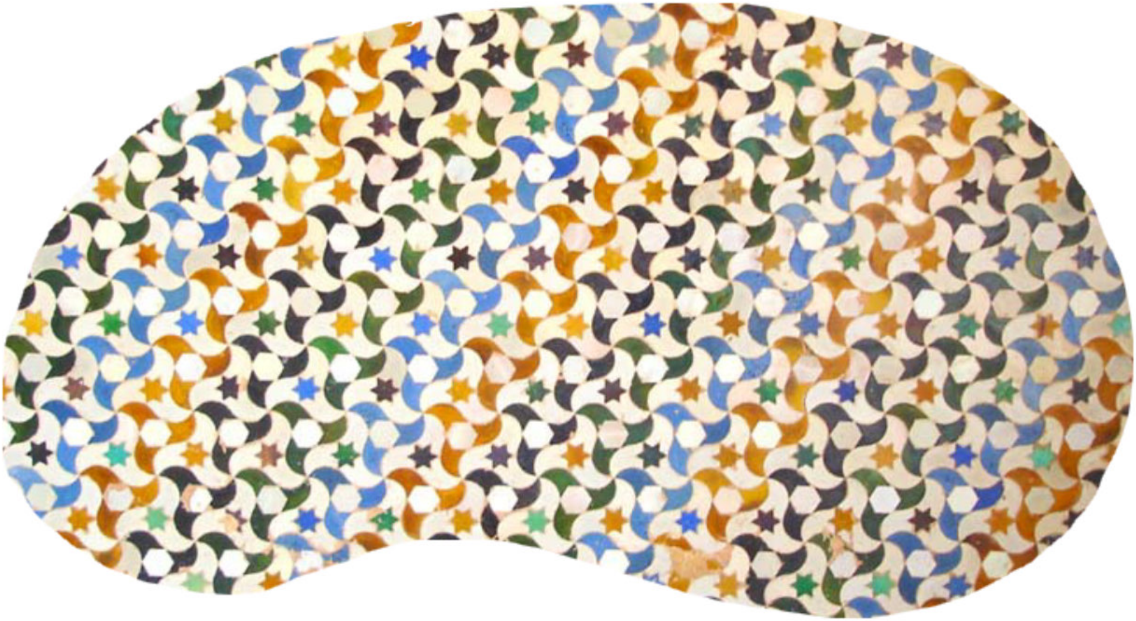
Duality, Descent & Defects II

Lecture I

"SYMMETRY MODELS & THE GODEMENT-GIVEN CRITERION"

2024/25



A FINITE REGION of THE MOORISH TILING in PATIO de ARRAYANES in ALHAMBRA
- A FINE INSTANTIATION of SYMMETRY OUTSIDE THE GROUP PARADIGM

CLASSIC PARADIGM of FUNDAMENTAL INTERACTIONS:

ARENA: (X, g) METRIC SPACETIME

$$\begin{cases} dF = J_2 \\ d*F = J_1 \end{cases}$$ SOURCES

CHARGE CURRENTS

$J.$

MESSENGER FIELDS

$A \rightsquigarrow F(A)$

GEODESIC DYNAMICS $+ \langle F(A), J \rangle$

LORENTZ-TYPE FORCES

REDUCTIONS

$(\phi, T\phi) \searrow ([\phi]_{\mathcal{S}}, \nabla_A [\phi]_{\mathcal{S}})$

$\phi \in \Gamma(\mathcal{F} \twoheadrightarrow X)$

$$\begin{array}{c} J^\alpha \\ \parallel\!\mid \end{array}$$
$$d *_g \left(\frac{\delta \mathcal{L}}{\delta T\phi} \left(T_e \alpha (\exp^{-1}(s))(\phi) \right) \right) = 0$$

CONSERVATION LAWS

$\{J, -\}_{P.B.} \; (\approx 0)$

AUTOMORPHISMS of STATE SPACE

EQUIVALENCES between DESCRIPTIONS of STATES

$A \circlearrowleft \mathcal{S}$

AUTOMORPHISMS of FIELDS

LOCAL SYMMETRIES of DYNAMICS $\mathcal{L}$

$\alpha : \mathcal{S} \longrightarrow \mathrm{Aut}(\mathcal{F} \twoheadrightarrow X)$

SIMPLE STANDARD MODEL:

COMPACT WORLD-VOLUME

$$J_\cdot = g \cdot \mathrm{Vol}(x(\Sigma)) \quad \text{for} \quad x : \Sigma \xrightarrow[C^\infty]{} X$$

$\dim \Sigma = p+1$

$$S[x] = \frac{\mu}{2} \int_\Sigma \mathrm{Vol}(\Sigma, x^* g) + g \int_{x(\Sigma)} A$$

$A \in \Omega^{p+1}(X)$, e.g., $p=0$ (of POINT-LIKE CHARGES) CURRENT $\quad A \in \Omega^1(X)$

$$A \sim A + i\gamma^{-1} d\gamma \ , \quad \gamma \in \mathcal{G}$$

U(1)-GAUGE TRANSFORMATION

$$\mathcal{G} = C^\infty(X, U(1))$$

U(1)-GAUGE GROUP

VARIATIONS:

I GEOMETRISATION

NON-TRIVIAL TOPOLOGY of X $\overset{(QM)}{\Longrightarrow}$ GEOMETRISATION of A

$\vee$

HIGHER-DIMENSIONALITY of Σ
& NON-TRIVIAL TOPOLOGY

(BUNDLES, p-GERBES
w/ CONNECTIVE STRUCTURE)

— x —

II GEOMETRISATION

CURRENTS
INTERPRETED AS LOCALISED
EXCITATIONS of CHARGED FIELDS $\rightsquigarrow$ FIBRATION of CHARGE DOFs over X

(CHARGE-FIELD BUNDLES
ASSOCIATED to GAUGE BUNDLES)

— x —

III GEOMETRISATION

MODELLING of $\mathcal{F}/\!\!/\mathcal{S}$ $\rightsquigarrow$ INTEGRATION over ISOCLASSES
of GAUGE BUNDLES

— x —

IV CATEGORIFICATION
of SYMMETRY MODEL $\mathcal{S}$ $\rightsquigarrow$ GROUPOIDS, BUNDLES,
SMOOTH CATEGORIES

WE START from...

IV.1 RECAPITULATION of STANDARD MODEL of SYMMETRY

DEF. 1. A LIE GROUP is a GROUP OBJECT in Man.

E.g.,

(i) $(\{\cdot\},\cdot,\cdot,\cdot)$

(ii) $\mathbb{S}^1 \simeq U(1) \subset \mathbb{C}^\times \; (\subset \mathbb{C})$

(iii) $\mathbb{S}^3 \simeq SU(2) \subset GL(2;\mathbb{C}) \; (\subset \mathbb{C}(2))$

(iv) $ISO(3,1) \equiv SO(3,1) \ltimes_{\mathrm{Vect}} \mathbb{R}^4$

THM. 2. [CARTAN'S CLOSED-SUBGROUP THEOREM] EVERY TOPOLOGICALLY CLOSED SUBGROUP of A LIE GROUP IS A LIE SUBGROUP (i.e., A SUBMANIFOLD).

PROP. 3. $\forall\, G$ – LIE GROUP : TG is LIE, AND – for $\ell_\cdot : G \to \mathrm{Diff}(G) : g \mapsto m(g,\cdot)$ –

$$T_e\ell_\cdot : G \ltimes_{T_e\mathrm{Ad}_\cdot} T_eG \simeq TG.$$

LEFT REGULAR ACTION

THE INDUCED MAP $L_{\cdot} : T_eG \longrightarrow \Gamma(TG) : X \longmapsto T_e l_{\cdot}(X)$ GIVES US

DEF. 4. LEFT-INVARIANT VECTOR FIELD ON LIE GROUP G:

$$L \in \Gamma(TG) \quad : \quad \forall g \in G : l_g^* L = L .$$

THESE COMPOSE THE SPACE OF LEFT-INVARIANT VECTOR FIELDS

$$\mathfrak{X}_L(G) = \{ L \in \Gamma(TG) \mid L \text{ is } LI \}$$

DEF. 5. A LIE ALGEBRA (OVER $\mathbb{R}$) IS A PAIR $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$

COMPOSED OF

(*) $\mathfrak{g} \in \mathrm{Vect}_{\mathbb{R}}^{(<\infty)}$ s.t. (LA1) $[\cdot,\cdot]_{\mathfrak{g}} = -[\cdot,\cdot] \circ \tau$;

(**) $[\cdot,\cdot]_{\mathfrak{g}} \in L(\mathfrak{g},\mathfrak{g};\mathfrak{g})$ (LA2) $\underbrace{\mathrm{Jac}_{\mathfrak{g}}}_{\text{JACOBIATOR}} \equiv 0$.

E.g., (i) $(\mathbb{R}, [\cdot,\cdot]_{\mathbb{R}}=0)$

(ii) $(\mathbb{R}^{\times 3}, [\cdot,\cdot]_{\wedge})$, where $[x_i, x_j] = \varepsilon_{ijk}\, x_k$, $i\in\{1,2,3\}$

PROP. 6. $\left(\mathfrak{X}_L(G), [\cdot,\cdot]_{\Gamma(TG)}\restriction_{\mathfrak{X}_L(G)\times\mathfrak{X}_L(G)}\right)$ is a LIE ALGEBRA, with $\mathfrak{X}_L(G) \xrightarrow[\mathrm{ev}_e]{\cong} T_eG$. $L \longmapsto L(e)$

$\Downarrow$

PROP. 7. / DEF. $\mathfrak{g} \equiv T_eG$ CARRIES a CANONICAL STRUCTURE of LIE ALGEBRA, with LIE BRACKET : $[\cdot,\cdot]_{\mathfrak{g}} : T_eG \times T_eG \longrightarrow T_eG$

$: (X, Y) \longmapsto \mathrm{ev}_e\left([L_X, L_Y]_{\Gamma(TG)}\right)$

WE CALL $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ THE TANGENT LIE ALGEBRA of G.

PROP. 8. THE ASSIGNMENT $G \to \mathfrak{g}$ EXTENDS TO A COVARIANT FUNCTOR $\mathbf{Lie} : \mathbf{LieGrp} \to \mathbf{LieAlg}_{\mathbb{R}}$, with

$$\mathbf{Lie} : \mathrm{Hom}_{\mathbf{LieGrp}}(G_1, G_2) \to \mathrm{Hom}_{\mathbf{LieAlg}_{\mathbb{R}}}(\mathrm{Lie}(G_1), \mathrm{Lie}(G_2))$$

$$: \chi \longmapsto T_e\chi .$$

PROP. 9. LI VECTOR FIELDS on LIE GROUP G ARE COMPLETE, with

$$\Phi_L : \mathbb{R} \times G \to G : (t,g) \longmapsto p_{\Phi_L(t,e)} ,$$

where

$$p : G \to \mathrm{Diff}(G) : g \longmapsto m(\cdot, g) .$$

RIGHT REGULAR ACTION

PROP. 10. / DEF. THE MAP $\breve{\lambda}_\cdot : \mathfrak{g} \times \mathbb{R} \to G : (X,t) \longmapsto \Phi_{L_X}(t,e)$ IS SMOOTH. IT YIELDS THE EXPONENTIAL MAP

$$\exp^G : \mathfrak{g} \to G : X \longmapsto \breve{\lambda}_X(1), \text{ with } T_{0_\mathfrak{g}} \exp^G = \mathrm{id}_\mathfrak{g} .$$

DEF. 11. LET $M \in \mathbf{Man}$ & $G \in \mathbf{LieGrp}$.

THE LEFT LOGARYTHMIC DERIVATIVE on $C^\infty(M,G)$ is a MAPPING

$$\delta_L \log : C^\infty(M,G) \longrightarrow \Omega^1(M) \otimes_{\mathbb{R}} \mathfrak{g} \quad \text{GIVEN by}$$

$$\delta_L \log(f) : TM \longrightarrow \mathfrak{g} : (v,x) \longmapsto T_{f(x)} \ell_{f(x)^{-1}} \circ T_x f(v).$$

PROP. 12. / DEF. THERE EXISTS a CANONICAL LEFT-INVARIANT $\mathfrak{g}$-VALUED 1-FORM on LIE GROUP G, GIVEN by $\theta_L := \delta_L \log(\mathrm{id}_G)$.

IT IS CALLED THE LI MAURER-CARTAN FORM on G.

PROP. 13. / DEF. THE MC FORM on LIE GROUP G SATISFIES THE MAURER-CARTAN EQUATIONS $d\theta_L = -\theta_L \wedge \theta_L$, & SO IT ENCODES THE STRUCTURE EQUATIONS of $\mathfrak{g}$:

$[t_A, t_B]_{\mathfrak{g}} = f_{ABC}\, t_C$ in ANY BASIS $\{t_A\}_{A \in \overline{1,\dim G}}$ of $\mathfrak{g}$.

DEF. 14. LET $G \in \mathrm{LieGrp}$. A (LEFT) G-MANIFOLD is A PAIR (M, λ) COMPOSED of

(*) $M \in \mathrm{Man}$

(**) $\lambda \in \mathrm{Hom}_{\mathrm{Grp}}(G, \mathrm{Diff}(M))$, CALLED THE ACTION,

s.t. THE INDUCED MAPPING $\underline{\lambda}: G \times M \longrightarrow M : (g, m) \longmapsto \lambda_g(m)$ IS SMOOTH.

GIVEN TWO G-MANIFOLDS: $(M_A, \lambda^A), A \in \{1, 2\}$, A G-EQUIVARIANT MAP IS ANY $F \in C^\infty(M_1, M_2)$ s.t. $\forall g \in G: \ F \circ \lambda^1_g = \lambda^2_g \circ F$.

WE DENOTE THE SET of SUCH MAPS as $\mathrm{Hom}_G(M_1, M_2)$.

PROP. 15. IF λ^1 IS TRANSITIVE, THEN F HAS CONSTANT RANK.

DEF. 16. AN ACTION $\lambda \in \mathrm{Hom}_{\mathrm{Grp}}(G, \mathrm{Diff}(M))$ of LIE GROUP G on MANIFOLD M IS CALLED PROPER IF $(\underline{\lambda}, \mathrm{pr}_2)$ IS PROPER (TOP.).

E.G.,

PROP. 17. AN ARBITRARY ACTION of A COMPACT LIE GROUP IS PROPER.

PROP. 18. THE RESTRICTION of THE REGULAR ACTION (ℓ OR p) OF A LIE GROUP $\mathbf{G}$ On ITSELF to AN ARBITRARY CLOSED (⇓ LIE) SUBGROUP IS PROPER.

PROP. 19. / DEF. AN ACTION $\lambda \in \mathrm{Hom}_{\mathrm{grp}}(G, \mathrm{Diff}(M))$ of LIE GROUP G on MANIFOLD $\mathbf{M}$ CANONICALLY INDUCES an INVOLUTIVE DISTRIBUTION $\mathcal{F}(\lambda) \subset TM$ SPANNED by THE FUNDAMENTAL VECTOR FIELDS

$$\mathcal{K} : ((\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}), T_e\mathrm{Ad}_\cdot) \longrightarrow ((\Gamma(TM), [\cdot,\cdot]_{\Gamma(TM)}), T\lambda_\cdot)$$

given by $\mathcal{K}_X(m) = -T_{(e,m)}\lambda\,(X, 0_{T_mM}) \equiv \frac{d}{dt}\Big|_{t=0} \lambda_{\exp^G(-t\triangleright X)}(m)$.

Prop. 20. THE INTEGRAL LEAVES of $\mathcal{F}(\lambda)$ ARE ORBITS of λ.

Thm. 21$^{(0)}$. FOR ANY FREE & PROPER ACTION $\lambda \in \mathrm{Hom}_{\mathrm{grp}}(G, \mathrm{Diff}(M))$ of LIE GROUP G on MANIFOLD M, THERE EXISTS a UNIQUE SMOOTH STRUCTURE on THE SPACE $M/\!/G$ OF ORBITS of λ s.t. THE CANONICAL PROJECTION $\pi : M \longrightarrow M/\!/G : m \longmapsto [m]_{\sim_\lambda}$ IS A SURJECTIVE SUBMERSION.

THE ABOVE IS A SPECIALISATION of A VERY GENERAL & POWERFUL RESULT, HISTORICALLY ASCRIBED to GODEMENT (by SERRE):

THM. 21. [GODEMENT CRITERION]

LET M BE A SMOOTH MANIFOLD & LET $\sim$ BE AN EQUIVALENCE RELATION on M. THERE EXISTS A SMOOTH STRUCTURE on $M/\sim$ COMPATIBLE with the QUOTIENT TOPOLOGY, SUCH THAT $\pi : M \to M/\sim$ is A SUBMERSION iff THE GRAPH $R \subset M \times M$ IS A CLOSED EMBEDDED SUBMANIFOLD & the RESTRICTION of the CANONICAL PROJECTION $pr_1 : M \times M \to M$ to R IS A SUBMERSION.

NB: A PROOF of THE THEOREM CAN BE FOUND in R.L. FERNANDES's LECTURE NOTES on "DIFFERENTIAL GEOMETRY".

IV.2 Oïdisation of the Standard Model . . .

Motivation: The group symmetry model for an infinite rectangular tiling of $\mathbb{R}^{\times 2}$
[after Weinstein]

VS

The non-group symmetry model for its finite region

Definition 22. *A* **groupoid** *is a small category with all morphisms invertible.* Thus, *it is a septuple* $\mathbf{Gr} = (\mathrm{Ob}\mathbf{Gr}, \mathrm{Mor}\mathbf{Gr}, s, t, \mathrm{Id}, \mathrm{Inv}, \mathrm{m} \equiv .)$ *composed of a pair of sets:*

- *the* ***object set*** $\mathrm{Ob}\mathbf{Gr}$*;*
- *the* ***arrow set*** $\mathrm{Mor}\mathbf{Gr}$*,*

and a quintuple of ***structure maps****:*

- *the* ***source map*** $s\colon \mathrm{Mor}\mathbf{Gr} \longrightarrow \mathrm{Ob}\mathbf{Gr}$*;*
- *the* ***target map*** $t\colon \mathrm{Mor}\mathbf{Gr} \longrightarrow \mathrm{Ob}\mathbf{Gr}$*;*
- *the* ***unit map*** $\mathrm{Id}\colon \mathrm{Ob}\mathbf{Gr} \longrightarrow \mathrm{Mor}\mathbf{Gr}\colon m \longmapsto \mathrm{Id}_m$*;*
- *the* ***inverse map*** $\mathrm{Inv}\colon \mathrm{Mor}\mathbf{Gr} \longrightarrow \mathrm{Mor}\mathbf{Gr}\colon g \longmapsto \mathrm{Inv}(g) \equiv g^{-1}$*;*
- *the* ***multiplication map*** $\mathrm{m}\colon \mathrm{Mor}\mathbf{Gr}\,{}_s\!\times_t \mathrm{Mor}\mathbf{Gr} \longrightarrow \mathrm{Mor}\mathbf{Gr}\colon (g,h) \longmapsto \mathrm{m}(g,h) \equiv g.h$*,*

defined in terms of the subset $\mathrm{Mor}\mathbf{Gr}\,{}_s\!\times_t \mathrm{Mor}\mathbf{Gr}$ *of composable morphisms,*

$$\mathrm{Mor}\mathbf{Gr}\,{}_s\!\times_t \mathrm{Mor}\mathbf{Gr} = \{\, (g,h) \in \mathrm{Mor}\mathbf{Gr} \times \mathrm{Mor}\mathbf{Gr} \quad | \quad s(g) = t(h) \,\} \equiv \mathrm{Mor}\mathbf{Gr} \times_{\mathrm{Ob}\mathbf{Gr}} \mathrm{Mor}\mathbf{Gr}\,,$$

and subject to the conditions (in force whenever the expressions are well-defined):

(i) $s(g.h) = s(h),\ t(g.h) = t(g)$*;*
(ii) $(g.h).k = g.(h.k)$*;*
(iii) $\mathrm{Id}_{t(g)}.g = g = g.\mathrm{Id}_{s(g)}$ *;*
(iv) $s(g^{-1}) = t(g),\ t(g^{-1}) = s(g),\ g.g^{-1} = \mathrm{Id}_{t(g)},\ g^{-1}.g = \mathrm{Id}_{s(\vec{g})}$*.*

Thus, a groupoid is a (small) category with all morphisms invertible.

A **morphism** *between two groupoids* $\mathbf{Gr}_A,\ A \in \{1,2\}$ *is a functor* $\Phi\colon \mathbf{Gr}_1 \longrightarrow \mathbf{Gr}_2$*.*

A **Lie groupoid** *is a groupoid whose object and arrow sets are smooth manifolds, whose structure maps are smooth, and whose source and target maps are surjective submersions. A morphism between two Lie groupoids is a functor between them with smooth object and morphism components.*

We shall represent a Lie groupoid $\mathbf{Gr}$ *by a (non-commutative) diagram*

$$\begin{tikzcd} \mathrm{Mor}\mathbf{Gr} \times_{\mathrm{Ob}\mathbf{Gr}} \mathrm{Mor}\mathbf{Gr} \arrow[r, "\mathrm{m}"] & \mathrm{Mor}\mathbf{Gr} \arrow[r, "\mathrm{Inv}"] & \mathrm{Mor}\mathbf{Gr} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathrm{Ob}\mathbf{Gr} \arrow[l, bend right, "\mathrm{Id}"'] \end{tikzcd}.$$

E.g.1

Example 23. *A Lie group* G *can be viewed as a Lie groupoid* $(\{\bullet\}, \mathrm{G}, \bullet, \bullet, \bullet \longmapsto e, \mathrm{Inv}_{\mathrm{G}}, \cdot)$ *with the singleton* $\{\bullet\}$ *as the object manifold.* $=: \mathcal{G}$

Example 24. *An important example of a Lie groupoid is provided by the* **pair groupoid** $\mathrm{Pair}(M)$ *of a smooth manifold* $\mathrm{Ob}\,(\mathrm{Pair}(M)) = M$*, with the object manifold* M *and the arrow manifold* $\mathrm{Mor}\,(\mathrm{Pair}(M)) = M \times M$*, the source map* $s = \mathrm{pr}_2$ *and the target map* $t = \mathrm{pr}_1$ *given by the canonical projections, the composition of morphisms*

$$\mathrm{m}\colon (M \times M)_{\mathrm{pr}_2} \times_{\mathrm{pr}_1} (M \times M) \longrightarrow M \times M \colon \big((m_3, m_2), (m_2, m_1)\big) \longmapsto (m_2, m_1)\,,$$

the unit map

$$\mathrm{Id}\colon M \longrightarrow M \times M \colon m \longmapsto (m, m)\,,$$

and the inversion map

$$\mathrm{Inv}\colon M \times M \longrightarrow M \times M \colon (m_2, m_1) \longmapsto (m_1, m_2)\,.$$

The pair groupoid contains, as a proper Lie subgroupoid, the **Lie groupoid of** M *obtained through restriction of the arrow manifold to the diagonal* $M \times_M M \equiv M$*.* – see. Ex.25.

Whenever M *is the total space of a fibre bundle* (M, Σ, F, π_M) *over a base* Σ*, the pair groupoid* $\mathrm{Pair}(M)$ *contains, as a proper Lie subgroupoid, the* **Σ-fibred pair groupoid** $\mathrm{Pair}_\Sigma(M)$ *of* M*, with the arrow manifold*

$$\mathrm{Mor}\,(\mathrm{Pair}_\Sigma(M)) = M \times_\Sigma M \equiv \{\, (m_1, m_2) \in M \times M \quad | \quad \pi_M(m_1) = \pi_M(m_2) \,\}\,.$$

Example 25. *Another Lie groupoid of relevance is the* **action groupoid** $\mathrm{G} \ltimes_\lambda M$ *associated with a smooth action*

$$\lambda\colon \mathrm{G} \times M \longrightarrow M \colon m \longmapsto \lambda_g(m) \equiv g \rhd m \equiv g.m$$

of a Lie group G *on a smooth manifold* M*, with the object manifold* $\mathrm{Ob}\,(\mathrm{G} \ltimes_\lambda M) = M$ *and the arrow manifold* $\mathrm{Hom}\,(\mathrm{G} \ltimes_\lambda M) = \mathrm{G} \times M$*, the source map* $s = \mathrm{pr}_2$ *given by the canonical projection, the target map* $t = \lambda$ *given by the smooth (left) action* λ*, the composition of morphisms*

$$\mathrm{m}\colon (\mathrm{G} \times M)_{\mathrm{pr}_2} \times_\lambda (\mathrm{G} \times M) \longrightarrow \mathrm{G} \times M \colon \big((h, g.m), (g, m)\big) \longmapsto (h \cdot g, m) =: (h, g.m).(g, m)\,,$$

the unit map ($e \in \mathrm{G}$ *is the group unit)*

$$\mathrm{Id}\colon M \longrightarrow \mathrm{G} \times M \colon m \longmapsto \mathrm{id}_m = (e, m)\,,$$

and – finally – the inversion map

$$\mathrm{Inv}\colon \mathrm{G} \times M \longrightarrow \mathrm{G} \times M \colon (g, m) \longmapsto \big(g^{-1}, g.m\big) =: \mathrm{Inv}(g, m) \equiv (g, m)^{-1}\,.$$

EXAMPLE 26. EVERY MANIFOLD M CAN BE VIEWED as A LIE GROUPOID $(M, M, \mathrm{id}_M, \mathrm{id}_M, \mathrm{id}_M, \mathrm{id}_M, \mathrm{id}_M) =: \widehat{M}$.

(with FIXED ENDPOINTS)

EXAMPLE 27. UPON ENDOWING THE SET $\Pi(M)$ of HOMOTOPY CLASSES of SMOOTH PATHS $\gamma : [0,1] \to M$ in $M \in \mathbf{Man}$ with SMOOTH STRUCTURE (∞-DIM.), WHICH WE SHALL NOT DISCUSS, ONE OBTAINS THE FUNDAMENTAL GROUPOID of M:

$$\left(M, \Pi(M), ev_0, ev_1, M \ni m \mapsto m \in \Pi(M), \Pi(M) \ni [\gamma] \mapsto [\gamma \circ (1-\cdot)], \text{CONCATENATION of PATHS}\right) =: \mathrm{Fund}(M).$$

EXAMPLE 28. FOR ANY LIE GROUPOID $Gr = (M, G, s, t, Id, Inv, m)$, WE HAVE A CANONICAL MORPHISM

$$
Gr : \begin{tikzcd} G \arrow[r, "(t,s)"] \arrow[d, shift left] \arrow[d, shift right] & M \times M \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, equal] & M \end{tikzcd} : Pair(M),
$$

CALLED THE GROUPOID ANCHOR.

EXAMPLE 29. GIVEN LIE GROUP G, THE DIVISION MAP

$$\delta : G \times G \to G : (g, h) \mapsto g \cdot h^{-1}$$

DEFINES A LIE-GROUPOID MORPHISM

$$
Pair(G) : \begin{tikzcd} G \times G \arrow[r, "\delta"] \arrow[d, shift left] \arrow[d, shift right] & G \arrow[d, shift left] \arrow[d, shift right] \\ G \arrow[r, dashed] & * \end{tikzcd} : G
$$

EXAMPLE 30. Given $G_1, G_2 \in \mathrm{LieGrp}$ & $M_1, M_2 \in \mathrm{Man}$ with $\lambda^A \in \mathrm{Hom}_{\mathrm{Grp}}(G_A, \mathrm{Diff}(M_A))$, $A \in \{1,2\}$, every pair (φ, f) composed of $\varphi \in \mathrm{Hom}_{\mathrm{LieGrp}}(G_1, G_2)$ & $f \in C^\infty(M_1, M_2)$ s.t. $\forall g \in G_1 : f \circ \lambda^1_g = \lambda^2_{\varphi(g)} \circ f$ is a Lie-groupoid morphism

$$\begin{tikzcd}
G_1 \times M_1 \arrow[r, "\varphi \times f"] \arrow[d, shift left] \arrow[d, shift right] & G_2 \times M_2 \arrow[d, shift left] \arrow[d, shift right] \\
M_1 \arrow[r, "f"'] & M_2
\end{tikzcd}$$