Čech this out!

Výklad XI

2021/22

Čech cohomology & rel'n to de Rham cohomology

M – manifold

①

multiplicative notation!

$\{\mathcal{O}_i\}_{i\in I}$ – open cover

with multiple intersections $\mathcal{O}_{i_0 i_1 \dots i_p} = \mathcal{O}_{i_0} \cap \mathcal{O}_{i_1} \cap \dots \cap \mathcal{O}_{i_p}$

$\mathcal{O}$

We consider local mappings of a fixed type T into an **abelian** group G (e.g., locally smooth, locally constant, ...) $\to$ p-COCHAIN GROUPS:

$$\check{C}^p(\mathcal{O};G) := \{ (f_{i_0 i_1 \dots i_p}) \mid f_{i_0 i_1 \dots i_p} \in \mathrm{Map}_T(\mathcal{O}_{i_0 i_1 \dots i_p}, G),\ \forall \sigma \in \mathfrak{S}_{0,p}: f_{i_{\sigma(0)} i_{\sigma(1)} \dots i_{\sigma(p)}} = \operatorname{sign}\sigma\, f_{i_0 i_1 \dots i_p} \}$$

These form a cochain complex

$$\check{C}^0(\mathcal{O};G) \xrightarrow{\check{\delta}^{(0)}} \check{C}^1(\mathcal{O};G) \xrightarrow{\check{\delta}^{(1)}} \cdots \xrightarrow{\check{\delta}^{(p-1)}} \check{C}^p(\mathcal{O};G) \xrightarrow{\check{\delta}^{(p+1)}} \cdots \quad (2)$$

with ČECH COBOUNDARY operators

$$\check{\delta}^{(p)} : \check{C}^p(\mathcal{O};G) \longrightarrow \check{C}^{p+1}(\mathcal{O};G)$$

$$: (f_{i_0 i_1 \dots i_p}) \longmapsto \prod_{k=0}^{p+1} \rho_{i_k}\left(\mathrm{Inv}^k \circ f_{i_0 i_1 \dots \hat{i}_k \dots i_{p+1}}\right)$$

where $\rho_{i_k} f_{i_0 i_1 \dots \hat{i}_k \dots i_{p+1}} := f_{i_0 i_1 \dots \hat{i}_k \dots i_{p+1}}\big|_{\mathcal{O}_{i_k} \cap \mathcal{O}_{i_0 i_1 \dots \hat{i}_k \dots i_{p+1}}}$

We define

p-COCYCLES : $\check{Z}^p(\mathcal{O};G) = \operatorname{Ker} \check{\delta}^{(p)}$ (3)

p-COBOUNDARIES : $\check{B}^p(\mathcal{O};G) \equiv \operatorname{Im} \check{\delta}^{(p-1)}$

& COHOMOLOGY GROUPS : $\check{H}^p(\mathcal{O};G) = \operatorname{Ker} \check{\delta}^{(p)} \big/ \operatorname{Im} \check{\delta}^{(p-1)}$

The Čech cohomology of M with values in G is the limit of $\check{H}^\bullet(\mathcal{O};G)$ over refinements of $\mathcal{O}$. (∃)

Take $\mathcal{O}$ to be GOOD → NEED NO REFINEMENT (Leray)! & locally finite (i.e., each point has a neighbourhood that intersects only a finite number of the $\mathcal{O}_i$'s). ④

Let $\{\chi_i\}_{i\in I}$ be the associated partition of unity, i.e.,

(PU0) $\forall i\in I:\ \chi_i\in C^\infty(M;\mathbb{R})$

(PU1) $\forall i\in I:\ 0\leq\chi_i\leq 1$

(PU2) $\forall i\in I:\ \operatorname{supp}\chi_i\subset\mathcal{O}_i$

(PU3) $\sum_{i\in I}\chi_i=1$

(These always exist on C^2 manifolds.)

For $\begin{cases} T = \text{"locally constant"}, \\ G = \mathbb{R} \end{cases}$

(5)

we have $\check{H}^p(\mathcal{O};\mathbb{R}) \simeq H^p_{dR}(M;\mathbb{R})$, induced

from $\omega_\cdot : \check{C}^p(\mathcal{O};\mathbb{R}) \to \Omega^p(M) : f \mapsto \omega_f$,

where

$$\omega_f := -\sum_{i_0 i_1 \dots i_p} f_{i_0 i_1 \dots i_p} \, \chi_{i_0} \, d\chi_{i_1} \wedge d\chi_{i_2} \wedge \dots \wedge d\chi_{i_p}$$

NB: Outside $\mathcal{O}_{i_0 i_1 \dots i_p}$, $\chi_{i_0} d\chi_{i_1} \wedge \dots \wedge d\chi_{i_p}$ vanishes $\Rightarrow$ this is well-defined (we may extend arbitrarily)

We calculate

$$d\omega_f = -\sum_{i_0 i_1 \dots i_p} f_{i_0 i_1 \dots i_p}\, dh_{i_0} \wedge dh_{i_1} \wedge \dots \wedge dh_{i_p} \qquad (6)$$

&

$$\omega_{\check{\delta}^{(p)} f} = -\sum_{i_0 i_1 \dots i_{p+1}} \sum_{k=0}^{p+1} \rho_{i_k} \ln v^k \circ f_{i_0 i_1 \dots \hat{i}_k \dots i_{p+1}}\, h_{i_0}\, dh_{i_1} \wedge dh_{i_2} \wedge \dots \wedge dh_{i_{p+1}}$$

$$= \left\{ \sum_{i_k} \rho_{i_k} h_{i_k} = 1 \right\} = -\sum_{i_0 i_1 \dots i_p} f_{i_1 i_2 \dots i_{p+1}}\, dh_{i_1} \wedge dh_{i_2} \wedge \dots \wedge dh_{i_{p+1}}$$

$$\equiv d\omega_f$$

Conclusions :

$$d \circ \omega = \omega_\delta,$$

or : ω is a cochain map (7)

$\Downarrow$

It induces a homomorphism of cohomology groups.

Surjectivity : We only demonstrate it for $p = \underline{2}$.

Let $\omega \in Z^2_{dR}(M)$.

$$\Downarrow$$

$$\omega\big|_{O_i} = d\theta_i \quad , \; i \in I$$ (8)

$$\theta_i \in \Omega^1(O_i)$$

$$(\theta_j - \theta_i)\big|_{O_{ij}} = d\gamma_{ij} \quad , \; i,j \in I \; , \; O_{ij} \neq \emptyset$$

$$\gamma_{ij} \in C^\infty(O_{ij}, \mathbb{R})$$

local constants

$$(\gamma_{jk} - \gamma_{ik} + \gamma_{ij})\big|_{O_{ijk}} = c_{ijk} \in \mathbb{R} \quad , \; i,j,k \in I \; , \; O_{ijk} \neq \emptyset$$

$$\simeq c_{[ijk]}$$

$$\check{\delta}^{(2)} c_\omega = 0 \implies c_\omega \in \check{Z}^2(\Theta, \mathbb{R})$$

We write, as postulated,

9

$$\underset{(p)}{\omega}_{c_\omega} = -\sum_{ijk} c_{ijk} X_i \, dX_j \wedge dX_k$$

But $\quad c_{ijk} = c_{jkl} - c_{kli} + c_{lij} \quad \forall l$

over Θ_{ijkl}

$$\Downarrow$$

$$\underset{(p)}{\omega}_{c_\omega}\Big|_{\Theta_l} = -\sum_{ijk} (c_{jkl} - c_{kli} + c_{lij}) X_i \, dX_j \wedge dX_k \Big|_{\Theta_l}$$

$$= -\sum_{ijk} c_{jkl} X_i \, dX_j \wedge dX_k \Big|_{\Theta_l}$$ drop out as $\sum_i X_i = 1$
$\equiv$ const

$$= -\sum_{jk} c_{ljk}\, dX_j \wedge dX_k \restriction O_l$$

$$= d\Big(-\sum_{jk} c_{ljk}\, X_j\, dX_k\Big) \restriction O_l \qquad (10)$$

The same multiplied by $X_j\, dX_k$ yields

$$\sum_{jk} c_{ijk}\, X_j\, dX_k\Big|_{O_{il}} = \sum_{jk}\big(c_{ijk} - c_{ljk} + \cancel{c_{i\cdot}}\big) X_j\, dX_k\Big|_{O_{il}} = d\Big(\sum_k c_{ilk}\, X_k\Big) + \sum_{jk} c_{ljk}\, X_j\, dX_k\Big|_{O_{il}}$$

But $\sum_k c_{ilk}\, X_k = \gamma_{il} + \sum_k \gamma_{lk}\, X_k - \sum_k \gamma_{ik}\, X_k$ on il

Therefore, it suffices to put

$$\eta_i := \theta_i + \sum_{jk} c_{ijk} x_j \, dx_k - d\Big(\sum_k \gamma_{ik} x_k\Big)$$

on θ_i

(11)

$\gamma_{jk} - \gamma_{ik} = -\gamma_{ij}$ $+ c_{ijk}$ on θ_{ijk}

to find θ

$$(\eta_j - \eta_i)\big|_{\theta_{ij}} = (\theta_j - \theta_i)\big|_{\theta_{ij}} + \sum_{k\ell} (c_{jk\ell} - c_{ik\ell}) x_k \, dx_\ell \big|_{\theta_{ij}}$$

$$- d\Big(\sum_k \gamma_{jk} x_k\Big)\Big|_{\theta_{ij}}$$

$$+ d\Big(\sum_k \gamma_{ik} x_k\Big)\Big|_{\theta_{ij}}$$

$$\overset{\shortparallel}{d\mu_{ij}}$$

$$\sum_k (\gamma_{jk} - \gamma_{ik}) x_k = \sum_k (\gamma_{jk} - \gamma_{ik})\big|_{\theta_{ijk}} x_k = -\sum_k \gamma_{ij} x_k = -\gamma_{ij}$$

$$= d\Big(\sum_k c_{ijk} x_k\Big)\Big|_{O_i} + \sum_{kl} (c_{jkl} - c_{ikl}) x_k \, dx_l \Big|_{O_{ij}} \quad (12)$$

$$= d\Big(\sum_k c_{ijk} x_k\Big) + \sum_{kl} (c_{jkl} - c_{ikl})\big|_{O_{ijkl}} x_k \, dx_l \Big|_{O_{ij}}$$

$$= d\Big(\sum_k c_{ijk} x_k\Big) + d\Big(\sum_l c_{jil} x_l\Big) = 0 \quad \checkmark$$

$$\Rightarrow \exists \eta \in \Omega^1(M) : \eta|_{O_i} = \eta_i .$$

Finally, we calculate:

$$d\eta_i = \omega\restriction_{\theta_i} + \sum_{jk} c_{ijk}\, dX_j \wedge dX_k$$

13

$$\equiv \omega\restriction_{\theta_i} - \omega_{c_\omega}\restriction_{\theta_i}, \text{ i.e.}$$

$$d\eta = \omega - \omega_{c_\omega}, \text{ or}$$

$$[\omega]_{dR} = [\omega_{c_\omega}]_{dR}.$$

Thus, $[c] \mapsto [\omega_{c_\beta}]_{dR}$ has a right inverse $[\omega] \mapsto [c_\omega]$

We still need to show that 14

$\omega + d\theta$ yields $[c_{\omega+d\theta}] = [c_\omega]$

$d\theta$ has $\theta_i = \theta \restriction \sigma_i$

$(\theta_j - \theta_i) \restriction \sigma_{ij} = 0 \Rightarrow$ we may take

$\gamma_{ij} = 0$

$\Rightarrow c_{ijk} = 0$ are ok

$\Downarrow$

$[c_{\omega+d\theta}] \equiv [c_\omega]$.

But we may also shift: 15

$\theta_i \mapsto \theta_i + d\psi_i$

$(\theta_j - \theta_i)\restriction_{O_{ij}} = d\gamma_{ij} + d(\psi_j - \psi_i)\restriction_{O_{ij}}$

we may next add $\gamma_{ij} \mapsto \gamma_{ij} + (\psi_j - \psi_i)\restriction_{O_{ij}}$
$+ \nu_{ij} \in \mathbb{R}$

$$\check{\delta}\gamma_{ijk} \mapsto \check{\delta}\gamma_{ijk} + \check{\delta}\nu_{ijk}$$

$$\overset{\shortparallel}{c_{ijk}} \qquad c_{ijk} + \check{\delta}\nu_{ijk}$$

$\Rightarrow$ it is only $[C_\omega]$ that is assigned to $[\omega]$

Conversely, let $c \in \check{Z}^2(\Theta; \mathbb{R})$

16

& consider

$$\underset{(p)}{\omega}_c = -\sum_{ijk} c_{ijk}\, X_i\, dX_j \wedge dX_k$$

Just using $\delta c = 0$, we obtain

$$\underset{(p)}{\omega}_c \restriction_{\Theta_i} = d\theta_i \quad , \quad \theta_i \underset{(e.g.)}{=} -\sum_{jk} c_{ijk}\, X_j\, dX_k$$

& $(\theta_j - \theta_i)\restriction_{\Theta_{ij}} = -\sum_{kl} (c_{jkl} - c_{ikl})\, X_k\, dX_l \Big/_{\Theta_{ij}}$

$$= -\sum_{kl} (c_{jkl} - c_{ikl})\restriction_{\Theta_{ijkl}} X_k\, dX_l$$

$$= -d\left(\sum_k c_{jik} X_k\right) = d\left(\sum_k c_{ijk} X_k\right)$$

$$\Longrightarrow \quad \gamma_{ij} = \sum_k c_{ijk} X_k \quad \text{(e.g.)}$$

So then

$$(\gamma_{jk} - \gamma_{ik} + \gamma_{ij})\big|_{\Theta_{ijk}} = \sum_l (c_{jkl} - c_{ikl} + c_{ijl}) X_l \big|_{\Theta_{ijk}}$$

$$= \sum_l (c_{jkl} - c_{ikl} + c_{ijl} - c_{ijk}) X_l \big|_{\Theta_{ijk}}$$

$$+ \sum_l c_{ijk} X_l \big|_{\Theta_{ijk}} = c_{ijk} \Longrightarrow$$

R

Once again,

$$[C_{\omega_{(\varphi)}C}] = [C]$$

?

other choices could correct $C_{\omega C}$ by sth. exact!

Thus, altogether, we obtain

$$[\omega_{(\varphi)}] : \check{H}^2(\mathcal{O};\mathbb{R}) \xrightarrow{\cong} H^2_{dR}(M).$$

Leray!

Next, we want to understand the rel$^{\underline{n}}$ between $\check{H}^p(\mathcal{O};\mathbb{Z})$ $(\equiv \check{H}^p(M;\mathbb{Z}))$ 19
Leray

& $H^p_{dR}(M;\mathbb{Z})$, i.e., the homology of the (co)chain complex

INTEGRATION!

$$\left(\mathrm{Hom}_{Grp}(C_p(M), \mathbb{Z}) = C^p_{dR}(M;\mathbb{Z}) \ , \ d_{dR} \right),$$

for the chain complex

$$(C_p(M), \partial_p)$$ (p-chains in M, ∂_p - boundary operator).

Why? Because short exact sequences of abelian groups

$$0 \to G \to H \to K \to 0$$

canonically induce long exact sequence in cohomology

BOCKSTEIN HOMO (CONNECTING)

$$\cdots \to H^p(C_\cdot;H) \to H^p(C_\cdot;K) \xrightarrow{\beta^{(p)}} H^{p+1}(C_\cdot;G) \to H^{p+1}(C_\cdot;H) \to \cdots$$

(at least for (torsion-)free abelian groups $C_\cdot$)

So we have, e.g., $0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \to 0$

Back to Čech vs de Rham ...

Once again, we consider $p = 2$ for illustration (21)

Take an arbitrary $\omega \in Z^2_{dR}(M)$ & integrate it over any $\sigma_2 \in Z_2(M)$ (a closed 2-submanifold).

To this end, we tessellate σ_2 and that each 2-cell p sits entirely in some $\mathcal{O}_{i_p} \in \mathcal{O}$, i.e., we have a map

$$\Delta \longrightarrow I \; : \; \tau \longmapsto i_\tau \; ,$$

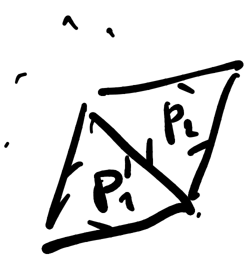
$\underset{\Delta_2}{\text{plaquettes}}$, $\underset{\Delta_1}{\text{edges}}$, $\underset{\Delta_0}{\text{vertices}}$

so that we obtain

$$\int_{\sigma_2}\omega = \sum_{p\in\Delta_2}\int_p \omega\Big|_{\theta_{i_p}} = \sum_{p\in\Delta_2}\int_p d\theta_{i_p} \qquad (22)$$

$$= \sum_{p\in\Delta_2}\int_{\partial p}\theta_{i_p} = \sum_{p\in\Delta_2}\sum_{e\in\partial p}\int_e \theta_{i_p}$$

induced orientation!

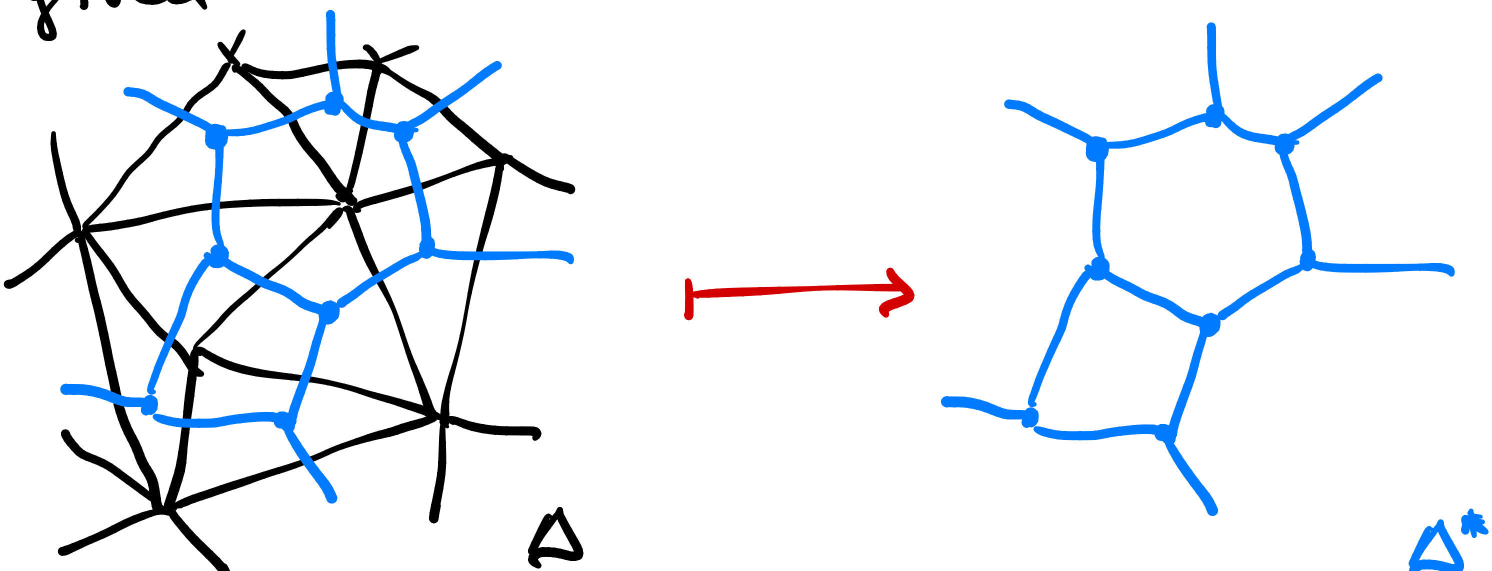


$$= \sum_{e\in\Delta_1}\int_e \left(\theta_{i_{p_+(e)}} - \theta_{i_{p_-(e)}}\right)\Big|_{\theta_{i_{p_+(e)}i_{p_-(e)}}}$$

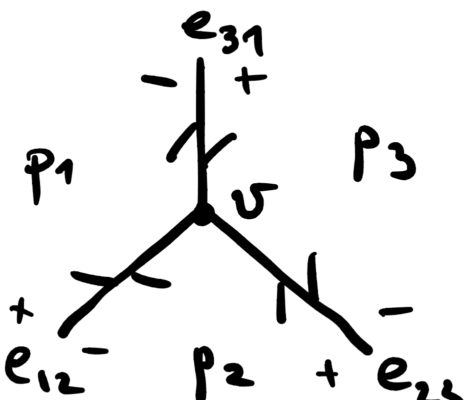
orientation induced from e ~~as~~ $\in \partial p_+$

$$= \sum_{e\in\Delta_1}\int_e d\gamma_{i_{p_-(e)}i_{p_+(e)}} = \sum_{e\in\Delta_1}\sum_{v\in\partial e}\gamma_{i_{p_-(e)}i_{p_+(e)}}(v)\Big|_{\theta_{i_{p_+(e)}i_{p_-(e)}i_v}}$$

In order to facilitate subsequent analysis, we assume Δ trivalent, which can always be achieved by dualising any given 23



It's clear that we end up with a sum over Δ_0. Let us derive the precise contribution of a given vertex $v \in \Delta_0$. (24)



$$\left(\gamma_{i_{p_2} i_{p_1}} + \gamma_{i_{p_3} i_{p_2}} + \gamma_{i_{p_1} i_{p_3}}\right)(v)$$

yields

$$= \left(-\gamma_{i_{p_2} i_{p_3}} + \gamma_{i_{p_1} i_{p_3}} - \gamma_{i_{p_1} i_{p_2}}\right)(v)$$

$$= -c_{i_{p_1} i_{p_2} i_{p_3}} \quad !$$

$$-123 = -(23 - 13 + 12)$$
$$= -23 + 13 - 12$$

Thus, altogether, $\int_{\sigma_2} \omega = - \sum_{v \in \Delta_0} c_{i_{p_1} i_{p_2} i_{p_3}}$,

& so $c \in \check{Z}^2(\mathcal{O};\mathbb{Z}) \Longrightarrow \omega \in Z^2_{dR}(M;\mathbb{Z})$ (25)

In fact, by moving considering all 2-cycles within M, we readily recover the converse statement

$$\omega \in Z^2_{dR}(M;\mathbb{Z}) \Longrightarrow c \in \check{Z}^2(\mathcal{O};\mathbb{Z})$$

Therefore, $H^2_{dR}(M;\mathbb{Z}) \simeq \check{H}^2(M;\mathbb{Z})$.

The way we have constructed our reasoning, it is clear that it generalises

to all $p \in \mathbb{N}$, good! (26)

$$H^p_{dR}(M;G) \simeq \check{H}^p(\mathcal{O};G) \simeq \check{H}^p(M;G),$$

$G \in \{\mathbb{R}, \mathbb{Z}\}$