] \int [23]\ \tilde{\delta}_{1.23}\ \sigma_{23} \int [56]\ P^+ \tilde{\delta}_{P.56}\ \psi(x_5, k_{56}^\perp)\ \delta_{\tilde{2}6}\ \delta_{15}\ \delta_{\tilde{1}1} \tag{G32}$$

$$= \int [23]\ \tilde{\delta}_{\tilde{1}.23}\ \sigma_{23} \int [34]\ P^+ \tilde{\delta}_{P.56}\ \psi(x_5, k_{56}^\perp)\ \delta_{\tilde{2}6}\ \delta_{\tilde{1}5} \tag{G33}$$

$$= \int [23]\ \tilde{\delta}_{\tilde{1}.23}\ \sigma_{23} \int [6]\ P^+ \tilde{\delta}_{P.\tilde{1}6}\ \psi(x_{\tilde{1}}, k_{\tilde{1}6}^\perp)\ \delta_{\tilde{2}6} \tag{G34}$$

$$= \int [23]\ \tilde{\delta}_{\tilde{1}.23}\ \sigma_{23}\ P^+ \tilde{\delta}_{P.\tilde{1}\tilde{2}}\ \psi(x_{\tilde{1}}, k_{\tilde{1}\tilde{2}}^\perp) \tag{G35}$$

$$= \int [x_2 k_{23}^\perp] \frac{1}{p_{\tilde{1}}^+} \sigma_{23}\ P^+ \tilde{\delta}_{P.\tilde{1}\tilde{2}}\ \psi(x_{\tilde{1}}, k_{\tilde{1}\tilde{2}}^\perp) \tag{G36}$$

$$= \frac{\Sigma_{\tilde{1}}}{p_{\tilde{1}}^+} \quad P^+ \tilde{\delta}_{P.\tilde{1}\tilde{2}}\ \psi(x_{\tilde{1}}, k_{\tilde{1}\tilde{2}}^\perp)\ . \tag{G37}$$

****

b. Scheme for evaluating exchanges in gauge theories

First there is the Hamiltonian term

$$H = \int_F -g_{\mu\nu} J^{a\mu} A^{a\nu} \; f \; , \tag{G38}$$

$$J^{a\mu} = -g\bar{\psi} T^a \gamma^\mu \psi \; , \tag{G39}$$

and the field operators

$$\psi = \sum_{c=1}^{3} \sum_{\sigma=1}^{2} \int [p] \left[u_{p\sigma} \chi_c b_{p\sigma c} e^{-ipx} + v_{p\sigma} \chi_c d^\dagger_{p\sigma c} e^{ipx} \right]_{x^+=0} \; , \tag{G40}$$

$$A^\mu = \sum_{c=1}^{8} \sum_{\sigma=1}^{2} \int [p] \left[\varepsilon^\mu_{p\sigma} T^c a_{p\sigma c} e^{-ipx} + \varepsilon^{\mu *}_{p\sigma} T^c a^\dagger_{p\sigma c} e^{ipx} \right]_{x^+=0} \; , \tag{G41}$$

$$\phi = \sum_{c=1}^{8} \int [p] \left[-iT^c a_{p3c} e^{-ipx} + iT^c a^\dagger_{p3c} e^{ipx} \right]_{x^+=0} \; . \tag{G42}$$

Hence,

$$H = -g_{\mu\nu}(-g) \int [123] \tilde{\delta}_{c.a} f \left[u_1 \chi_1 b_1 + v_1 \chi_1 d^\dagger_1 \right]^\dagger \gamma^0 \gamma^\mu \left[\varepsilon^\nu_3 T^3 a_3 + \varepsilon^{\nu *}_3 T^3 a^\dagger_3 \right] \left[u_2 \chi_2 b_2 + v_2 \chi_2 d^\dagger_2 \right] . \tag{G43}$$

Exchange means no pairs and no self-interaction.

$$H = -g_{\mu\nu} \int [123] \; \tilde{\delta}_{c.a} \; X^{\mu\nu} \; , \tag{G44}$$

$$X^{\mu\nu} = (-g) f \left[\bar{u}_1 \chi^\dagger_1 \gamma^\mu T^3 u_2 \chi_2 \; b^\dagger_1 b_2 - \bar{v}_1 \chi^\dagger_1 \gamma^\mu T^3 v_2 \chi_2 \; d^\dagger_2 d_1 \right] \left[\varepsilon^\nu_3 a_3 + \varepsilon^{\nu *}_3 a^\dagger_3 \right] \; . \tag{G45}$$

Rewritten,

$$X^{\mu\nu} = f \left[-g\bar{u}_1 \gamma^\mu u_2 \; \chi^\dagger_1 T^3 \chi_2 \; b^\dagger_1 b_2 + g\bar{v}_1 \gamma^\mu v_2 \; \chi^\dagger_1 T^3 \chi_2 \; d^\dagger_2 d_1 \right] \left[\varepsilon^\nu_3 a_3 + \varepsilon^{\nu *}_3 a^\dagger_3 \right] \tag{G46}$$

$$= f \left[j^{3\mu}_{12} \; b^\dagger_1 b_2 + \bar{j}^{3\mu}_{12} \; d^\dagger_2 d_1 \right] \left[\varepsilon^\nu_3 a_3 + \varepsilon^{\nu *}_3 a^\dagger_3 \right] \; , \tag{G47}$$

with

$$j^{3\mu}_{12} = -g\bar{u}_1 \gamma^\mu u_2 \; \chi^\dagger_1 T^3 \chi_2 \; , \; \bar{j}^{3\mu}_{12} = g\bar{v}_1 \gamma^\mu v_2 \; \chi^\dagger_1 T^3 \chi_2 \; . \tag{G48}$$

Stop 20241220 20:30 pia San Dimas Start 20241221 10:38 sob San Dimas I simplify H.

$$H = -\int [123] \; \tilde{\delta}_{c.a} f \left[j^3_{12} \varepsilon_3 \; b^\dagger_1 b_2 a_3 + j^3_{12} \varepsilon^*_3 \; a^\dagger_3 b^\dagger_1 b_2 + \bar{j}^3_{12} \varepsilon_3 \; d^\dagger_2 d_1 a_3 + \bar{j}^3_{12} \varepsilon^*_3 \; a^\dagger_3 d^\dagger_2 d_1 \right] \; . \tag{G49}$$

The exchange occurs in the square of H. Limiting H^2 to the exchange between a fermion and an anti-fermion, I have

$$\begin{aligned} H^2 &= \int [1'2'3'] \; \tilde{\delta}_{c'.a'} \\ &\times \; f \left[j^{3'}_{1'2'} \varepsilon_{3'} \; b^\dagger_{1'} b_{2'} a_{3'} + j^{3'}_{1'2'} \varepsilon^*_{3'} \; a^\dagger_{3'} b^\dagger_{1'} b_{2'} + \bar{j}^{3'}_{1'2'} \varepsilon_{3'} \; d^\dagger_{2'} d_{1'} a_{3'} + \bar{j}^{3'}_{1'2'} \varepsilon^*_{3'} \; a^\dagger_{3'} d^\dagger_{2'} d_{1'} \right] \\ &\times \int [1''2''3''] \; \tilde{\delta}_{c''.a''} \\ &\times \; f \left[j^{3''}_{1''2''} \varepsilon_{3''} \; b^\dagger_{1''} b_{2''} a_{3''} + j^{3''}_{1''2''} \varepsilon^*_{3''} \; a^\dagger_{3''} b^\dagger_{1''} b_{2''} + \bar{j}^{3''}_{1''2''} \varepsilon_{3''} \; d^\dagger_{2''} d_{1''} a_{3''} + \bar{j}^{3''}_{1''2''} \varepsilon^*_{3''} \; a^\dagger_{3''} d^\dagger_{2''} d_{1''} \right] \\ &\to \int [1'2'3'] \; \tilde{\delta}_{1'.2'3'} \; \left[j^{3'}_{1'2'} \varepsilon_{3'} \; b^\dagger_{1'} b_{2'} a_{3'} \right] \int [1''2''3''] \; \tilde{\delta}_{3''2''.1''} \; \left[\bar{j}^{3''}_{1''2''} \varepsilon^*_{3''} \; a^\dagger_{3''} d^\dagger_{2''} d_{1''} \right] \\ &+ \int [1'2'3'] \; \tilde{\delta}_{2'.1'3'} \; \left[\bar{j}^{3'}_{1'2'} \varepsilon_{3'} \; d^\dagger_{2'} d_{1'} a_{3'} \right] \int [1''2''3''] \; \tilde{\delta}_{3''1''.2''} \; \left[j^{3''}_{1''2''} \varepsilon^*_{3''} \; a^\dagger_{3''} b^\dagger_{1''} b_{2''} \right] \; . \end{aligned} \tag{G50}$$

Contraction of $3'$ with $3''$ yields

$$\begin{aligned} H^2 \;\to\; & \int [1'2'3']\; \tilde{\delta}_{1'.2'3'} \; \left[j^{3'}_{1'2'}\varepsilon_{3'} \; b^\dagger_{1'} b_{2'}\right] \int [1''2'']\; \tilde{\delta}_{3'2''.1''} \; \left[\bar{j}^{3'}_{1''2''}\varepsilon^*_{3'} \; d^\dagger_{2''} d_{1''}\right] ff \\ + & \int [1'2'3']\; \tilde{\delta}_{2'.1'3'} \; \left[\bar{j}^{3'}_{1'2'}\varepsilon_{3'} \; d^\dagger_{2'} d_{1'}\right] \int [1''2'']\; \tilde{\delta}_{3'1''.2''} \; \left[j^{3'}_{1''2''}\varepsilon^*_{3'} \; b^\dagger_{1''} b_{2''}\right] ff \; . \end{aligned} \tag{G51}$$

Integration over $3'$ yields
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$$\begin{aligned} H^2 \;\to\; & \sum_{3'} \int [1'2']\; \frac{\theta(1'-2')}{p^+_{1'} - p^+_{2'}} \left[j^{3'}_{1'2'}\varepsilon_{3'} \; b^\dagger_{1'} b_{2'}\right] \int [1''2'']\; \tilde{\delta}_{c.a} \; \left[\bar{j}^{3'}_{1''2''}\varepsilon^*_{3'} \; d^\dagger_{2''} d_{1''}\right] ff \\ + & \sum_{3'} \int [1'2']\; \frac{\theta(2'-1')}{p^+_{2'} - p^+_{1'}} \left[\bar{j}^{3'}_{1'2'}\varepsilon_{3'} \; d^\dagger_{2'} d_{1'}\right] \int [1''2'']\; \tilde{\delta}_{c.a} \; \left[j^{3'}_{1''2''}\varepsilon^*_{3'} \; b^\dagger_{1''} b_{2''}\right] ff \; . \end{aligned} \tag{G52}$$

Change of notation.

$$\begin{aligned} H^2 \;\to\; & \sum_{c\sigma} \int [121'2']\; \frac{\theta_{1-1'}\tilde{\delta}_{12.1'2'}}{p^+_{\overline{1}} - p^+_{\underline{1}}} \; j^{\mu c}_{11'}\varepsilon_{\sigma\mu} \; \bar{j}^{\nu c}_{2'2}\varepsilon^*_{\sigma\nu} \; b^\dagger_1 d^\dagger_2 d_{2'} b_{1'} ff \\ + & \sum_{c\sigma} \int [121'2']\; \frac{\theta_{1'-1}\tilde{\delta}_{12.1'2'}}{p^+_{\overline{1}} - p^+_{\underline{1}}} \; j^{\nu c}_{11'}\varepsilon_{\sigma\nu} \; \bar{j}^{\mu c}_{2'2}\varepsilon^*_{\sigma\mu} \; b^\dagger_1 d^\dagger_2 d_{2'} b_{1'} ff \; . \end{aligned} \tag{G53}$$

Sum over the gluon 3 polarizations.

$$\begin{aligned} H^2 \;\to\; & \sum_{c} \int [121'2']\; \frac{\theta_{1-1'}\tilde{\delta}_{12.1'2'}}{p^+_{\overline{1}} - p^+_{\underline{1}}} \; d_{\mu\nu} \; j^{\mu c}_{11'} \; \bar{j}^{\nu c}_{2'2} \; b^\dagger_1 d^\dagger_2 d_{2'} b_{1'} ff \\ + & \sum_{c} \int [121'2']\; \frac{\theta_{1'-1}\tilde{\delta}_{12.1'2'}}{p^+_{\overline{1}} - p^+_{\underline{1}}} \; d_{\mu\nu} \; j^{\mu c}_{11'} \; \bar{j}^{\nu c}_{2'2} \; b^\dagger_1 d^\dagger_2 d_{2'} b_{1'} ff \; . \end{aligned} \tag{G54}$$

Knowing that the Dirac fermion currents,

$$j^{c\mu}_{12} \;=\; -g\bar{u}_1\gamma^\mu u_2 \; \chi^\dagger_1 T^c \chi_2 \; , \; \bar{j}^{c\mu}_{12} \;=\; g\bar{v}_1\gamma^\mu v_2 \; \chi^\dagger_1 T^c \chi_2 \; , \tag{G55}$$

are conserved and using FF rules of kinematic momentum conservation,

$$q_{m_g} \;=\; p_{\overline{1}} - p_{\underline{1}} + \frac{1}{2}\,\eta(q^-_{m_g} + p_{\underline{1}} - p_{\overline{1}}) \;=\; p_{\overline{2}} - p_{\underline{2}} + \frac{1}{2}\,\eta(q^-_{m_g} + p_{\underline{2}} - p_{\overline{2}}) \; , \tag{G56}$$

$$q_1 \;=\; p_{\overline{1}} - p_{\underline{1}} \; , \; q_2 \;=\; p_{\overline{2}} - p_{\underline{2}} \; , \tag{G57}$$

$$d^{\mu\nu} \;=\; -g^{\mu\nu} + \frac{\eta^\mu q^\nu_{m_g} + q^\mu_{m_g}\eta^\nu}{q^+_{m_g}} \tag{G58}$$

$$\equiv\; -g^{\mu\nu} + \frac{1}{2q^+_{m_g}}\eta^\mu\eta^\nu(q^-_{m_g} + p_{\underline{1}} - p_{\overline{1}} + q^-_{m_g} + p_{\underline{2}} - p_{\overline{2}}) \tag{G59}$$

$$=\; -g^{\mu\nu} + \frac{1}{2q^+_{m_g}}\eta^\mu\eta^\nu \left(\frac{\mathcal{M}^2_{g\underline{1}} - m^2_1}{p^+_{\overline{1}}} + \frac{\mathcal{M}^2_{g\underline{2}} - m^2_2}{p^+_{\overline{2}}}\right) \tag{G60}$$

$$=\; -g^{\mu\nu} + \frac{1}{2q^{+2}_{m_g}}\eta^\mu\eta^\nu \left(m^2_g - q^2_1 + m^2_g - q^2_2\right) \; . \tag{G61}$$

I universally have the $Q\bar{Q}$ gluon exchange term in the form

$$H^2 \;\to\; \sum_c \int [121'2']\; \tilde{\delta}_{12.1'2'} \; \frac{d_{\mu\nu}}{q^+_{m_g}} \; j^{\mu c}_{11'} \; \bar{j}^{\nu c}_{2'2} \;\; b^\dagger_1 d^\dagger_2 d_{2'} b_{1'} \; f_1 f_2 \; . \tag{G62}$$

This confirms my earlier results and conventions and ends the scheme of evaluating the Hamiltonian term of gluon exchange between Dirac fermions.

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$$\sigma_{122'1'} = \frac{d_{\mu\nu}}{q^+_{m_g}} f_1 f_2 \;\; j^{\mu c}_{11'} \; \bar{j}^{\nu c}_{2'2} \,, \tag{G63}$$

$$\langle\tilde{1}\tilde{2}| \int [122'1'] \; \tilde{\delta}_{12.2'1'} \; \sigma_{122'1'} \; b^\dagger_1 d^\dagger_2 d_{2'} b_{1'} \int [56] \; P^+ \tilde{\delta}_{P.56} \; \psi_{56}(x_5, k^\perp_{56}) |56\rangle \tag{G64}$$

$$= \int [122'1'] \; \tilde{\delta}_{12.2'1'} \; \sigma_{122'1'} \; \int [56] \; P^+ \tilde{\delta}_{P.56} \; \psi_{56}(x_5, k^\perp_{56})$$

$$\times \; \langle 0| d_{\tilde{2}} b_{\tilde{1}} \; b^\dagger_1 d^\dagger_2 d_{2'} b_{1'} \; b^\dagger_5 d^\dagger_6 |0\rangle \tag{G65}$$

$$= \int [122'1'] \; \tilde{\delta}_{12.2'1'} \; \sigma_{122'1'} \; \int [56] \; P^+ \tilde{\delta}_{P.56} \; \psi_{56}(x_5, k^\perp_{56}) \; \delta_{1'5} \; \delta_{2'6} \; \delta_{\tilde{1}1} \; \delta_{\tilde{2}2} \tag{G66}$$

$$= \int [2'1'] \; \tilde{\delta}_{\tilde{1}\tilde{2}.2'1'} \; \sigma_{\tilde{1}\tilde{2}2'1'} \; P^+ \tilde{\delta}_{P.1'2'} \; \psi_{1'2'}(x_{1'}, k^\perp_{1'2'}) \tag{G67}$$

$$= \left\{ \int [2'1'] = \int [P_{1'2'}] \int [x_{1'} k^\perp_{1'2'}] \right\} \; \tilde{\delta}_{\tilde{1}\tilde{2}.2'1'} \; \sigma_{\tilde{1}\tilde{2}2'1'} \; P^+ \tilde{\delta}_{P.1'2'} \; \psi_{1'2'}(x_{1'}, k^\perp_{1'2'}) \tag{G68}$$

$$= \int [P_{1'2'}] \int [x_{1'} k^\perp_{1'2'}] \; \tilde{\delta}_{\tilde{1}\tilde{2}.2'1'} \; \sigma_{\tilde{1}\tilde{2}2'1'} \; P^+ \tilde{\delta}_{P.1'2'} \; \psi_{1'2'}(x_{1'}, k^\perp_{1'2'}) \tag{G69}$$

$$= P^+ \tilde{\delta}_{P.\tilde{1}\tilde{2}} \int [x_{1'} k^\perp_{1'2'}] \; \frac{\sigma_{\tilde{1}\tilde{2}2'1'}}{P^+} \; \psi_{1'2'}(x_{1'}, k^\perp_{1'2'}) \tag{G70}$$

Appendix H: Evaluation of W denominators

Hypothesis:

$$\left[P^-_{12} - P^-_3\right]^{-1} + \left[P^-_{1'2'} - P^-_3\right]^{-1} \; = \; \left[q^-_1 - q^-_{m_G}\right]^{-1} + \left[q^-_2 - q^-_{m_G}\right]^{-1} \;\; . \tag{H1}$$

Check:

$$\left[q^-_1 - q^-_{m_G}\right]^{-1} + \left[q^-_2 - q^-_{m_G}\right]^{-1} \tag{H2}$$

$$= \; \left[p^-_{\tilde{1}} - p^-_{\underline{1}} - q^-_{m_G}\right]^{-1} + \left[p^-_{\tilde{2}} - p^-_{\underline{2}} - q^-_{m_G}\right]^{-1} \tag{H3}$$

$$= \; \theta_z \left[\frac{1}{p^-_{1'} - p^-_1 - q^-_{m_G}} + \frac{1}{p^-_2 - p^-_{2'} - q^-_{m_G}} \right]$$

$$+ \; \theta_{-z} \left[\frac{1}{p^-_1 - p^-_{1'} - q^-_{m_G}} + \frac{1}{p^-_{2'} - p^-_2 - q^-_{m_G}} \right] \tag{H4}$$

$$= \; \theta_z \left[\frac{1}{p^-_{1'} + p^-_{2'} - p^-_1 - p^-_{2'} - q^-_{m_G}} + \frac{1}{p^-_1 + p^-_2 - p^-_1 - p^-_{2'} - q^-_{m_G}} \right]$$

$$+ \; \theta_{-z} \left[\frac{1}{p^-_1 + p^-_2 - p^-_{1'} - p^-_2 - q^-_{m_G}} + \frac{1}{p^-_{1'} + p^-_{2'} - p^-_{1'} - p^-_2 - q^-_{m_G}} \right] \tag{H5}$$

$$= \; \theta_z \left[\frac{1}{P^-_{1'2'} - p^-_{\underline{1}} - p^-_{\underline{2}} - q^-_{m_G}} + \frac{1}{P^-_{12} - p^-_{\underline{1}} - p^-_{\underline{2}} - q^-_{m_G}} \right]$$

$$+ \; \theta_{-z} \left[\frac{1}{P^-_{12} - p^-_{\underline{1}} - p^-_{\underline{2}} - q^-_{m_G}} + \frac{1}{P^-_{1'2'} - p^-_{\underline{1}} - p^-_{\underline{2}} - q^-_{m_G}} \right] \tag{H6}$$

$$= \; \left[\frac{1}{P^-_{12} - p^-_{\underline{1}} - p^-_{\underline{2}} - q^-_{m_G}} + \frac{1}{P^-_{1'2'} - p^-_{\underline{1}} - p^-_{\underline{2}} - q^-_{m_G}} \right] \tag{H7}$$

$$= \; \left[P^-_{12} - P^-_3\right]^{-1} + \left[P^-_{1'2'} - P^-_3\right]^{-1} \tag{H8}$$

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$$\left[\frac{1}{q^+_{m_g}} = \frac{2}{2q^+_{m_g}}\right] \left[\frac{1}{q_1^- - q^-_{m_G}} + \frac{1}{q_2^- - q^-_{m_G}}\right] \tag{H9}$$

$$= \left[\frac{1}{\frac{1}{2}(q_1^+ + q^+_{m_G})(q_1^- - q^-_{m_G})} + \frac{1}{\frac{1}{2}(q_2^+ + q^+_{m_G})(q_2^- - q^-_{m_G})}\right] \tag{H10}$$

$$= \left[\frac{1}{(q_1 + q_{m_G})(q_1 - q_{m_G})} + \frac{1}{(q_2 + q_{m_G})(q_2 - q_{m_G})}\right] \tag{H11}$$

$$= \left[\frac{1}{q_1^2 - q^2_{m_G}} + \frac{1}{q_2^2 - q^2_{m_G}}\right] = \left[\frac{1}{q_1^2 - m_G^2} + \frac{1}{q_2^2 - m_G^2}\right] \tag{H12}$$

$$= -\left[\frac{1}{\rho_1 + (m_G^2 - m_g^2)} + \frac{1}{\rho_2 + (m_G^2 - m_g^2)}\right] . \tag{H13}$$

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1. Gluon mass in bound states

Moreover, on the basis of [K. Serafin, M. Gomez-Rocha, J. More and S. D. Glazek, *Dynamics of heavy quarks in the Fock space*, in preparation], one can expect that the results for quarkonium states are not significantly sensitive to the value of m_G already when it exceeds ~ 0.5 GeV for $c\bar{c}$ or ~ 0.8 GeV for $b\bar{b}$ states. However, the Hamiltonian H of Eq. (**??**) depends on the gluon mass parameter m_g, which requires consideration of the limit $m_g \to 0$.

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Appendix I: COMMENTS to include on technical issues

Estimates of error: symmetries of the spectrum, such as rotational symmetry. Dependence on $s > 0$. Convergence of eigenvalues like in papers with Wilson and Mlynik. Fate in QCD allows one to check “the magnitude of errors” by comparison with data.

RGPEP does more for us than the Melosh transformation did, *cf.* [67, 68], because it provides the relation between the current and constituent quarks no matter in what frame of reference one works.

We have opened access to small-x logarithmic effects that point out how to look for confinement on the Hilbert space of QCD. How is this related to logarithmic running of the coupling constant?

Hamiltonian approach and relativity: The role of the fixed point.

Binding in quantum mechanics requires mass defect and I secure the mass defect by positive mss squared counterterms that tend to infinity when $m_g \to 0$.

Relativity and counterterms, why would the results be relativistic? Only for relativistic value of the coupling constant?

The issue has a long history, starting with the pre-QCD treatment light quarks [67, 68].

The transition amplitudes, obtained using perturbation theory, involve the quadratic, linear (for sharp cutoffs), and logarithmic dependence on Λ.

Jump to Sec. ??, Eq. ??). Also: Explain the relationship to AdS/QCD inspired phenomenology via $1/y$ in the integral over y that results from writing $l^\perp = \sqrt{|y|}\, u^\perp$, or using the transverse momentum transfer variable $u^\perp = l^\perp \sqrt{|y|}$.

The effective theory cannot be derived in perturbative expansion by any procedure that does not neutralizes small denominators because small-x and large $k^\perp$ divergences are sensitive to the small denominators.

$$H_s^{(2)} = f_s H_r^{(2)} + C^{(2)}_{r\,\mathrm{m}} + (f_s - f_s f_s)\Delta\, H_r^{(1)} H_r^{(1)} \tag{I1}$$

$$= f_s H_r^{(2)} + C^{(2)}_{r\,\mathrm{m}} + (f_s - f_s f_s)\Delta\, H_r^{(1)} H_r^{(1)} - ff) H_r^{(2)} \tag{I2}$$

$$= ff H_r^{(2)} + C^{(2)}_{r\,\mathrm{m}} + (f - ff)\left[\Delta\, H^{(1)} H^{(1)} + H_r^{(2)}\right] , \tag{I3}$$

$$H_r^{(2)} = H^{(2)}_{r\,\mathrm{se}} + C^{(2)}_{r\,\mathrm{ex}} + C^{(2)}_{r\,\mathrm{se}} . \tag{I4}$$

$$
\begin{aligned}
H_s^{(2)} &= f_s H_r^{(2)} + C_{\rm m}^{(2)} + (f_s - f_s f_s)\Delta\, H_r^{(1)} H_r^{(1)} && \text{(I5)}\\
&= f_s H_r^{(2)} + C_{\rm m}^{(2)} + (f_s - f_s f_s)\left[\Delta\, H_r^{(1)} H_r^{(1)} + H_r^{(2)}\right] - (f - ff) H_r^{(2)} && \text{(I6)}\\
&= ff H_r^{(2)} + C_{\rm m}^{(2)} + (f - ff)\,\left[\Delta\, H^{(1)} H^{(1)} + H_r^{(2)}\right]\ , && \text{(I7)}\\
H_r^{(2)} &= H_{\rm se}^{(2)} + C_{\rm ex}^{(2)} + C_{\rm se}^{(2)}\ . && \text{(I8)}
\end{aligned}
$$

For $C_{\rm ex}^{(2)} = -C_{\rm se}^{(2)}$,

$$
\begin{aligned}
H^{(2)} &= f H_{\rm se}^{(2)} + C_{\rm m}^{(2)} + (f - ff)\Delta\, H^{(1)} H^{(1)} && \text{(I9)}\\
&= f H_{\rm se}^{(2)} + C_{\rm m}^{(2)} + (f - ff)\left[\Delta\, H^{(1)} H^{(1)} + H_{\rm se}^{(2)}\right] - (f - ff) H_{\rm se}^{(2)} && \text{(I10)}\\
&= ff H_{\rm se}^{(2)} + C_{\rm m}^{(2)} + (f - ff)\,\left[\Delta\, H^{(1)} H^{(1)} + H_{\rm se}^{(2)}\right]\ . && \text{(I11)}
\end{aligned}
$$

Once the severe small-x singularities $\sim 1/x^2$ are under control one can begin studies of the logarithmic effects due to terms $\sim 1/x$. An example of such effects is provided in the case of heavy quarkonia where the logarithmic divergences are capable to describe confinement in the following way: in the limit $m_g \to 0$ the quark self-interaction terms diverge but the exchange cancels the divergence in colorless states.

(More readable exposition, explanation of boost to IMF = larger P^+ means less evolution from bare theory, which explains Bjorken scaling and its violation.) The infrared singularities in the limit of vanishing gluon mass are handled using the general features of the RGPEP. Namely, all small denominators are absent in perturbative computation of renormalized Hamiltonians. The mechanism is somehwat similar to the one found in models in 1+1 dimensional QCD of a large number of colors, except that the transverse momenta are included and taken care of by the RGPEP factors. The distinct logarithmic small-x divergences, specific to the front form of QCD, can thus be identified and shown to cancel out. The cancellation occurs for globally colorless states and hints at a computationally verifiable scenario for describing confinement of color. Unlike in the lattice or perturbative forms of QCD, the scenario applies to the renormalized Hamiltonian operator acting in the space of states of virtual quarks and gluons.

The separation of the small-x divergences from genuine infrared problems with emission or absorption of infrared gluons is facilitated by the RGPEP design which eliminates small energy denominators from perturbative computation of renormalized Hamiltonian operators.

The key to the cancellation of severe small-x divergences, $\sim 1/x^2$, using gluon mass and 3 polarization states, is that the $\eta\eta$ part of the gluon exchange can be written as a sum of the term with $(\rho_1 - \rho_2)^2$ and a term opposite to the seagull. The part with $(\rho 1 - \rho_2)^2$ provides the factor x^2 in the numerator. It also contains a small denominator. But the small denominator is canceled by the RGPEP factor $f - ff$. The seagull and exchange divergences $\sim 1/x^2$ cancel each other exactly. One could introduce counterterms for both. These would be opposite to each other. But they would also include small denominators with $k^{\perp 2} + m_g^2$. Such denominators would make the counterterms infrared divergent when $m_g \to 0$. Using my trick with the 3rd gluon, I avoid the infrared transverse denominator that could cause logarithmic infrared divergence in the limit $m_g \to 0$, because of the restoration of the seagull-exchange cancellation of the $\eta\eta$ terms.

Gluon mass m_g is not used to regulate the infrared divergences, such $(\vec q^{\,2} + m_g^2)^{-2}$ for small gluon momentum $\vec q$. These will be overcome using the RGPEP. But the gluon mass is helpful in handling the small-x singularities that appear at all scales simultaneously.

The degree of evolution depends on P^+. The larger P^+ the less evolved the effective particles.

't Hooft model is no accident: it works because the leading singularities in QCD cancel out and produce the same output.

Separation of the most singular terms from the logarithmic dynamics is possible because of the identities implied by the current conservation, RGPEP analogs of the Slavnov-Taylor identities (Craig Roberts).

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[1] A. Casher and L. Susskind, Phys. Lett. B **44**, 171 (1973).
[2] J. B. Kogut and L. Susskind, Phys. Rept. **8**, 75 (1973).
[3] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **147**, 385 (1979).
[4] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **147**, 448 (1979).
[5] X. Gao, A. D. Hanlon, S. Mukherjee, P. Petreczky, P. Scior, S. Syritsyn, and Y. Zhao, Phys. Rev. Lett. **128**, 142003 (2022), arXiv:2112.02208 [hep-lat].
[6] X. Ji, Nucl. Phys. B **1007**, 116670 (2024), arXiv:2408.03378 [hep-ph].
[7] E. Shuryak and I. Zahed, (2026), arXiv:2601.15085 [hep-ph].
[8] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).
[9] S. Navas et al. (Particle Data Group), Phys. Rev. D **110**, 030001 (2024).
[10] K. G. Wilson, T. S. Walhout, A. Harindranath, W.-M. Zhang, R. J. Perry, and S. D. Glazek, Phys. Rev. D **49**, 6720 (1994), arXiv:hep-th/9401153.
[11] D. E. Soper, Phys. Rev. D **4**, 1620 (1971).
[12] T.-M. Yan, Phys. Rev. D **7**, 1760 (1973).
[13] T.-M. Yan, Phys. Rev. D **7**, 1780 (1973).
[14] G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980).
[15] J. M. Namyslowski, Prog. Part. Nucl. Phys. **14**, 49 (1985).
[16] M. Burkardt, Adv. Nucl. Phys. **23**, 1 (1996), arXiv:hep-ph/9505259.
[17] S. J. Brodsky, H.-C. Pauli, and S. S. Pinsky, Phys. Rept. **301**, 299 (1998), arXiv:hep-ph/9705477.
[18] J. Carbonell, B. Desplanques, V. A. Karmanov, and J. F. Mathiot, Phys. Rept. **300**, 215 (1998), arXiv:nucl-th/9804029.
[19] S. J. Brodsky, G. F. de Teramond, H. G. Dosch, and J. Erlich, Phys. Rept. **584**, 1 (2015), arXiv:1407.8131 [hep-ph].
[20] X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang, and Y. Zhao, Rev. Mod. Phys. **93**, 035005 (2021), arXiv:2004.03543 [hep-ph].
[21] G. P. Lepage, S. J. Brodsky, T. Huang, and P. B. Mackenzie, in Banff Summer Institute in Particles and Fields: QCD, Gauge Theories ar (1982) pp. 83–141.
[22] E. C. G. Stueckelberg, Helv. Phys. Acta **11**, 299 (1938).
[23] S. D. Głazek, Phys. Rev. D **101**, 034005 (2020), arXiv:2001.08261 [hep-th].
[24] T. Kunimasa and T. Goto, Prog. Theor. Phys. **37**, 452 (1967).
[25] H. Fritzsch and M. Gell-Mann, eConf **C720906V2**, 135 (1972), arXiv:hep-ph/0208010.
[26] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).
[27] S. Weinberg, The Quantum theory of fields. Vol. 1, Foundations (Cambridge University Press, 1995).
[28] A. Harindranath and J. P. Vary, Phys. Rev. D **37**, 3010 (1988).
[29] E. A. Bartnik and S. D. Glazek, Phys. Rev. D **39**, 1249 (1989).
[30] S. D. Glazek, Phys. Rev. D **38**, 3277 (1988).
[31] S. D. Glazek, Acta Phys. Polon. B **29**, 1979 (1998), arXiv:hep-th/9712188.
[32] S. D. Glazek, Acta Phys. Polon. B **41**, 1937 (2010), arXiv:1006.4132 [hep-th].
[33] S. D. Glazek, Acta Phys. Polon. B **41**, 2669 (2010).
[34] J. D. Bjorken, Prog. Math. Phys. **4**, 423 (1979).

[35] K. G. Wilson, Phys. Rev. **140**, B445 (1965).
[36] K. G. Wilson, Phys. Rev. D **2**, 1438 (1970).
[37] S. D. Głazek 10.1140/epjs/s11734-026-02127-y (2026).
[38] S. D. Glazek and K. G. Wilson, Phys. Rev. D **48**, 5863 (1993).
[39] S. D. Glazek and K. G. Wilson, Phys. Rev. D **49**, 4214 (1994).
[40] S. D. Glazek and K. G. Wilson, Phys. Rev. D **57**, 3558 (1998), arXiv:hep-th/9707028.
[41] S. D. Glazek, Phys. Rev. D **75**, 025005 (2007), arXiv:hep-th/0611015.
[42] F. Wegner, Annalen Phys. **506**, 77 (1994).
[43] A. Mielke, Eur. Phys. J. B **5**, 605 (1998), arXiv:quant-ph/9803040.
[44] F. Wegner, Journal of Physics A: Mathematical and General **39**, 8221 (2006).
[45] S. Kehrein, The flow equation approach to many-particle systems, Vol. 217 (Springer, 2006).
[46] M. Vogl, P. Laurell, A. D. Barr, and G. A. Fiete, Phys. Rev. X **9**, 021037 (2019).
[47] P. Deift, T. R. Nanda, and C. Tomei, SIAM Journal on Numerical Analysis **20**, 1 (1983).
[48] M. T. Chu, SIAM Review **30**, 375 (1988), https://doi.org/10.1137/1030090.
[49] R. Brockett, Linear Algebra and its Applications **146**, 79 (1991).
[50] S. D. Glazek and J. Mlynik, Phys. Rev. D **67**, 045001 (2003), arXiv:hep-th/0210110.
[51] S. D. Glazek and J. Mlynik, Acta Phys. Polon. B **35**, 723 (2004), arXiv:hep-th/0307207.
[52] S. D. Głazek, Phys. Rev. D **103**, 014021 (2021), arXiv:2101.02789 [hep-th].
[53] S. D. Glazek, Acta Phys. Polon. B **43**, 1843 (2012), arXiv:1204.4760 [hep-th].
[54] K. Serafin, M. Gómez-Rocha, J. More, and S. D. Głazek, Phys. Rev. D **109**, 016017 (2024), arXiv:2310.00365 [hep-ph].
[55] T. Appelquist and H. D. Politzer, Phys. Rev. Lett. **34**, 43 (1975).
[56] T. Appelquist, A. De Rujula, H. D. Politzer, and S. L. Glashow, Phys. Rev. Lett. **34**, 365 (1975).
[57] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T.-M. Yan, Phys. Rev. D **17**, 3090 (1978), [Erratum: Phys.Rev.D 21, 313 (1980)].
[58] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T.-M. Yan, Phys. Rev. D **21**, 203 (1980).
[59] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[60] N. Brambilla et al., Eur. Phys. J. C **71**, 1534 (2011), arXiv:1010.5827 [hep-ph].
[61] S. D. Glazek, Phys. Rev. D **90**, 045020 (2014), arXiv:1406.0127 [hep-th].
[62] S. D. Glazek, Phys. Rev. D **69**, 065002 (2004), arXiv:hep-th/0307064.
[63] I. Tamm, J. Phys. (USSR) **9**, 449 (1945).
[64] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).
[65] R. J. Perry, A. Harindranath, and K. G. Wilson, Phys. Rev. Lett. **65**, 2959 (1990).
[66] R. J. Perry and A. Harindranath, Phys. Rev. D **43**, 4051 (1991).
[67] H. J. Melosh, Phys. Rev. D **9**, 1095 (1974).
[68] M. Gell-Mann, in 3rd International Symposium on the History of Particle Physics: The Rise of the Standard Model (1992) pp. 625–633.